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THE EXTENSION OF SOME ORLICZ SPACE RESULTS TO THE THEORY OF OPTIMAL MEASURE

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Abstract. We extend some fundamental results about the Orlicz space L^{Φ} (such as the Jensen and Hölder inequalities, the L^{Φ} Banach spaces and their duals) to Optimal Measure Theory.

1. Introduction

Let Φ be a *convex Young function*, i.e.

$$\Phi(x) = \int_0^x \varphi(t)\,dt,\; x \in \mathbb{R}_+,$$

where $\varphi : (0,\infty) \to (0,\infty)$ is a right-continuous and increasing function such that $\varphi(0) \geq 0$ and $\varphi(\infty) = \infty$.

The *conjugate Young functions* are defined as follows:

For $t \in (0,\infty)$ put $\psi(t) := \sup\{x > 0 : \varphi(x) < t\}$ and let $\psi(0) = 0$. It can be easily checked that ψ satisfies all the conditions imposed on φ and we trivially have $\psi(\varphi(x)) \leq x \leq \psi(\varphi(x) + 0)$, whenever $x \in (0,\infty)$.

The convex Young function

$$\Psi(x) := \int_0^x \psi(t)\,dt,\; x \in [0,\infty),$$

is said to be *conjugate* to Φ and the pair (Φ, Ψ) is referred to as *mutually conjugate convex Young functions*.

Every pair (Φ, Ψ) of mutually conjugate convex Young functions satisfies the fundamental Young inequality

$$xy \leq \Phi(x) + \Psi(y) \tag{1.1}$$

for all $x, y \in [0,\infty)$, and the Young equality

$$xy = \Phi(x) + \Psi(y) \tag{1.2}$$

if and only if $y \in [\varphi(x), \varphi(x+0)]$ or $x \in [\psi(y), \psi(y+0)]$.

For more about convex Young functions, see [8].

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A function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *concave Young function* if for all $x \geq 0$ it is defined by

$$\Gamma(x) = \int_0^x \gamma(t)\, dt,$$

where $\Gamma(0) = 0$ and $\gamma : (0, \infty) \to (0, \infty)$ is a decreasing right-continuous function which is integrable on every finite interval $(0, x)$. We also assume that $\Gamma(\infty) = \infty$. For more about concave Young-functions see [9], say.

In [1, 2] we initiated the optimal measure and optimal average to mimic the Lebesgue measure and integral respectively. The corresponding monotone convergence theorems are established (with some restriction in the case of decreasing sequences of measurable functions). By means of optimal measure and average we were able to characterize various notions of well-known convergence such as the notions of discrete, equally, uniform and pointwise convergence of sequences of measurable functions. The boundedness of sequences of measurable functions can be characterized using the same tools. The corresponding Banach spaces of measurable functions are also obtained, see [1]. As part of volume rendering the ambient occlusion is a technique to compute shadows. Csaba Barcsák (cf. [7]) used the algorithm in [5] to select exactly one (the "best") among many ambient occlusion samples generated randomly.

We note that many of the results obtained in [1–5] can be found in [6].

A set function $p : \mathcal{F} \to [0, 1]$ is called *optimal measure* if it satisfies the following three axioms ([1]):

Axiom 1. $p(\Omega) = 1$ and $p(\varnothing) = 0$.

Axiom 2. $p(B \cup E) = p(B) \vee p(E)$ for all measurable sets B and E.

Axiom 3. p is continuous from above, i.e. whenever $(E_n) \subset \mathcal{F}$ is a decreasing sequence, then $p\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} p(E_n) = \bigwedge\limits_{n=1}^{\infty} p(E_n)$.

A p-atom H is *decomposable* if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p\left(H \cap \overline{B}\right)$. If no such subatom exists, we shall say that H is *indecomposable.*

The following fundamental result about optimal measures (cf. [2]) is worth being mentioned.

The Structure Theorem. *Let p be an optimal measure defined on the measurable space $(\Omega, \mathcal{F})$. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms, where J is some countable (i.e., finite or countably infinite) index-set such that for any measurable set B, with $p(B) > 0$, we have that*

$$p(B) = \max\{p(B \cap H_n) : n \in J\}.$$

Moreover, the only limit point of the set $\{p(H_n) : n \in J\}$ is 0 provided that J is a countably infinite set.

In the Structure Theorem the set $\mathcal{H}(p)$ is referred to as a *p-generating system*, and we write $\mathcal{P}_{<\infty}$ (resp. $\mathcal{P}_{\infty}$) for the collection of all optimal measures p with finite (resp. countably infinite) generating systems. We let $\mathcal{P}$ denote the union of $\mathcal{P}_{\infty}$ and $\mathcal{P}_{<\infty}$.

The remark down after, which is immediate from the Structure Theorem, is worth been noted.

Remark 1.1. *Let be given any optimal measure p with $\mathcal{H}(p) = \{H_n : n \in J\}$ its generating system and a measurable set $A \in \mathcal{F}$. Then $p(A) = 0$ if and only if $p(A \cap H) = 0$ for every $H \in \mathcal{H}(p)$.*

The optimal average of $s \in \mathbb{S}^+$ is defined by

$$\int_\Omega sdp := \max\{b_i p(B_i) : i = 1, \ldots, n\},$$

where $\mathbb{S}^+$ is the collection of all non-negative measurable simple functions on $(\Omega, \mathcal{F})$. This non-linear operator was introduced in the image of the well-known Lebesgue integral of $s \in \mathbb{S}^+$.

We note that the optimal average of any $s \in \mathbb{S}^+$ does not depend on the decompositions of s (cf. [1], Theorem 1.0, page 135) just like in the case of Lebesgue integral.

