Received 10/02/12

A NOTE CONCERNING SEVERAL HERMITE- HADAMARD INEQUALITIES FOR DIFFERENT TYPES OF CONVEX FUNCTIONS

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ABSTRACT. The goal of this paper is to see which will be the forms of Hermite-Hadamard inequalities for different types of convex functions if in its proofs we consider the Holder′s inequality having the form from Theorem 2. Also a Lemma and left-Hadamard inequality for double differentiable m- convex functions and (α, m)-convex functions will be presented.

1. INTRODUCTION

We need to recall the well-known Holder′s integral inequality which can be stated as follows, see [14], [10] and then Theorem 2.1, see [10].

Theorem 1. *If $f(x) \geq 0$, $g(x) \geq 0$ and $f(x) \in L^p[a,b]$, $g(x) \in L^q[a,b]$ and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{1}$$

Theorem 2. *If the conditions of Theorem 1 are satisfied and $t > 0$ then*

$$\int_a^b f(x)g(x)dx \leq C(p,t) \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{2}$$

where $C(p,t) = \frac{1}{p} t^{\frac{1}{p}-1} + (1 - \frac{1}{p}) t^{\frac{1}{p}}$.

We also need to recall the definition of s-convex functions in the second sense and Theorem 2.3 from [1].

Definition 1. *A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$ is said to be s-convex on I if the inequality $f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$ holds for all x, $y \in I$ and α, $\beta \in [0,1]$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$.*

Date: July 20, 2011.
2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. s-convex functions, quasi-convex functions, Hermite-Hadamard inequalities.

Theorem 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f^{'} \in L[a,b]$, where $a,\ b \in I$ with $a < b$. If $|f^{'}|^{p/(p-1)}$, $(p > 1)$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx| \leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}[((2^{1-s}+s+1)|f^{'}(a)|^q+$$

$$+2^{1-s}|f^{'}(b)|^q)^{\frac{1}{q}}+(2^{1-s}|f^{'}(a)|^q+(2^{1-s}+s+1)|f^{'}(b)|^q)^{\frac{1}{q}}], \quad (3)$$

where p is the conjugate of q, $q = p/(p-1)$.

Now we recall the notion of quasi-convex functions which also generalizes the notion of convex function and then we present Theorem 2.3 from [2].

Definition 2. *A function $f : [a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if*

$$f(\lambda x + (1-\lambda)y) \leq \sup\{f(x), f(y)\},$$

for all $x,\ y \in [a,b]$ and $\lambda \in [0,1]$.

Theorem 4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f^{'} \in L[a,b]$, where $a,\ b \in I$ with $a < b$. If $|f^{'}|^{p/(p-1)}$ is quasi-convex on $[a,b]$ for $p > 1$ then we have:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}[(\sup\{|f^{'}(\frac{a+b}{2})|^{\frac{p}{p-1}}, |f^{'}(b)|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}}+$$

$$+(\sup\{|f^{'}(\frac{a+b}{2})|^{\frac{p}{p-1}}, |f^{'}(a)|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}}].$$

Definition 3. *The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex where $m \in [0,1]$ if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have:*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

As a generalization of previous notion we have the following:

Definition 4. *The function $f : [0,b] \to \mathbb{R},\ b > 0$ is said to be (α, m) -convex, where $(\alpha, m) \in [0,1]^2$ if we have*

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Definition 5. *A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$ if for every $x,\ y \in S$ and $t \in [0,1]$*

$$y + t\eta(x,y) \in S.$$

Definition 6. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Then the function $f : S \to \mathbb{R}$ is said to be prequasiinvex with respect to η if for every $x,\ y \in S$ and $t \in [0,1]$*

$$f(y + t\eta(x,y)) \leq \max\{f(x), f(y)\}.$$

Definition 7. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function (or belongs to the class SX(h,I))) if f is non-negative and*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

for all $x,\ y \in I$ and $t \in (0, 1)$.

Definition 8. *Let $I \subseteq \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ is said to belong to the class $P(I)$ (or P- convex) if it is nonnegative and for all $x,\ y \in I$ and $\lambda \in [0, 1]$ satisfies the inequality*

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y).$$

2. Hermite-Hadamard's type inequalities for s-convex functions in the second sense

Theorem 5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a,\ b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$, $(p > 1)$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}[C(p,l_1)((2^{1-s}+s+1)\cdot$$

$$\cdot|f'(a)|^q + 2^{1-s}|f'(b)|^q)^{\frac{1}{q}} + C(p,l_2)(2^{1-s}|f'(a)|^q + (2^{1-s}+s+1)|f'(b)|^q)^{\frac{1}{q}}],$$

where p is the conjugate of q, $q = p/(p-1)$, and $C(p, l) = \frac{1}{p}l^{\frac{1}{p}-1} + (1-\frac{1}{p})l^{\frac{1}{p}}$, $l > 0$.

