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BOUNDS FOR THE RIEMANN–STIELTJES INTEGRAL VIA s–CONVEX INTEGRAND OR INTEGRATOR

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Abstract. Several bounds in approximating the Riemann–Stieltjes integral in terms of s-convex integrands or integrator are given.

1. Introduction

A function $f:\mathbb{R}^{+}\to\mathbb{R}$, where $\mathbb{R}^{+}=[0,\infty)$, is said to be s–convex in the second sense if

$$f\left(\alpha x+\beta y\right)\le\alpha^{s}f\left(x\right)+\beta^{s}f\left(y\right) \tag{1.1}$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\ge0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$. This class of functions is denoted by K_{s}^{2}. It can be easily seen that for $s=1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$ (see [6]).

In [2], Cerone and Dragomir have prove some error bounds in approximating the Riemann–Stieltjes integral in terms of some moments of the integrand are given. Among others, they proved the following result:

Theorem 1. *Let u be p–convex with $p>0$, f be monotonic nondecreasing on $[a,b]$ and such that the Riemann–Stieltjes integral $\int_a^b f\left(t\right)du\left(t\right)$ and the Riemann integrals $\int_a^b\left(t-a\right)^{p-1}f\left(t\right)dt$, $\int_a^b\left(b-t\right)^{p-1}f\left(t\right)dt$ exist. Then*

$$\int_a^b f\left(t\right)du\left(t\right) \ge\frac{p}{\left(b-a\right)^{p}}\left[u\left(b\right)\int_a^b\left(t-a\right)^{p-1}f\left(t\right)dt-u\left(a\right)\int_a^b\left(b-t\right)^{p-1}f\left(t\right)dt\right] \tag{1.2}$$

For other results concerning different bounds for the Riemann–Stieltjes integral under various assumptions on f and u, see the recent papers [1]–[5] and the references therein.

In this paper, several inequalities for the Riemann–Stieltjes integral $\int_a^b f\left(x\right)dg\left(x\right)$ are proved. Namely, the integrand f is assumed to be s-convex (s-concave) and the integrator g is monotonic increasing, bounded and s-convex (s-concave).

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2. Inequalities for s-convex integrands or integrands

We may start with the following result:

Theorem 2. *Let $f,g:[a,b]\subseteq\mathbb{R}^{+}\to\mathbb{R}$ be such that f is s–convex on $[a,b]$, g is monotonically increasing on $[a,b]$ and the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ and the Riemann integrals $\int_a^b (x-a)^{s-1} g(x)\,dt$, $\int_a^b (b-x)^{s-1} g(x)\,dt$ exist. Then we have the inequality*

$$\int_a^b f(x)\,dg(x) \le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}\,dx\right] + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}\,dx\right] \le [g(b)-g(a)]\cdot[f(a)+f(b)] \tag{2.1}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists.

Proof. Since f is s–convex on $[a,b]$ and by using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\le \int_a^b \left[\left(\frac{x-a}{b-a}\right)^s f(b) + \left(\frac{b-x}{b-a}\right)^s f(a)\right] dg(x)\\ &= f(b)\int_a^b \left(\frac{x-a}{b-a}\right)^s dg(x) + f(a)\int_a^b \left(\frac{b-x}{b-a}\right)^s dg(x)\\ &\le \frac{f(b)}{(b-a)^s}\int_a^b (x-a)^s\,dg(x) + \frac{f(a)}{(b-a)^s}\int_a^b (b-x)^s\,dg(x)\\ &= \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}\,dx\right]\\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}\,dx\right],\end{aligned} \tag{2.2}$$

which proves the first inequality in (2.1). To prove the second inequality in (2.1) and on utilizing the monotonicity of g on $[a,b]$, we get

$$\int_a^b (x-a)^{s-1} g(x)\,dx \ge g(a)\int_a^b (x-a)^{s-1}\,dx = \frac{1}{s} g(a)(b-a)^s,$$

and

$$\int_a^b (b-x)^{s-1} g(x)\,dx \le g(b)\int_a^b (b-x)^{s-1}\,dx = \frac{1}{s} g(b)(b-a)^s.$$

