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NEW INEQUALITIES OF OSTROWSKI TYPE FOR CO-ORDINATED CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper an identity similar to an identity proved in [32] for fractional integrals for functions of two variables is established. Some new Ostrowski type inequalities for Riemann-Liouville fractional integrals of functions of two variables are established as well. The results established in this paper generalize those results proved in [32].

1. Introduction

In 1938, A. Ostrowski proved the following interesting inequality [34]:

Theorem 1. [34] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_{\infty}:=\sup_{t\in(a,b)}|f'(t)|<\infty$. The we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_{\infty}, \qquad (1.1)$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

The inequality (1.1) can be rewritten in equivalent form as:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\|f'\|_{\infty}.$$

Since 1938 when A. Ostrowski proved his famous inequality, many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc.

Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings, quasi convex mappings and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [2]-[4], [9], [11], [15]-[19], [28], [37]-[40], [45] and [46] and the references therein.

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Let us consider now a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le \lambda f(x,y) + (1-\lambda) f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$. The mapping f is said to be concave on the co-ordinates on Δ if the above inequality holds in reversed direction, for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A modification for convex (concave) functions on Δ, which are also known as co-ordinated convex (concave) functions, was introduced by S. S. Dragomir [12] as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex (concave) on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are convex (concave) where defined for all $x \in [a,b], y \in [c,d]$.

A formal definition for co-ordinated convex (concave) functions may be stated in:

Definition 1. [30] *A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$\begin{aligned} & f(tx + (1-t)y, ru + (1-r)w) \\ & \le trf(x,u) + t(1-r)f(x,w) + r(1-t)f(y,u) + (1-t)(1-r)f(y,w), \end{aligned} \tag{1.2}$$

holds for all $t,r \in [0,1]$ and $(x,u),(y,w) \in \Delta$. The mapping f is concave on the co-ordinates on Δ if the inequality (1.2) holds in reversed direction for all $t,r \in [0,1]$ and $(x,u),(y,w) \in \Delta$.

Clearly, every convex (concave) mapping $f : \Delta \to \mathbb{R}$ is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [12]).

Here we also quote the following result form [12] to be used in the sequel of the paper:

Theorem 2. [12] *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ & \le \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\,dydx \\ & \le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\ & \left. \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\ & \le \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned} \tag{1.3}$$

The above inequalities are sharp. The inequalities in (1.3) hold in reverse direction if the mapping f is concave.

Alomari et al. [6, 7] defined the co-ordinated s-conxity in the second sense as follows:

Definition 2. [6, 7]*Let* $\Delta =: [a,b] \times [c,d] \subseteq [0,\infty)^2$ *with* $a < b$ *and* $c < d$. *A mapping* $f : \Delta \to \mathbb{R}$ *is said to be* s*-convex in the second sense on* Δ *if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x,y) + (1-\lambda)^s f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$, $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$. *The mapping* f *is said to be* s*-concave on the co-ordinates on* Δ *if the above inequality holds in reversed direction, for all* $(x,y),(z,w) \in \Delta$, $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be s-convex (s-concave) in the seond senses on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are s-convex (s-concave) in the second sense where defined for all $x \in [a,b], y \in [c,d]$ for some fixed $s \in (0,1]$.

A formal definition for co-ordinated s-convex (s-concave) functions in the second sense may be stated in:

Definition 3. *A mapping* $f : \Delta \to \mathbb{R}$ *is said to be* s*-convex in the second sense on the co-ordinates on* Δ *if the inequality*

$$f(tx + (1-t)y, ru + (1-r)w) \tag{1.4}$$
$$\leq t^s r^s f(x,u) + t^s(1-r)^s f(x,w) + r^s(1-t)^s f(y,u) + (1-t)^s(1-r)^s f(y,w), \tag{1.5}$$

holds for all $t, r \in [0,1]$, $(x,u),(y,w) \in \Delta$ *and for some fixed* $s \in (0,1]$. *The mapping* f *is* s*-concave in the second sense on the co-ordinates on* Δ *if the inequality (1.4) holds in reversed direction for all* $t, r \in [0,1]$, $(x,u),(y,w) \in \Delta$ *for some fixed* $s \in (0,1]$.

It is also proved in [6, 7] that every s-convex mapping $f : \Delta \to \mathbb{R}$ is s-convex on the co-ordinates on Δ. Furthermore, there exists co-ordinated s-convex function which is not s-convex, (see for instance [6, 7]).