Proof. As in the proof of Theorem 2.3 from [1], we have

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{b-a}{4}[\int_0^1 t|f'(t\frac{a+b}{2} + (1-t)a)|dt+$$

$$+\int_0^1 (1-t)|f'(tb + (1-t)\frac{a+b}{2})|dt].$$

Using now Theorem 2 and then the fact that $|f'|^q$ is s-convex we shall obtain

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{b-a}{4}[C(p,l_1)(\int_0^1 t^p dt)^{\frac{1}{p}}\cdot(\int_0^1 |f'(t\frac{a+b}{2}+(1-t)a)|^q dt)^{\frac{1}{q}}+$$

$$+C(p,l_2)(\int_0^1 (1-t)^p dt)^{\frac{1}{p}}(\int_0^1 |f'(tb + (1-t)\frac{a+b}{2})|^q dt)^{\frac{1}{q}}].$$

By calculus using that if $|f'|^q$ is s-convex on $[a, b]$, for any $t \in [0, 1]$ then $2^{s-1}|f'(\frac{a+b}{2})| \leq \frac{|f'(a)|+|f'(b)|}{s+1}$, see [1] we obtain the desired inequality.
∎

Using the above theorem we can find the following result:

Corollary 1. *If the conditions of Corollary 2.1, see* [1] *are satisfied then the following inequality holds:*

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx| \le (\frac{b-a}{4})\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}[(C(p,l_1)(2^{1-s}+s+1)^{\frac{1}{q}}+$$

$$+C(p,l_2)2^{(1-s)/q})|f^{'}(a)|+(C(p,l_1)2^{(1-s)/q}+C(p,l_2)(2^{1-s}+s+1)^{\frac{1}{q}})|f^{'}(b)|],$$

where $l_1, l_1 > 0$ *and* $C(p,l)$ *is as in Theorem 2.*

Proof. Applying inequality from Theorem 5 and using the fact

$$\sum_{i=1}^{n}(a_i+b_i)^r \le \sum_{i=1}^{n}a_i^r + \sum_{i=1}^{n}b_i^r$$

when $0 < r < 1,\ a_1,\ a_2,...a_n \ge 0,\ b_1,\ b_2,...,b_n \ge 0$ for $a_1 = C(p,l_1)^q(2^{1-s}+s+1)|f^{'}(a)|^q,\ b_1 = C(p,l_1)^q 2^{1-s}|f^{'}(b)|^q,\ a_2 = C(p,l_2)^q 2^{1-s}|f^{'}(a)|^q$ and $b_2 = C(p,l_2)^q(2^{1-s}+s+1)|f^{'}(b)|^q$ we have the inequalities

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx| \le (\frac{b-a}{4})\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}[C(p,l_1)((2^{1-s}+s+1)\cdot$$

$$\cdot|f^{'}(a)|^q+2^{1-s}|f^{'}(b)|^q)^{\frac{1}{q}}+C(p,l_2)(2^{1-s}|f^{'}(a)|^q+(2^{1-s}+s+1)|f^{'}(b)|^q)^{\frac{1}{q}}] \le$$

$$\le (\frac{b-a}{4})\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}[(C(p,l_1)(2^{1-s}+s+1)^{\frac{1}{q}}+C(p,l_2)2^{(1-s)/q})|f^{'}(a)|+$$

$$+(C(p,l_1)2^{(1-s)/q}+C(p,l_2)(2^{1-s}+s+1)^{\frac{1}{q}})|f^{'}(b)|].$$

∎

As an application to Special Means and also an analogue of Proposition 3.2, see [1], we have:

Remark 1. *If* $a,b \in I^{\circ},\ a < b$ *and* $0 < s < 1$ *then for all* $q > 1$ *the following inequality hold:*

$$|L_s^s(a,b)-A^s(a,b)| \le s(\frac{b-a}{4})\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}[(C(p,l_1)(2^{1-s}+s+1)^{\frac{1}{q}}+$$

$$+C(p,l_2)2^{(1-s)/q})|a|^{s-1}+(C(p,l_1)2^{(1-s)/q}+C(p,l_2)(2^{1-s}+s+1)^{\frac{1}{q}})|b|^{s-1}],$$

where $A(\alpha,\beta) = \frac{\alpha+\beta}{2},\ \alpha,\ \beta \in \mathbb{R}$ *is arithmetic mean and* $L_n(\alpha,\beta) = [\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}]^{\frac{1}{n}},\ n \in \mathbb{Z}-\{-1,0\},\ \alpha,\ \beta \in \mathbb{R}, \alpha \ne \beta$ *is generalized log-mean.*

Remark 2. *If we take above* $l_1 = l_2 = 1$ *we obtain Proposition 3.2,* [1].

An analogue of Theorem 1, see [13] will be presented here, using Theorem 2 instead of well-known Holder′s inequality.

Consequence 1. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f^{'} \in L^1[a,b]$*, where* $a,\ b \in I$ *with* $a < b$*. If* $|f^{'}|^q$ *is s-convex on* $[a,b]$*, for some fixed* $s \in (0,1)$ *and* $q > 1$*, then the following inequality holds:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \le$$

$$\leq C(p,l)\frac{(b-a)}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{s+\left(\frac{1}{2}\right)^{s}}{(s+1)(s+2)}\right]^{\frac{1}{q}}(|f^{'}(a)|^{q}+|f^{'}(b)|^{q})^{\frac{1}{q}},$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Now, we will give a variant of Theorem 2, see [13] for concave functions.