Therefore by (2.2), we get

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}\,dx\right]\\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}\,dx\right]\\ &\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - g(a)(b-a)^s\right]\\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + g(b)(b-a)^s\right]\\ &= [g(b)-g(a)]\cdot[f(a)+f(b)],\end{aligned}$$

which proves the second inequality in (2.2), and thus the theorem is proved. □

The following result holds:

Theorem 3. *Let $g:[a,b]\subseteq\mathbb{R}^+\to\mathbb{R}$ be a monotonically increasing on $[a,b]$.*

(1) *If $f:[a,b]\to\mathbb{R}^+$ is convex on $[a,b]$, then we have the inequality*

$$\int_a^b f(x)\,dg(x) \le \begin{cases} \frac{1}{2}\left[f(a)+f(b)+|f(a)-f(b)|\right]\cdot[g(b)-g(a)] \\ \left[\frac{g(b)-g(a)}{2} + \left|\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx\right|\right]\cdot[f(a)+f(b)] \end{cases} \tag{2.3}$$

(2) *If f is concave, then we have the inequality*

$$\int_a^b f(x)\,dg(x) \ge \begin{cases} \frac{1}{2}\left[f(a)+f(b)-|f(a)-f(b)|\right]\cdot[g(b)-g(a)] \\ \left[\frac{g(b)-g(a)}{2} - \left|\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx\right|\right]\cdot[f(a)+f(b)] \end{cases} \tag{2.4}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists

Proof. (1) In (2.2), set $s=1$, then we get

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\le \int_a^b \left[\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a)\right]dg(x)\\
&= \frac{f(b)}{b-a}\left[(b-a)\,g(b)-\int_a^b g(x)\,dx\right]+\frac{f(a)}{b-a}\left[-(b-a)\,g(a)+\int_a^b g(x)\,dx\right]\\
&= f(b)\left[g(b)-\frac{1}{b-a}\int_a^b g(x)\,dx\right]+f(a)\left[\frac{1}{b-a}\int_a^b g(x)\,dx-g(a)\right]\\
&\le \begin{cases}\max\{f(a),f(b)\}\,[g(b)-g(a)]\\ \max\left\{\left[g(b)-\frac{1}{b-a}\int_a^b g(x)\,dx\right],\left[\frac{1}{b-a}\int_a^b g(x)\,dx-g(a)\right]\right\}[f(a)+f(b)]\end{cases}\\
&= \begin{cases}\frac{1}{2}\left[f(a)+f(b)+|f(a)-f(b)|\right]\cdot[g(b)-g(a)]\\ \left[\frac{g(b)-g(a)}{2}+\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(x)\,dx\right|\right]\cdot[f(a)+f(b)]\end{cases}.\end{aligned}$$

which proves (2.3).

(2) If f is concave, then we have

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\ge \int_a^b \left[\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a)\right]dg(x)\\
&= \frac{f(b)}{b-a}\left[(b-a)\,g(b)-\int_a^b g(x)\,dx\right]+\frac{f(a)}{b-a}\left[-(b-a)\,g(a)+\int_a^b g(x)\,dx\right]\\
&= f(b)\left[g(b)-\frac{1}{b-a}\int_a^b g(x)\,dx\right]+f(a)\left[\frac{1}{b-a}\int_a^b g(x)\,dx-g(a)\right]\\
&\ge \begin{cases}\min\{f(a),f(b)\}\,[g(b)-g(a)]\\ \min\left\{\left[g(b)-\frac{1}{b-a}\int_a^b g(x)\,dx\right],\left[\frac{1}{b-a}\int_a^b g(x)\,dx-g(a)\right]\right\}[f(a)+f(b)]\end{cases}\\
&= \begin{cases}\frac{1}{2}\left[f(a)+f(b)-|f(a)-f(b)|\right]\cdot[g(b)-g(a)]\\ \left[\frac{g(b)-g(a)}{2}-\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(x)\,dx\right|\right]\cdot[f(a)+f(b)]\end{cases}\end{aligned}$$