The following Hermite-Hadamard type inequalities were proven in [7]:

Theorem 3. [7]*Suppose* $f : \Delta =: [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ *with* $a < b$ *and* $c < d$ *is* s*-convex on the co-ordinates on* Δ. *The one has the inequalities:*

$$\begin{aligned}
&4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad \leq 2^{s-2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
&\quad \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
&\quad \leq \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
&\qquad \left.\frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
&\quad \leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{(s+1)^2}.
\end{aligned} \tag{1.6}$$

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [5]-[8], [12, 13, 27], [29]-[32], [36] and [43] (see also the references therein). Alomari et al. [5]-[7], proved several Hermite-Hadamard type inequalities for co-ordinated s-convex functions. Bakula et. al [8], proved Jensen's inequality for convex functions on the co-ordinates from the rectangle from the plan. Dragomir [12], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [27], also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated to the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [29]-[32], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex function, product of two co-ordinated convex mappings, for co-ordinated h-convex mappings and some Ostrowski type inequalities for co-ordinated convex mappings. Özdemir et. al [36], proved Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions. Sarikaya, et. al proved Hermite-Hadamard type inequalities for differentiable co-ordinated convex function. For more inequalities on co-ordinated convex functions see also the references in the above cited papers.

In the present paper, we establish new Ostrowski type inequalities for co-ordinated convex functions similar to those from [32] but via Riemann-Liouville fractional integral.

2. Main Results

We give first some necessary definitions and mathematical preliminaries of fractional calculus theory which are used in this sections.

Definition 4. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$. It is to be noted that $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case $\alpha = 1$, the fractional integral reduces to the classical integral.

For further properties and results concerning this operator we refer the intrusted reader to [1], [10], [20]-[25], [33], [41] and [42].

For the sake of convenience, we will use the following notation throughout this section:

$$\begin{aligned} A = {} & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)(d-c)} \Big[J^{\alpha,\beta}_{x-,y-} f(a,c) + J^{\alpha,\beta}_{x-,y+} f(a,d) + J^{\alpha,\beta}_{x+,y-} f(b,c) \\ & + J^{\alpha,\beta}_{x+,y+} f(b,d) \Big] \\ & - \frac{\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right]\Gamma(\beta+1)}{(b-a)(d-c)} \Big[J^{\beta}_{y-} f(x,c) + J^{\beta}_{y+} f(x,d) \Big] \\ & - \frac{\left[(y-c)^{\beta} + (d-y)^{\beta}\right]\Gamma(\alpha+1)}{(b-a)(d-c)} \left[J^{\alpha}_{x-} f(a,y) + J^{\alpha}_{x+} f(b,y) \right], \end{aligned}$$

where

$$\begin{aligned}
&J_{a+,c+}^{\alpha,\beta}f(x,y)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-u)^{\alpha-1}(y-v)^{\beta-1}f(u,v)dvdu, x>a, y>c,\\
&J_{b-,d-}^{\alpha,\beta}f(x,y)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (u-x)^{\alpha-1}(v-y)^{\beta-1}f(u,v)ddvdu, x<b, y<d,\\
&J_{a+,d-}^{\alpha,\beta}f(x,y)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-u)^{\alpha-1}(v-y)^{\beta-1}f(u,v)ddvdu, x>a, y<d,\\
&J_{b-,c+}^{\alpha,\beta}f(x,y)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (u-x)^{\alpha-1}(y-v)^{\beta-1}f(u,v)ddvdu, x<b, y>c,
\end{aligned}$$

and Γ is the Euler Gamma function.

To establish our main results we need the following identity:

Lemma 1. *Let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° with $a<b$, $c<d$. If $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$ and $\alpha,\beta>0$, $a,c\geq 0$, then the following identity holds:*

$$\begin{aligned}
&\frac{\left[(b-x)^\alpha+(x-a)^\alpha\right]\left[(d-y)^\beta+(y-c)^\beta\right]}{(b-a)(d-c)}f(x,y)+A\\
&=\frac{(x-a)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^\beta t^\alpha\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a, ry+(1-r)c)\,drdt\\
&-\frac{(x-a)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^\beta t^\alpha\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a, ry+(1-r)d)\,drdt\\
&-\frac{(b-x)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^\beta t^\alpha\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b, ry+(1-r)c)\,drdt\\
&+\frac{(b-x)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^\beta t^\alpha\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b, ry+(1-r)d)\,drdt,
\end{aligned}\tag{2.1}$$

for all $(x,y)\in\Delta$.