Theorem 6. *Let* $f: I \to \mathbb{R},\ I \in \mathbb{R}$ *be a differentiable function on* I° *such that* $f^{'} \in L^{1}[a,b]$, *where* $a,\ b \in I$ *with* $a<b$. *If* $|f^{'}|^{q}$ *is concave on* $[a,b]$ *for some* $q>1$ *then:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}(C(p,l_1)|f^{'}(\frac{a+3b}{4})|+C(p,l_2)|f^{'}(\frac{3a+b}{4})|),$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

3. Hermite-Hadamard's type inequalities for quasi-convex functions

The following result is an analogue of Theorem 2.3, [2].

Theorem 7. *Let* $f: I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f^{'} \in L[a,b]$, *where* $a,\ b \in I$ *with* $a<b$. *If* $|f^{'}|^{p/(p-1)}$ *is quasi-convex on* $[a,b]$ *for* $p>1$ *then we have:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}[C(p,l_1)(\sup\{|f^{'}(\frac{a+b}{2})|^{\frac{p}{p-1}},|f^{'}(b)|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}}$$

$$+C(p,l_2)(\sup\{|f^{'}(\frac{a+b}{2})|^{\frac{p}{p-1}},|f^{'}(a)|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}}],$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Proof. As in the proof of Theorem 2.3, [2] we use Lemma 2.1 ([2]) and Theorem 2. ∎

Remark 3. *If* f *is as in above theorem, and:*
(i) $|f^{'}|^{p/(p-1)}$ *is increasing then we have*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}[C(p,l_1)|f^{'}(b)|+C(p,l_2)|f^{'}(\frac{a+b}{2})|].$$

(ii) $|f^{'}|^{p/(p-1)}$ *is decreasing then we have*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}[C(p,l_1)|f^{'}(a)|+C(p,l_2)|f^{'}(\frac{a+b}{2})|].$$

As a consequence of the above theorem and an analogue of Proposition 4.2 [2] we can also present an application to special means of real numbers for quasi-convex functions.

Remark 4. *Let $a, b \in \mathbb{R}, a < b$ and 0 is not in the interval $[a, b]$. Then for all $p > 1$ we obtain*

$$L^{-1}(a,b) - A(a^{-1}, b^{-1}) \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \{[C(p,l_1) \sup(|\frac{a+b}{2}|^{-\frac{2p}{p-1}}, |a|^{-\frac{2p}{p-1}})]^{\frac{p-1}{p}} +$$

$$+C(p,l_2)[\sup(|\frac{a+b}{2}|^{-\frac{2p}{p-1}}, |b|^{-\frac{2p}{p-1}})]^{\frac{p-1}{p}}\},$$

where $L(\alpha, \beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}$, $|\alpha| \neq |\beta|$, $\alpha, \beta \neq 0$, $\alpha, \beta \in \mathbb{R}$ *is logarithmic mean.*

4. Hermite-Hadamard′s type inequalities for m-convex functions

We study the forms of several right-hand side of Hermite-Hadamard inequalities for $q > 1$ for functions whose absolute values of second derivatives raised to positive real powers are m-convex given in [15], Theorem 2, Theorem 3, Theorem 4.

If in the proof of these three theorems we shall apply Theorem 2 instead of well-known Holder′s inequality and of course we shall use Lemma 1, see[15] we shall obtain the next three results.

Consequence 2. *Let $f : I^\circ \to \mathbb{R}$, $I^\circ \subset [0, \infty)$ be a twice differentiable function on I° such that $f^{''} \in L[a,b]$, where a, $b \in I$, $a < b$. If $|f^{''}|^q$ is m-convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q > 1$ then the following inequality holds:*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq C(p,l)\frac{(b-a)^2}{12}\left[\frac{|f^{''}(a)|^q + m|f^{''}(\frac{b}{m})|^q}{2}\right]^{\frac{1}{q}}$$

where $C(p, l)$, $l > 0$ is as in Theorem 2.

Consequence 3. *Let $f : I^\circ \to \mathbb{R}$, $I^\circ \subset [0, \infty)$ be a twice differentiable function on I° such that $f^{''} \in L[a,b]$, where a, $b \in I$, $a < b$. If $|f^{''}|^q$ is m-convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q > 1$ then the following inequality holds:*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq C(p,l)\frac{(b-a)^2}{8}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left[\frac{|f^{''}(a)|^q + m|f^{''}(\frac{b}{m})|^q}{2}\right]^{\frac{1}{q}}$$

where $C(p, l)$, $l > 0$ is as in Theorem 2.

Consequence 4. *With the assumptions of the above theorem we have the inequality:*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq C(p,l)\frac{(b-a)^2}{2}\left[|f^{''}(a)|^q + m(q+1)|f^{''}(\frac{b}{m})|^q\right]^{\frac{1}{q}},$$

where $C(p, l)$, $l > 0$ is as in Theorem 2.