which proves (2.4). □

Theorem 4. *Let $f,g:[a,b]\subseteq\mathbb{R}^+\to\mathbb{R}$ be respectively s_1,s_2–convex functions on $[a,b]$, $s_1,s_2\in(0,1]$. Then we have the inequality*

$$\begin{aligned}&\int_a^b f(x)\,dg(x)\\ &\quad\le \frac{s_2}{s_1+s_2}\left[f(b)\,g(b)-f(a)\,g(a)\right]+s_1\cdot\beta(s_2+1,s_1)\left[f(a)\,g(b)-f(b)\,g(a)\right]\end{aligned}\tag{2.5}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists. If f,g are s–concave then the inequality (2.5) is reversed.

Proof. Since f is s–convex on $[a,b]$ and by using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\begin{aligned}
\int_a^b f(x)\,dg(x) &\le \int_a^b \left[\left(\frac{x-a}{b-a}\right)^{s_1} f(b) + \left(\frac{b-x}{b-a}\right)^{s_1} f(a)\right] dg(x) \\
&= f(b)\int_a^b \left(\frac{x-a}{b-a}\right)^{s_1} dg(x) + f(a)\int_a^b \left(\frac{b-x}{b-a}\right)^{s_1} dg(x) \\
&\le \frac{f(b)}{(b-a)^{s_1}}\int_a^b (x-a)^{s_1}\,dg(x) + \frac{f(a)}{(b-a)^{s_1}}\int_a^b (b-x)^{s_1}\,dg(x) \\
&= \frac{f(b)}{(b-a)^{s_1}}\left[(b-a)^{s_1} g(b) - s_1\int_a^b g(x)(x-a)^{s_1-1}\,dx\right] \qquad (2.6)\\
&\quad + \frac{f(a)}{(b-a)^{s_1}}\left[-(b-a)^{s_1} g(a) + s_1\int_a^b g(x)(b-x)^{s_1-1}\,dx\right].
\end{aligned}$$

Since $g(x)$ is s_2-convex on $[a,b]$, then we have

$$g(x) \le \left[\left(\frac{x-a}{b-a}\right)^{s_2} g(b) + \left(\frac{b-x}{b-a}\right)^{s_2} g(a)\right]$$

which follows by (2.6), that

$$\begin{aligned}
&\int_a^b f(x)\,dg(x) \\
&\le \frac{f(b)}{(b-a)^{s_1}}\left[(b-a)^{s_1} g(b) - s_1\int_a^b \left[\left(\frac{x-a}{b-a}\right)^{s_2} g(b) + \left(\frac{b-x}{b-a}\right)^{s_2} g(a)\right](x-a)^{s_1-1}\,dx\right] \\
&\quad + \frac{f(a)}{(b-a)^{s_1}}\left[-(b-a)^{s_1} g(a) + s_1\int_a^b \left[\left(\frac{x-a}{b-a}\right)^{s_2} g(b) + \left(\frac{b-x}{b-a}\right)^{s_2} g(a)\right](b-x)^{s_1-1}\,dx\right] \\
&\le \frac{f(b)}{(b-a)^{s_1}}\left[(b-a)^{s_1} g(b) - s_1\frac{g(b)}{(b-a)^{s_2}}\int_a^b (x-a)^{s_1+s_2-1}\,dx\right. \\
&\quad \left. - s_1\frac{g(a)}{(b-a)^{s_2}}\int_a^b (b-x)^{s_2}(x-a)^{s_1-1}\,dx\right] \\
&\quad + \frac{f(a)}{(b-a)^{s_1}}\left[-(b-a)^{s_1} g(a) + s_1\frac{g(b)}{(b-a)^{s_2}}\int_a^b (x-a)^{s_2}(b-x)^{s_1-1}\,dx\right. \\
&\quad \left. + s_1\frac{g(a)}{(b-a)^{s_2}}\int_a^b (b-x)^{s_1+s_2-1}\,dx\right].
\end{aligned}$$

Simple calculations yield that

$$\int_a^b (b-x)^{s_2}(x-a)^{s_1-1}\,dx = \beta(s_1, s_2+1),$$

and

$$\int_a^b (x-a)^{s_2}(b-x)^{s_1-1}\,dx = \beta(s_2+1, s_1),$$

where,

$$\int_a^b (x-a)^p (b-x)^q \, dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q dt = (b-a)^{p+q+1} \beta(p+1, q+1)$$

and $\beta(\cdot,\cdot)$ is the Euler Beta function.