Proof. By integration by parts and by the change of the variables $u=tx+(1-t)a$, $v=ry+(1-r)c$, we have

$$\frac{(x-a)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^\beta t^\alpha \frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\,dsdt$$

$$=\frac{(x-a)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\int_0^1 t^\alpha\left[r^\beta\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\,ds\right]dt$$

$$=\frac{(x-a)^{\alpha}(y-c)^{\beta}}{(b-a)(d-c)}f(x,y)$$

$$-\frac{\beta(x-a)^\alpha}{(b-a)(d-c)}\int_c^y(v-c)^{\beta-1}f(x,v)\,dv-\frac{\alpha(y-c)^\beta}{(b-a)(d-c)}\int_a^x(u-a)^{\alpha-1}f(u,y)\,du$$

$$+\frac{\alpha\beta}{(b-a)(d-c)}\int_a^x\int_c^y(u-a)^{\alpha-1}(v-c)^{\beta-1}f(u,v)\,dudv$$

$$=\frac{(x-a)^{\alpha}(y-c)^{\beta}}{(b-a)(d-c)}f(x,y)-\frac{\beta(x-a)^\alpha\Gamma(\beta)}{(b-a)(d-c)}\frac{1}{\Gamma(\beta)}\int_c^y(v-c)^{\beta-1}f(x,v)\,dv$$

$$-\frac{\alpha(y-c)^\beta\Gamma(\alpha)}{(b-a)(d-c)}\frac{1}{\Gamma(\alpha)}\int_a^x(u-a)^{\alpha-1}f(u,y)\,du$$

$$+\frac{\alpha\beta\Gamma(\alpha)\Gamma(\beta)}{(b-a)(d-c)}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y(u-a)^{\alpha-1}(v-c)^{\beta-1}f(u,v)\,dvdu$$

$$=\frac{(x-a)^{\alpha}(y-c)^{\beta}}{(b-a)(d-c)}f(x,y)-\frac{(x-a)^\alpha\Gamma(\beta+1)}{(b-a)(d-c)}J^\beta_{y-}f(x,c)$$

$$-\frac{(y-c)^\beta\Gamma(\alpha+1)}{(b-a)(d-c)}J^\alpha_{x-}f(a,y)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)(d-c)}J^{\alpha,\beta}_{x-,y-}f(a,c) \quad (2.2)$$

Similarly, by integration by parts, we also have

$$\frac{(x-a)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^\beta t^\alpha \frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\,drdt$$

$$=-\frac{(x-a)^{\alpha}(d-y)^{\beta}}{(b-a)(d-c)}f(x,y)+\frac{(x-a)^\alpha\Gamma(\beta+1)}{(b-a)(d-c)}J^\beta_{y+}f(x,d)$$

$$+\frac{(d-y)^\beta\Gamma(\alpha+1)}{(b-a)(d-c)}J^\alpha_{x-}f(a,y)-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)(d-c)}J^{\alpha,\beta}_{x-,y+}f(a,d), \quad (2.3)$$

$$\frac{(b-x)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^\beta t^\alpha \frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\,drdt$$

$$=-\frac{(b-x)^{\alpha}(y-c)^{\beta}}{(b-a)(d-c)}f(x,y)+\frac{(b-x)^\alpha\Gamma(\beta+1)}{(b-a)(d-c)}J^\beta_{y-}f(x,c)$$

$$+\frac{(y-c)^\beta\Gamma(\alpha+1)}{(b-a)(d-c)}J^\alpha_{x+}f(b,y)-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)(d-c)}J^{\alpha,\beta}_{x+,y-}f(b,c) \quad (2.4)$$

and

$$\frac{(b-x)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^{\beta}t^{\alpha}\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\,dsdt$$
$$=\frac{(b-x)^{\alpha}(d-y)^{\beta}}{(b-a)(d-c)}f(x,y)-\frac{(b-x)^{\alpha}\Gamma(\beta+1)}{(b-a)(d-c)}J_{y+}^{\beta}f(x,d)$$
$$-\frac{(d-y)^{\beta}\Gamma(\alpha+1)}{(b-a)(d-c)}J_{x+}^{\alpha}f(b,y)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)(d-c)}J_{x+,y+}^{\alpha,\beta}f(b,d). \quad (2.5)$$