Lemma 1. *Let $f : I^{\circ} \to \mathbb{R},\ I^{\circ} \subset [0,\infty)$ be a twice differentiable function on I° where $a,\ b \in I,\ a < b$. If $f^{''} \in L[a,b]$, then the following equality holds:*

$$-f(\frac{a+b}{2}) + \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{16}[\int_0^1 t^2 f^{''}(t\frac{a+b}{2} + (1-t)a)dt +$$

$$+\int_0^1 (t-1)^2 f^{''}(tb + (1-t)\frac{a+b}{2})dt].$$

Proof. We note that

$$I_1 = \int_0^1 \frac{t^2}{2} f^{''}(t\frac{a+b}{2} + (1-t)a)dt = \frac{1}{b-a}f^{'}(\frac{a+b}{2}) - \frac{4}{(b-a)^2}f(\frac{a+b}{2}) +$$

$$+\frac{4}{(b-a)^2}f(t\frac{a+b}{2} + (1-t)a)dt = \frac{1}{b-a}f^{'}(\frac{a+b}{2}) - \frac{4}{(b-a)^2}f(\frac{a+b}{2}) +$$

$$+\frac{8}{(b-a)^3}\int_a^{\frac{a+b}{2}} f(x)dx$$

where we used the substitution $x = t\frac{a+b}{2} + (1-t)a$ and $dx = \frac{b-a}{2}dt$. Similarly, we can obtain

$$I_2 = \int_0^1 \frac{(t-1)^2}{2} f^{''}(tb + (1-t)\frac{a+b}{2})dt = -\frac{1}{b-a}f^{'}(\frac{a+b}{2}) - \frac{4}{(b-a)^2}f(\frac{a+b}{2}) +$$

$$+\frac{4}{(b-a)^2}f(tb + (1-t)\frac{a+b}{2})dt = -\frac{1}{b-a}f^{'}(\frac{a+b}{2}) - \frac{4}{(b-a)^2}f(\frac{a+b}{2}) +$$

$$+\frac{8}{(b-a)^3}\int_{\frac{a+b}{2}}^{b} f(x)dx$$

where we used substitution $x = tb + (1-t)\frac{a+b}{2}$ and $dx = \frac{b-a}{2}dt$.

Therefore

$$I_1 + I_2 = -\frac{8}{(b-a)^2}f(\frac{a+b}{2}) + \frac{8}{(b-a)^3}\int_a^b f(x)dx$$

or

$$\frac{(b-a)^2}{8}(I_1 + I_2) = -f(\frac{a+b}{2}) + \frac{1}{b-a}\int_a^b f(x)dx$$

and that completes the proof.

∎

Now we will give a left Hermite-Hadamard inequality and some consequences for double differentiable m-convex functions.

Theorem 8. *Let $f : I^{\circ} \to \mathbb{R},\ I^{\circ} \subset \mathbb{R}$ be a twice differentiable function on I° such that $f^{''} \in L^1[a,b]$, where $a,\ b \in I,\ a < b$. If $|f^{''}|$ is m-convex on $[a,b]$ for some fixed $m \in (0,1]$ and $q > 1$ then the following inequality holds:*

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \le \frac{(b-a)^2}{64}\left[2|f^{''}(\frac{a+b}{2})| + \frac{m}{3}\left(|f^{''}(\frac{a}{m})| + |f^{''}(\frac{b}{m})|\right)\right].$$

Proof. From Lemma 1 we have

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx| \le \frac{(b-a)^2}{16}[\int_0^1 t^2|f^{''}(t\frac{a+b}{2}+(1-t)a)|dt+$$

$$+\int_0^1 (t-1)^2|f^{''}(tb+(1-t)\frac{a+b}{2})|dt] \le \frac{(b-a)^2}{16}[\int_0^1 t^2(t|f^{''}(\frac{a+b}{2})|+m(1-t)|f^{''}(\frac{a}{m})|)dt$$

$$+\int_0^1 (t-1)^2(tm|f^{''}(\frac{b}{m})|+(1-t)|f^{''}(\frac{a+b}{2})|)dt].$$

By calculuses we will obtain the right member of the inequality from theorem. ∎

We recall inequality (1.6) from Theorem 3([19]):

$$f(\frac{a+b}{2})(b-a) \le \frac{(b-a)}{8}[f(a)+f(b)+2m(f(\frac{a}{m})+f(\frac{b}{m}))+m^2(f(\frac{a}{m^2})+f(\frac{b}{m^2}))]$$

If we apply this inequality, under the conditions of Theorem 3, taking into account that $|f^{''}|$ is m-convex then we have:

Consequence 5. *Let $f : I^\circ \to \mathbb{R}$, $I^\circ \subset [0,\infty)$ be a twice differentiable function on I° such that $f^{''} \in L^1[a,b]$, where a, $b \in I$, $a < b$. If $|f^{''}|$ is m-convex on $[a,b]$ for some fixed $m \in (0,1]$ and $q > 1$ then the following inequality holds:*

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx| \le \frac{(b-a)^2}{64}[\frac{1}{4}\left(|f^{''}(a)|+|f^{''}(b)|\right)+$$

$$+\frac{5m}{6}\left(|f^{''}(\frac{a}{m})|+|f^{''}(\frac{b}{m})|\right)+\frac{m^2}{4}\left(|f^{''}(\frac{a}{m^2})|+|f^{''}(\frac{b}{m^2})|\right)].$$

If we apply the inequality from Theorem 4, see [11],

$$f(\frac{a+b}{2}) \le \frac{m}{4}[\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}],$$

when $0 \le a < b \le \infty$, $f \in L^1[a,b]$ then we obtain:

Consequence 6. *Let $f : I^\circ \to \mathbb{R}$, $I^\circ \subset [0,\infty)$ be a twice differentiable function on I° such that $f^{''} \in L^1[a,b]$, where a, $b \in I$, $a < b$. If $|f^{''}|$ is m-convex on $[a,b]$ for some fixed $m \in (0,1]$ and $q > 1$ then the following inequality holds:*

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx| \le \frac{(b-a)^2}{64}[\frac{m+1}{4}(|f^{''}(a)|+|f^{''}(b)|)+$$

$$+\frac{m(3m+7)}{12}\cdot(|f^{''}(\frac{a}{m})|+|f^{''}(\frac{b}{m})|)].$$

5. Hermite-Hadamard's type inequalities for (α, m)-convex functions

Theorem 9. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, b]$ where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f''|$ is (α, m) -convex on $[a, b]$ for $(\alpha, m) \in [0, 1] \times [0, 1]$ then the following inequality holds:*

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{16}\min\{A_1, A_2\}$$

where

$$A_1 = C(p, l_1)\left(\frac{1}{\alpha+3}|f''(\frac{a+b}{2})| + m(\frac{1}{3} - \frac{1}{\alpha+3})|f''(\frac{a}{m})|\right) + C(p, l_2)\cdot$$

$$\cdot\left(m(\frac{1}{\alpha+3} - \frac{2}{\alpha+2} + \frac{1}{\alpha+1})|f''(\frac{b}{m})| + (\frac{1}{3} - \frac{1}{\alpha+3} + \frac{2}{\alpha+2} - \frac{1}{\alpha+1})|f''(\frac{a+b}{2})|\right)$$

$$A_2 = C(p, l_1)\left(\frac{m}{\alpha+3}|f''(\frac{a+b}{2m})| + (\frac{1}{3} - \frac{1}{\alpha+3})|f''(a)|\right) + C(p, l_2)\cdot$$

$$\cdot\left(\frac{1}{\alpha+3} - \frac{2}{\alpha+2} + \frac{1}{\alpha+1}|f''(b)| + m(\frac{1}{3} - \frac{1}{\alpha+3} + \frac{2}{\alpha+2} - \frac{1}{\alpha+1})|f''(\frac{a+b}{2m})|\right)$$

,$\frac{1}{p} + \frac{1}{q} = 1$ and $C(p, l)$, $l > 0$ is as in Theorem 2.

Theorem 10. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, b]$ where $a, b \in I$ with $a < b, b^* > 0$. If $|f''|^{\frac{p}{p-1}}$ is (α, m) -convex on $[a, b]$ for $(\alpha, m) \in [0, 1] \times [0, 1]$ and $p > 1$ then the following inequality holds:*

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\min\{B_1, B_2\}$$

where

$$B_1 = C(p, l_1)\left(|f''(\frac{a+b}{2})|^q + m\alpha|f''(\frac{a}{m})|^q\right)^{\frac{1}{q}} + C(p, l_2)\left(m|f''(\frac{b}{m})|^q + \alpha|f''(\frac{a+b}{2})|^q\right)^{\frac{1}{q}}$$

$$B_2 = C(p, l_1)\left(m|f''(\frac{a+b}{2m})|^q + \alpha|f''(a)|^q\right)^{\frac{1}{q}} + C(p, l_2)\left(|f''(b)|^q + \alpha m|f''(\frac{a+b}{2m})|^q\right)^{\frac{1}{q}}$$

,$\frac{1}{p} + \frac{1}{q} = 1$ and $C(p, l)$, $l > 0$ is as in Theorem 2.

Proof. Using Lemma 1 as in the proof of Theorem 8 and applying then Theorem 2 we obtain:

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{16}[C(p, l_1)(\int_0^1 t^{2p}dt)^{\frac{1}{p}}(\int_0^1 |f''(t\frac{a+b}{2} + (1-t)a)|^q dt)^{\frac{1}{q}}$$

$$+C(p, l_2)(\int_0^1 (t-1)^{2p}dt)^{\frac{1}{p}}(\int_0^1 |f''(tb + (1-t)\frac{a+b}{2})|^q dt)^{\frac{1}{q}}] \leq$$

$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}[C(p, l_1)(\int_0^1 (t^\alpha|f''(\frac{a+b}{2}|^q + m(1-t^\alpha)|f''(\frac{a}{m})|^q)dt)^{\frac{1}{q}} +$$

$$+C(p, l_2)(\int_0^1 (mt^\alpha|f''(\frac{b}{m})|^q + (1-t^\alpha)|f''(\frac{a+b}{2})|^q)dt)^{\frac{1}{q}}] = \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}(\frac{1}{\alpha+1})^{\frac{1}{q}}B_1.$$

Analogously, we will obtain

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}(\frac{1}{\alpha+1})^{\frac{1}{q}}B_2.$$

∎

Now if we apply Theorem 2 in the proof of previous theorem as below we will obtain the following result:

Theorem 11. *Let $f:I\subset[0,b^*]\to\mathbb{R}$ be a twice differentiable function on I° such that $f^{''}\in L[a,b]$ where $a,b\in I$ with $a<b,b^*>0$. If $|f^{''}|^{\frac{p}{p-1}}$ is (α,m) -convex on $[a,b]$ for $(\alpha,m)\in[0,1]\times[0,1]$ and $p>1$ then the following inequality holds:*