It follows that

$$\begin{aligned}
&\int_a^b f(x)\, dg(x) \\
&\le \frac{f(b)}{(b-a)^{s_1}} \left[(b-a)^{s_1} g(b) - s_1 \frac{g(b)}{(b-a)^{s_2}} \frac{(b-a)^{s_1+s_2}}{s_1+s_2} - s_1 \frac{(b-a)^{s_1+s_2} g(a)}{(b-a)^{s_2}} \beta(s_1, s_2+1) \right] \\
&\quad + \frac{f(a)}{(b-a)^{s_1}} \left[-(b-a)^{s_1} g(a) + s_1 \frac{(b-a)^{s_1+s_2} g(b)}{(b-a)^{s_2}} \beta(s_2+1, s_1) + s_1 \frac{g(a)}{(b-a)^{s_2}} \frac{(b-a)^{s_1+s_2}}{s_1+s_2} \right] \\
&= f(b) g(b) - \frac{s_1}{s_1+s_2} f(b) g(b) - s_1 \cdot \beta(s_1, s_2+1) f(b) g(a) \\
&\quad - f(a) g(a) + s_1 \cdot \beta(s_2+1, s_1) f(a) g(b) + \frac{s_1}{s_1+s_2} f(a) g(a) \\
&= \frac{s_2}{s_1+s_2} \left[f(b) g(b) - f(a) g(a) \right] + s_1 \cdot \beta(s_2+1, s_1) \left[f(a) g(b) - f(b) g(a) \right],
\end{aligned}$$

since $\beta(s_1, s_2+1) = \beta(s_2+1, s_1)$, which proves the first inequality in (2.5). □

Corollary 1. *In Theorem 4, if $s_1 = s_2 = 1$, i,e, f, g are two convex functions on $[a,b]$, then we have*

$$\int_a^b f(x)\, dg(x) \le \frac{f(a) + f(b)}{2} \cdot [g(b) - g(a)]. \tag{2.7}$$

Theorem 5. *Let $f, g : [a,b] \subseteq \mathbb{R}^+ \to \mathbb{R}$ be such that g satisfies that $\phi \le g(t) \le \Phi$ for all $t \in [a,b]$.*

(1) *If f is s–convex on $[a,b]$, then we have the inequality*

$$\int_a^b f(x)\, dg(x) \le f(b) [g(b) - \phi] + f(a) [\Phi - g(a)]. \tag{2.8}$$

(2) *If f is s–concave, then we have the inequality*

$$\int_a^b f(x)\, dg(x) \ge f(b) [g(b) - \Phi] + f(a) [\phi - g(a)]. \tag{2.9}$$

Provided that the Riemann–Stieltjes integral $\int_a^b f(x)\, dg(x)$ exists.

Proof. (1) From (2.2), we get

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}dx\right]\\&\quad+\frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}dx\right]\\&\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\cdot\phi\int_a^b (x-a)^{s-1}dx\right]\\&\quad+\frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\cdot\Phi\int_a^b (b-x)^{s-1}dx\right]\\&\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - \phi\cdot(b-a)^s\right]\\&\quad+\frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + \Phi\cdot(b-a)^s\right]\\&= f(b)\left[g(b)-\phi\right] + f(a)\left[\Phi - g(a)\right]\end{aligned}$$

which proves (2.8).