From (2.2)-(2.5), we get (2.1). This completes the proof. □

Theorem 4. *Let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° with $a<b$, $c<d$, $a,c\geq 0$ such that $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|$ is s-covex on the co-ordinates on Δ and $\left|\frac{\partial^2}{\partial y\partial x}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, then the following inequality for fractional integrals with $\alpha,\beta>0$ holds:*

$$\left|\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{(b-a)(d-c)}f(x,y)+A\right|$$
$$\leq\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{b-a}\right]\left[\frac{(d-y)^{\beta+1}+(y-c)^{\beta+1}}{d-c}\right]K, \quad (2.6)$$

for all $(x,y)\in\Delta$, where

$$K=\frac{M}{(\alpha+s+1)(\beta+s+1)}+\frac{M\Gamma(s+1)\Gamma(\beta+1)\Gamma(\alpha+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}$$
$$+\frac{M\Gamma(s+1)\Gamma(\alpha+1)\Gamma(\beta+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}+\frac{M(\Gamma(s+1))^2\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}$$

and Γ is the Euler Gamma function.

Proof. From Lemma 1, we have that the following inequality holds for all $(x,y)\in\Delta$:

$$\left|\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{(b-a)(d-c)}f(x,y)+A\right|$$
$$\leq\frac{(x-a)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^{\beta}t^{\alpha}\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|drdt$$
$$+\frac{(x-a)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^{\beta}t^{\alpha}\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|drdt$$
$$+\frac{(b-x)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^{\beta}t^{\alpha}\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|drdt$$
$$+\frac{(b-x)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\int_0^1\int_0^1 r^{\beta}t^{\alpha}\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|drdt. \quad (2.7)$$

By the convexity of $\left|\frac{\partial^2 f}{\partial r \partial t}\right|$ on the co-ordinates on Δ and $\left|\frac{\partial^2}{\partial y \partial x} f(x,y)\right| \leq M$, $(x,y) \in \Delta$, we get the follwing inequalities:

$$\int_0^1 \int_0^1 r^\beta t^\alpha \left| \frac{\partial^2}{\partial r \partial t} f\left(tx + (1-t)a, ry + (1-r)c\right) \right| drdt$$
$$\leq M \int_0^1 \int_0^1 r^{\beta+s} t^{\alpha+s} dsdt + M \int_0^1 \int_0^1 t^{\alpha+s} r^\beta (1-r)^s drdt$$
$$+ M \int_0^1 \int_0^1 r^{\beta+s} t^\alpha (1-t)^s drdt + M \int_0^1 \int_0^1 t^\alpha (1-t)^s r^\beta (1-r)^s drdt$$
$$= \frac{M}{(\alpha+s+1)(\beta+s+1)} + \frac{M\Gamma(s+1)\Gamma(\beta+1)\Gamma(\alpha+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}$$
$$+ \frac{M\Gamma(s+1)\Gamma(\alpha+1)\Gamma(\beta+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)} + \frac{M(\Gamma(s+1))^2\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}. \quad (2.8)$$

Analogously, we also have the following inequalities:

$$\int_0^1 \int_0^1 r^\beta t^\alpha \left| \frac{\partial^2}{\partial r \partial t} f\left(tx + (1-t)a, ry + (1-r)d\right) \right| drdt$$
$$\leq \frac{M}{(\alpha+s+1)(\beta+s+1)} + \frac{M\Gamma(s+1)\Gamma(\beta+1)\Gamma(\alpha+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}$$
$$+ \frac{M\Gamma(s+1)\Gamma(\alpha+1)\Gamma(\beta+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)} + \frac{M(\Gamma(s+1))^2\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}, \quad (2.9)$$

$$\int_0^1 \int_0^1 r^\beta t^\alpha \left| \frac{\partial^2}{\partial r \partial t} f\left(tx + (1-t)b, ry + (1-r)c\right) \right| drdt$$
$$\leq \frac{M}{(\alpha+s+1)(\beta+s+1)} + \frac{M\Gamma(s+1)\Gamma(\beta+1)\Gamma(\alpha+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}$$
$$+ \frac{M\Gamma(s+1)\Gamma(\alpha+1)\Gamma(\beta+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)} + \frac{M(\Gamma(s+1))^2\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)} \quad (2.10)$$

and

$$\int_0^1 \int_0^1 r^\beta t^\alpha \left| \frac{\partial^2}{\partial r \partial t} f\left(tx + (1-t)b, ry + (1-r)d\right) \right| drdt$$
$$\leq \frac{M}{(\alpha+s+1)(\beta+s+1)} + \frac{M\Gamma(s+1)\Gamma(\beta+1)\Gamma(\alpha+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}$$
$$+ \frac{M\Gamma(s+1)\Gamma(\alpha+1)\Gamma(\beta+s+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)} + \frac{M(\Gamma(s+1))^2\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}. \quad (2.11)$$

By using (2.8)-(2.11) in (2.7), we get the desired inequality (2.6). This completes the proof of the theorem. □

Remark 1. *In Theorem 1, if we take $\alpha = \beta = 1$ and $s = 1$, then the inequality (2.6) reduces to the inequality established in* [32, Theorem 3].