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{(b-a)^2}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\min\{C_1,C_2\}$$

where

$$C_1=C(p,l_1)\left(\frac{1}{\alpha+2}|f^{''}(\frac{a+b}{2})|^q+m(\frac{1}{2}-\frac{1}{\alpha+2})|f^{''}(\frac{a}{m})|^q\right)^{\frac{1}{q}}+C(p,l_2)\cdot$$

$$\cdot\left(m(\frac{1}{\alpha+1}-\frac{1}{\alpha+2})|f^{''}(\frac{b}{m})|^q+(1-\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{\alpha+2})|f^{''}(\frac{a+b}{2})|^q\right)^{\frac{1}{q}}$$

$$C_2=C(p,l_1)\left(m\frac{1}{\alpha+2}|f^{''}(\frac{a+b}{2m})|^q+(1-\frac{1}{\alpha+1})|f^{''}(a)|^q\right)^{\frac{1}{q}}+C(p,l_2)\cdot$$

$$\cdot\left((\frac{1}{2}-\frac{1}{\alpha+2})|f^{''}(b)|^q+m(1-\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{\alpha+2})|f^{''}(\frac{a+b}{2m})|^q\right)^{\frac{1}{q}}$$

, $\frac{1}{p}+\frac{1}{q}=1$ and $C(p,l),\ l>0$ is as in Theorem 2.

6. Hermite-Hadamard's type inequalities for P-convex functions

Theorem 12. *Let $f:I\to\mathbb{R}$ be a differentiable function on I°. Assume that $p\in\mathbb{R},\ p>1$ such that $|f^{''}|^{\frac{p}{p-1}}|$ is a P - convex function on I. Suppose that $a,\ b\in I^\circ$ with $a<b$ and $f^{''}\in L^1[a,b]$. Then we have:*

$$|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{b-a}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}[C(p,l_1)(|f^{''}(\frac{a+b}{2})|^q+|f^{''}(a)|^q)^{\frac{1}{q}}+$$

$$+C(p,l_2)|f^{''}(\frac{a+b}{2})|^q+|f^{''}(b)|^q)^{\frac{1}{q}}]$$

where $\frac{1}{p}+\frac{1}{q}=1$, and $C(p,l),\ l>0$.

Proof. We apply Lemma 1 and Theorem 2.

∎

7. Other consequences for different type of convexities

In the following we will state several consequences which results by using a generalization of Holder′s inequality given in Theorem 2 or see [10].

Consequence 7. *Under the conditions of Theorem 2([21]) we have:*

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(b-a)}{4}[C(p,l_1)L_q(|f^{'}(a)|, \frac{|f^{'}(a)|+m|f^{'}(\frac{b}{m})|}{2})+$$

$$+C(p,l_2)L_q(\frac{|f^{'}(a)|+m|f^{'}(\frac{b}{m})|}{2}, m|f^{'}(\frac{b}{m})|)],$$

where $|f^{'}(a)| \neq m|f^{'}(\frac{b}{m})|$, $\frac{b}{m} < b^{*}$, L_q *is a q-logarithmic mean of positive real numbers and* $C(p,l)$, $l>0$ *is as in Theorem 2.*

Consequence 8. *Under the conditions of Theorem 3([21]) we have:*

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(b-a)}{4}\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}[C(p,l_1)(|f^{'}(a)|^q+$$

$$+3m|f^{'}(\frac{b}{m})|^q)^{\frac{1}{q}} + C(p,l_2)(3|f^{'}(a)|^q + m|f^{'}(\frac{b}{m})|^q)^{\frac{1}{q}}],$$

where $C(p,l)$, $l>0$ *is as in Theorem 2.*

A variant of a Theorem 8 ([20]), a Simpson′s type inequality based on s-convexity, is the following:

Consequence 9. *Under the conditions of Theorem 8([20]) we have:*

$$|\frac{1}{6}[f(a)+4f(\frac{a+b}{2})+f(b)] - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\cdot$$

$$\cdot[C(p,l_1)(\frac{|f^{'}(b)|^q+|f^{'}(\frac{a+b}{2})|^q}{s+1})^{\frac{1}{q}} + C(p,l_2)(\frac{|f^{'}(a)|^q+|f^{'}(\frac{a+b}{2})}{s+1}|^q)^{\frac{1}{q}}],$$

where $C(p,l)$, $l>0$ *is as in Theorem 2.*

A variant of Theorem 9 ([20]) which is another version of the Simpson′s type inequality for s-convex functions is given below:

Consequence 10. *Under the conditions of Theorem 9([20]) we have:*

$$|\frac{1}{6}[f(a)+4f(\frac{a+b}{2})+f(b)] - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\cdot$$

$$\cdot[C(p,l_1)(\frac{(2^{s+1}-1)|f^{'}(b)|^q+|f^{'}(a)|^q}{2^s(s+1)})^{\frac{1}{q}} + C(p,l_2)(\frac{(2^{s+1}-1)|f^{'}(a)|^q+|f^{'}(b)|^q}{2^s(s+1)})^{\frac{1}{q}}],$$

where $C(p,l)$, $l>0$ *is as in Theorem 2.*

Consequence 11. *Under the conditions of Theorem 9([16]) we have:*

$$|f(a)\int_a^x g(u)du+f(b)\int_x^b g(u)du-\int_a^b f(u)g(u)du|\leq$$

$$\leq ||g||_\infty C(p,l)[\frac{(x-a)^{p+1}+(b-x)^{p+1}}{p+1}]^{\frac{1}{q}}[2^{s-1}(b-a)|f^{'}(\frac{a+b}{2})|^q]^{\frac{1}{q}}$$

and

$$|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du|\leq$$

$$\leq ||g||_\infty C(p,l)[\frac{(x-a)^{p+1}+(b-x)^{p+1}}{p+1}]^{\frac{1}{q}}[2^{s-1}(b-a)|f^{'}(\frac{a+b}{2})|^q]^{\frac{1}{q}}$$

where $C(p,l),\ l>0$ is as in Theorem 2.

Consequence 12. *Under the conditions of Theorem 4([3]) we have:*

$$|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx|\leq$$

$$\leq \frac{|\theta(a,b)|C(p,l)}{2(p+1)^{\frac{1}{p}}}[\sup\{|f^{'}(a)|^{\frac{p}{p-1}},|f^{'}(b)|^{\frac{p}{p-1}}\}]^{\frac{p-1}{p}}$$

where $C(p,l),\ l>0$ is as in Theorem 2.

Consequence 13. *Under the conditions of Theorem 7([12]) we have:*

$$|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b dx|\leq$$

$$\leq \frac{(b-a)(\lambda+\mu)^{-1}]}{(p+1)(\lambda+\mu)^{\frac{1}{p}}}[C(p,l_1)\mu^{1+\frac{1}{p}}\left(\int_0^{\frac{\mu}{\lambda+\mu}}\{|f^{'}(a)|^q h(1-t)+|f^{'}(b)|^q h(t)\}dt\right)^{\frac{1}{q}}+$$

$$+C(p,l_2)\lambda^{1+\frac{1}{p}}\left(\int_{\frac{\mu}{\lambda+\mu}}^1\{|f^{'}(a)|^q h(1-t)+|f^{'}(b)|^q h(t)\}dt\right)^{\frac{1}{q}}]$$

where $C(p,l),\ l>0$ is as in Theorem 2.

Consequence 14. *Under the conditions of Theorem 2.2([4]) we have:*

$$|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx|\leq$$

$$\leq \frac{|\theta(a,b)|C(p,l)}{2(p+1)^{\frac{1}{p}}}[|f^{'}(a)|^{\frac{p}{p-1}}+|f^{'}(b)|^{\frac{p}{p-1}}]^{\frac{p-1}{p}}$$

where $C(p,l),\ l>0$ is as in Theorem 2.

Consequence 15. *Under the conditions of Theorem 2.2(*[5]*) we have:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a)}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}[C(p,l_1)(\max\{|f^{''}(a)|^q,|f^{''}(\frac{a+b}{2})|^q\})^{\frac{1}{q}}+$$

$$+C(p,l_2)(\max\{|f^{''}(b)|^q,|f^{''}(\frac{a+b}{2})|^q\})^{\frac{1}{q}}]$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Consequence 16. *Under the conditions of Theorem 2.2(*[6]*) we have:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a)}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)}{24}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}[C(p,l_1)(|f^{''}(a)|^q+|f^{''}(\frac{a+b}{2})|^q\})^{\frac{1}{q}}+$$

$$+C(p,l_2)(|f^{''}(b)|^q+|f^{''}(\frac{a+b}{2})|^q\})^{\frac{1}{q}}]$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

A variant of a Simpson′s type inequality for functions whose third derivatives in the absolute values are P- convex, see [7], is given below.

Consequence 17. *Under the conditions of Theorem 2.2(*[7]*) we have:*

$$|\int_a^b f(x)dx-\frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]|\leq\frac{2^{-\frac{1}{p}}(b-a)^4}{96}\left(\frac{\Gamma(1+p)\Gamma(1+2p)}{\Gamma(2+3p)}\right)^{\frac{1}{p}}\cdot$$

$$\cdot\{C(p,l_1)(|f^{'''}(a)|^q+|f^{'''}(\frac{a+b}{2}|^q)^{\frac{1}{q}}+C(p,l_2)(|f^{'''}(b)|^q+|f^{'''}(\frac{a+b}{2}|^q)^{\frac{1}{q}}\}$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Consequence 18. *Under the conditions of Theorem 2.4(*[8]*) we have:*

$$|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du|\leq\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot$$

$$\cdot\{C(p,l_1)\frac{(x-a)^2}{b-a}(|f^{'}(x)|^q+|f^{'}(a)|^q)^{\frac{1}{q}}+C(p,l_2)\frac{(b-x)^2}{b-a}(|f^{'}(x)|^q+|f^{'}(b)|^q)^{\frac{1}{q}}\}$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Consequence 19. *Under the conditions of Theorem 2(*[9]*) we have:*

$$|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(u)du|\leq\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right)\cdot$$