(2) If f is concave, then

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\ge \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}dx\right]\\&\quad+\frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}dx\right]\\&\ge \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\cdot\Phi\int_a^b (x-a)^{s-1}dx\right]\\&\quad+\frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\cdot\phi\int_a^b (b-x)^{s-1}dx\right]\\&\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - \Phi\cdot(b-a)^s\right]\\&\quad+\frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + \phi\cdot(b-a)^s\right]\\&= f(b)\left[g(b)-\Phi\right] + f(a)\left[\phi - g(a)\right]\end{aligned}$$

which proves (2.9).

□

Remark 1. *In Theorem 5, if we set* $-\infty < \phi := \inf_{x\in[a,b]} g(x)$ *and* $\sup_{x\in[a,b]} g(x) := \Phi < \infty$, *therefore we may re-write (2.8) and (2.9) respectively as follows:*

(1) *If f is s–convex on $[a,b]$, then we have the inequality*

$$\int_a^b f(x)\,dg(x) \le f(b)\left[g(b) - \inf_{x\in[a,b]} g(x)\right] + f(a)\left[\sup_{x\in[a,b]} g(x) - g(a)\right]. \tag{2.10}$$

(2) *If f is s–concave, then we have the inequality*

$$\int_a^b f(x)\,dg(x) \geq f(b)\left[g(b) - \sup_{x\in[a,b]} g(x)\right] + f(a)\left[\inf_{x\in[a,b]} g(x) - g(a)\right] \tag{2.11}$$

Remark 2. *Define the mapping $g : [a,b] \to \mathbb{R}^+$, $g(t) = \int_a^t u(s)ds$. Then g is differentiable on (a,b) and $g'(t) = u(t)$. Using the properties of the Riemann–Stieltjes integral, we have*

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,u(x)\,dx.$$

Therefore, we point out some results for the Riemann integral of a product.

(1) *Under the assumptions of Theorem 2, we have*

$$\int_a^b f(x)\,u(x)\,dx \leq [f(a) + f(b)] \cdot \int_a^b u(x)dx \tag{2.12}$$

(2) *Under the assumptions of Theorem 3, we have*

(a) *If $f : [a,b] \to \mathbb{R}^+$ is convex on $[a,b]$, then we have the inequality*

$$\int_a^b f(x)\,u(x)\,dx \leq \begin{cases} \frac{1}{2}\left[f(a) + f(b) + |f(a) - f(b)|\right] \cdot \int_a^b u(x)dx \\ \frac{1}{2}\left[\int_a^b u(x)dx + \left|\int_a^b u(x)dx - \frac{2}{b-a}\int_a^b\int_a^x u(t)dtdx\right|\right] \cdot [f(a) + f(b)] \end{cases} \tag{2.13}$$

(3) *If f is concave, then we have the inequality*

$$\int_a^b f(x)\,dg(x) \geq \begin{cases} \frac{1}{2}\left[f(a) + f(b) - |f(a) - f(b)|\right] \cdot \int_a^b u(x)dx \\ \frac{1}{2}\left[\int_a^b u(x)dx - \left|\int_a^b u(x)dx - \frac{2}{b-a}\int_a^b\int_a^x u(t)dtdx\right|\right] \cdot [f(a) + f(b)] \end{cases} \tag{2.14}$$

(4) *Under the assumptions of Theorem 4, we have*

$$\int_a^b f(x)\,u(x)\,dx \leq \left[\frac{s_2}{s_1 + s_2} f(b) + s_1 \cdot \beta(s_2 + 1, s_1)\, f(a)\right] \cdot \int_a^b u(x)\,dx \tag{2.15}$$

(5) *Under the assumptions of Theorem 5, we have*

(a) *If f is s–convex on $[a,b]$, then we have the inequality*

$$\int_a^b f(x)\,u(x)\,dx \leq f(b)\left[\int_a^b u(x)\,dx - \phi\right] + \Phi \cdot f(a). \tag{2.16}$$

(b) *If f is s–concave, then we have the inequality*

$$\int_a^b f(x)\,u(x)\,dx \geq f(b)\left[\int_a^b u(x)\,dx - \Phi\right] + \phi \cdot f(a) \tag{2.17}$$

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