The next result is about the powers of the absolute value of the partial derivatives.

Theorem 5. *Let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° with $a<b$, $c<d$, $a,c\geq 0$ such that $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-convex on the co-ordinates on Δ, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\left|\frac{\partial^2}{\partial y\partial x}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, then the following inequality for fractional integrals with $\alpha,\beta>0$ holds:*

$$\left|\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{(b-a)(d-c)}f(x,y)+A\right|$$
$$\leq M\left(\frac{2}{s+1}\right)^{\frac{2}{q}}\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{(b-a)(\alpha p+1)^{\frac{1}{p}}}\right]\left[\frac{(d-y)^{\beta+1}+(y-c)^{\beta+1}}{(d-c)(\beta p+1)^{\frac{1}{p}}}\right], \quad (2.12)$$

for all $(x,y)\in\Delta$.

Proof. From Lemma 1 and the Hölder inequality, we have that the following inequality holds, for all $(x,y)\in\Delta$:

$$\left|\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{(b-a)(d-c)}f(x,y)+A\right|\leq\left(\int_0^1\int_0^1 t^{\alpha p}r^{\beta p}drdt\right)^{\frac{1}{p}}$$
$$\left[\frac{(x-a)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right]. \quad (2.13)$$

By the co-ordinated convexity of f and $\left|\frac{\partial^2}{\partial y\partial x}f(x,y)\right|\leq M$, for all $(x,y)\in\Delta$, we have that the following inequality holds:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\leq\frac{4M^q}{(s+1)^2}.$$

Similarly, we also have the following inequalities:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\leq\frac{4M^q}{(s+1)^2},$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\leq\frac{4M^q}{(s+1)^2}$$

and

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\leq\frac{4M^q}{(s+1)^2}.$$

Using the fact

$$\int_0^1\int_0^1 t^{\alpha p}r^{\alpha p}drdt=\frac{1}{(\alpha p+1)(\beta p+1)}$$

and using the last four inequalities in (2.13), we obtain (2.12). This completes the proof of the theorem. □

Remark 2. *In Theorem5, if we take $\alpha=\beta=1$, then the inequality (2.12) becomes the inequality proved in* [32, Theorem 4].

A different appraoch leads us to the following result:

Theorem 6. *Let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° with $a<b$, $c<d$, $a,c\geq 0$ such that $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-convex on the co-ordinates on Δ, $q\geq 1$ and $\left|\frac{\partial^2}{\partial y\partial x}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, then the following inequality for fractional integrals with $\alpha,\beta>0$ holds:*

$$\left|\frac{[(b-x)^\alpha+(x-a)^\alpha]\left[(d-y)^\beta+(y-c)^\beta\right]}{(b-a)(d-c)}f(x,y)+A\right|$$
$$\leq\frac{1}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{b-a}\right]\left[\frac{(d-y)^{\beta+1}+(y-c)^{\beta+1}}{d-c}\right]M, \tag{2.14}$$

for all $(x,y)\in\Delta$..

Proof. From Lemma 1 and the power mean inequality, we have that the following inequality holds, for all $(x,y)\in\Delta$:

$$\left|\frac{[(b-x)^\alpha+(x-a)^\alpha]\left[(d-y)^\beta+(y-c)^\beta\right]}{(b-a)(d-c)}f(x,y)+A\right|\leq\left(\int_0^1\int_0^1 trdrdt\right)^{1-\frac{1}{q}}$$
$$\left[\frac{(x-a)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right]. \tag{2.15}$$

By the co-ordinated convexity of f and $\left|\frac{\partial^2}{\partial y\partial x}f(x,y)\right| \leq M$, for all $(x,y)\in\Delta$, we have that the following inequality holds:

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,ry+(1-r)c\right)\right|^q drdt \leq M^q\int_0^1\int_0^1 r^{s+1}t^{s+1}drdt$$
$$+M^q\int_0^1\int_0^1 r^{s+1}t\left(1-t\right)^s drdt+M^q\int_0^1\int_0^1 t^{s+1}r\left(1-r\right)^s drdt$$
$$+M^q\int_0^1\int_0^1 r\left(1-r\right)^s t\left(1-t\right)^s drdt \leq \frac{M^q}{\left(s+1\right)^2}.$$