$$\cdot\{C(p,l_1)(|f^{'}(\frac{3a+b}{4})|^q+|f^{'}(a)|^q)^{\frac{1}{q}}+C(p,l_2)(|f^{'}(\frac{a+b}{2})|^q+|f^{'}(\frac{3a+b}{4})|^q)^{\frac{1}{q}}+$$

$$+C(p,l_3)(|f^{'}(\frac{a+3b}{4})|^q+|f^{'}(\frac{a+b}{2})|^q)^{\frac{1}{q}}+C(p,l_4)(|f^{'}(\frac{a+3b}{4})|^q+|f^{'}(b)|^q)^{\frac{1}{q}}\}$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Consequence 20. *Under the conditions of Theorem 4([9]) we have:*

$$|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(u)du|\leq\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{16}\right)\cdot$$

$$\cdot\{C(p,l_1)|f^{'}(\frac{7a+b}{8})|+C(p,l_2)|f^{'}(\frac{5a+3b}{8})|+C(p,l_3)(|f^{'}(\frac{3a+5b}{8})|+C(p,l_4)|f^{'}(\frac{a+7b}{8})|$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Some variants of two new inequalities related to right hand side of Hermite-Hadamard inequality for the classes of functions whose derivatives of absolute values are m- convex and (α,m) - convex, see [17] are presented.

Consequence 21. *Under the conditions of Theorem 7([17]) we have:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a)}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)}{4}\left(\frac{q-1}{2q-p-1)}\right)^{\frac{q-1}{q}}[C(\frac{p}{q},l_1)(\frac{2p+3}{2(p+1)(p+2)}|f^{'}(a)|^q+\frac{m}{2(p+1)(p+2)}|f^{'}(\frac{b}{m})|^q\})^{\frac{1}{q}}+$$

$$+C(\frac{p}{q},l_2)(\frac{2p+3}{2(p+1)(p+2)}|f^{'}(b)|^q+\frac{m}{2(p+1)(p+2)}|f^{'}(\frac{a}{m})|^q\})^{\frac{1}{q}}]$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Consequence 22. *Under the conditions of Theorem 9([17]) we have:*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a)}\int_a^b f(x)dx|\leq$$

$$\leq\frac{(b-a)}{4}\left(\frac{1}{p+1)}\right)^{\frac{1}{p}}\left(\frac{2^{\alpha+1}-1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}[C(p,l_1)(|f^{'}(a)|^q+m|f^{'}(\frac{b}{m})|^q\})^{\frac{1}{q}}+$$

$$+C(p,l_2)(|f^{'}(b)|^q+m|f^{'}(\frac{a}{m})|^q\})^{\frac{1}{q}}]$$

where $C(p,l),\ l>0$ *is as in Theorem 2.*

Consequence 23. *Under the conditions of Theorem 5([18]) we have:*

$$|f(\frac{a+b}{2})-\frac{1}{b-a)}\int_a^b f(x)dx|\leq\frac{b-a}{4(p+1)^{\frac{1}{p}}}\min\{Z_1,Z_2\}$$

where

$$Z_1=C(p,l_1)(\frac{1}{\alpha+2}|f^{'}(\frac{a+b}{2})|^q+m(\frac{1}{2}-\frac{1}{\alpha+2})|f^{'}(\frac{a}{m})|^q)^{\frac{1}{q}}+C(p,l_2)\cdot$$

$$\cdot(\frac{1}{(\alpha+2)(\alpha+2)}|f^{'}(\frac{a+b}{2})|^q+m(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)})|f^{'}(\frac{b}{m})|^q)^{\frac{1}{q}}$$

$$Z_2=C(p,l_1)(\frac{1}{\alpha+2}|f^{'}(\frac{a}{)}|^q+m(\frac{1}{2}-\frac{1}{\alpha+2})|f^{'}(\frac{a+b}{2m})|^q)^{\frac{1}{q}}+C(p,l_2)\cdot$$

$$\cdot(\frac{1}{(\alpha+2)(\alpha+2)}|f^{'}(b)|^q+m(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)})|f^{'}(\frac{a+b}{2m})|^q)^{\frac{1}{q}}$$

and $C(p,l),\ l>0$ *is as in Theorem 2.*

Theorem 13. *If there is $f: I \subset [0,\infty) \to \mathbb{R}$ a differentiable mapping on I° such that $f'' \in L[a,b]$, where a, $b \in I$ with $a < b$. If $|f''|^{p/(p-1)}$, $(p > 1)$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \le \left(\frac{b-a}{16}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{q+s+1}\right)^{\frac{1}{q}}[C(p,l_1)\cdot$$

$$\cdot(\frac{2^{1-s}}{s+1}(|f''(a)|^q + |f''(b)|^q) + \frac{sq}{q+s}B(s,q)|f''(a)|^q)^{\frac{1}{q}} + C(p,l_2)(\frac{2^{1-s}}{s+1}(|f''(a)|^q +$$

$$+|f''(b)|^q) + \frac{sq}{q+s}B(s,q)|f''(b)|^q)^{\frac{1}{q}}]$$

where p is the conjugate of q, $q = p/(p-1)$ and $B(p_1,q_1) = \int_0^1 x^{p_1-1}(1-x)^{q_1-1}dx$.

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