In a similarly way, we also have the following inequalities:

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,ry+(1-r)d\right)\right|^q dsdt \leq \frac{M^q}{\left(s+1\right)^2},$$
$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,ry+(1-r)c\right)\right|^q dsdt \leq \frac{M^q}{\left(s+1\right)^2}$$

and

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,ry+(1-r)d\right)\right|^q dsdt \leq \frac{M^q}{\left(s+1\right)^2}.$$

Using the fact

$$\int_0^1\int_0^1 trdrdt=\frac{1}{4}$$

and the last four inequalities, we obtain from (2.15) the inequality (2.14). This completes the proof of the theorem. □

Remark 3. *In Theorem6, if we take $\alpha=\beta=1$, then the inequality (2.14) becomes the inequality proved in* [32, Theorem 5].

Now we drive some results with co-ordinated concavity property instead of co-ordinated convexity.

Theorem 7. *Let $f:\Delta\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is concave on the co-ordinates on Δ and $p,\ q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the inequality*

$$\left|\frac{\left[(b-x)^\alpha+(x-a)^\alpha\right]\left[(d-y)^\beta+(y-c)^\beta\right]}{(b-a)(d-c)}f(x,y)+A\right|$$
$$\leq\frac{4^{\frac{s-1}{q}}}{(1+\alpha p)^{\frac{1}{p}}(1+\beta p)^{\frac{1}{p}}(b-a)(d-c)}\left[(x-a)^{\alpha+1}(y-c)^{\beta+1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{y+c}{2}\right)\right|\right.$$
$$+(x-a)^{\alpha+1}(d-y)^{\beta+1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{d+y}{2}\right)\right|$$
$$+(b-x)^{\alpha+1}(y-c)^{\beta+1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},\frac{y+c}{2}\right)\right|$$
$$\left.+(b-x)^{\alpha+1}(d-y)^{\beta+1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},\frac{d+y}{2}\right)\right|\right],\quad(2.16)$$

hods for all $(x,y)\in\Delta$, where .

Proof. From Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\left|\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{(b-a)(d-c)}f(x,y)+A\right| \leq \left(\int_0^1\int_0^1 r^p t^p drdt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(x-a)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}(y-c)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^{\alpha+1}(d-y)^{\beta+1}}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right], \tag{2.17}$$

for all $(x,y)\in\Delta$.
Since $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is concave on the co-ordinates on Δ, so an application of (1.6) with inequalities in reversed direction, gives us the following inequalities:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt$$
$$\leq\frac{1}{2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{y+c}{2}\right)\right|^q dt\right.$$
$$\left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},ry+(1-r)c\right)\right|^q dr\right]$$
$$\leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{y+c}{2}\right)\right|^q, \tag{2.18}$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q dsdt$$
$$\leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{d+y}{2}\right)\right|^q dt+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},ry+(1-r)c\right)\right|^q dr\right]$$
$$\leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{d+y}{2}\right)\right|^q, \tag{2.19}$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt$$
$$\leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{y+c}{2}\right)\right|^q dt+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},ry+(1-r)c\right)\right|^q dr\right]$$
$$\leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+a}{2},\frac{y+c}{2}\right)\right|^q \tag{2.20}$$

and

$$\int_0^1 \int_0^1 \left|\frac{\partial^2}{\partial r \partial t} f\left(tx+(1-t)b, ry+(1-r)d\right)\right|^q drdt$$
$$\leq 2^{s-2}\left[\int_0^1 \left|\frac{\partial^2}{\partial r \partial t} f\left(tx+(1-t)b, \frac{d+y}{2}\right)\right|^q dt\right.$$
$$\left.+\int_0^1 \left|\frac{\partial^2}{\partial r \partial t} f\left(\frac{b+x}{2}, ry+(1-r)d\right)\right|^q dr\right]$$
$$\leq 4^{s-1}\left|\frac{\partial^2}{\partial r \partial t} f\left(\frac{b+x}{2}, \frac{d+y}{2}\right)\right|^q. \quad (2.21)$$

By making use of (2.18)-(2.21) in (2.17), we obtain (2.16). Thus the proof of the theorem is complete. □

Remark 4. *If we take $\alpha=\beta=1$, in Theorem 7, we get the inequalities proved in* [32, Theorem 5].

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