The optimal average of a non-negative measurable function f is defined by

$$\int_\Omega f\,dp := \sup\left\{\int_\Omega s\,dp : s \in \mathbb{S}^+,\ 0 \le s \le f\right\}. \tag{1.3}$$

Let f be any non-negative measurable function and p any optimal measure with generating system $\mathcal{H}(p) = \{H_n : n \in J\}$. Then it is stated (in [2], Proposition 2.6 and [3], Proposition 2.1) that

$$\int_\Omega fdp = \sup_{n \in J} \int_{H_n} fdp,$$

and moreover, if the quantity (1.3) is finite, then

$$\int_\Omega fdp = \sup\{c_n \cdot p(H_n) : n \in J\},$$

where $c_n = f(\omega)$ for almost all $\omega \in H_n$, $n \in J$.

Throughout the communication we shall be dealing with an arbitrary but fixed optimal measure space $(\Omega, \mathcal{F}, p)$, the pair $(\Omega, \mathcal{F})$ is a measurable space and on it an optimal measure $p \in \mathcal{P}$.

Let $f : \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ be any measurable function. We shall say that f belongs to $\mathcal{A}^\alpha$ if

$$\int_\Omega |f|^\alpha dp < \infty, \quad \alpha \in [1, \infty).$$

For any $\alpha \in [1, \infty)$, the space $\mathcal{A}^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, defined by

$$\|f\|_{\mathcal{A}^\alpha} = \sqrt[\alpha]{\int_\Omega |f|^\alpha dp}, \quad \text{if } f \in \mathcal{A}^\alpha,\ \alpha \in [1, \infty)$$

is a Banach space, cf. [1] for more details.

Our major aim in the present communication is to extend some fundamental results about the Orlicz space L^Φ in Measure Theory to the framework of Optimal Measure Theory, i.e. we generalize the space $\mathcal{A}^\alpha$ to the space $\mathcal{A}^\Phi$, where Φ is a convex Young function. Some set of non-linear functionals $F : \mathcal{A}^\Phi \to [0, \infty]$ is also investigated likewise the dual space of the Orlicz space L^Φ. We note that in our case the max or sup operators replaces the usual addition operation in the case of

Measure Theory. In contrast to the dual space of L^{Φ} the collection of all such non-linear functionals will be referred to as the "laud" space of $\mathcal{A}^{\Phi}$. Before all that, we shall establish the corresponding Jensen inequalities for convex as well as concave Young functions, and later on the corresponding Hölder inequality and some other well-known inequalities.

2. The Main Results

In the sequel (Φ, Ψ) will stand for a pair of mutually conjugate convex Young functions.

Definition 2.1. *We say that a measurable function f belongs to $\mathcal{A}^{\Phi}$ if there is a constant $c \in (0, \infty)$ such that*

$$\bigvee_{\Omega} \Phi\left(\frac{|f|}{c}\right) dp \leq 1. \tag{2.1}$$

In the image of the Luxemburg norm define on $\mathcal{A}^{\Phi}$ the operator $\|\cdot\|_{\mathcal{A}^{\Phi}}$ by

$$\|f\|_{\mathcal{A}^{\Phi}} = \inf\left\{c \in (0, \infty) : \bigvee_{\Omega} \Phi\left(\frac{|f|}{c}\right) dp \leq 1\right\}, \tag{2.2}$$

and $\|f\|_{\mathcal{A}^{\Phi}} = \infty$ if there is no $c \in (0, \infty)$ such that (2.1) holds.

Note that if $\Phi(t) = \dfrac{t^{1+\alpha}}{1+\alpha}$, $t \in [0, \infty)$ and $\alpha \in (0, \infty)$, then $\mathcal{A}^{\Phi} = \mathcal{A}^{1+\alpha}$.

Theorem 2.1. *Let $\Phi : [0, \infty) \to [0, \infty)$ be any function and f a non-negative finite measurable function. Then the inequality*

$$\Phi\left(\bigvee_{\Omega} f dp\right) \leq \bigvee_{\Omega} \Phi(f)\, dp$$

holds, and is referred to as the Optimal Jensen inequality, provided that Φ is a convex Young function. Furthermore, the inequality is reversed if Φ is a concave Young function.

Definition 2.2. *Let $\mathcal{A}_{+}^{\Phi} := \left\{f \in \mathcal{A}^{\Phi} : f \geq 0\right\}$. We say that a functional $F : \mathcal{A}_{+}^{\Phi} \to [0, \infty]$ belongs to $\widetilde{\mathcal{A}^{\Phi}}$ if the following conditions hold true simultaneously:*

1. *For all $f, h \in \mathcal{A}_{+}^{\Phi}$, and $\alpha, \beta \in [0, \infty)$ we have*

$$F(\alpha f \vee \beta h) = \alpha F(f) \vee \beta F(h).$$

2. *F is continuous from below, i.e. if $(f_n) \subset \mathcal{A}_{+}^{\Phi}$ is an increasing sequence, then*

$$\lim_{n\to\infty} F(f_n) = F\left(\lim_{n\to\infty} f_n\right).$$

3. *There is some constant $C > 0$ for which*

$$F(f) \leq C\|f\|_{\mathcal{A}^{\Phi}}, \text{ whenever } f \in \mathcal{A}_{+}^{\Phi}.$$

We extend Definition 2.2 to the entire $\mathcal{A}^{\Phi}$ space as follows.

Definition 2.3. *A functional $F \circ |\cdot| : \mathcal{A}^{\Phi} \to [0, \infty]$ is said to belong to $\widetilde{\mathcal{A}^{\Phi}}$ if the following conditions hold true simultaneously:*

1. *For all* $f,\, h \in \mathcal{A}^{\Phi}$, *and* $\alpha,\, \beta \in [0,\, \infty)$ *we have*

$$F\left(\alpha\left|f\right| \vee\, \beta\left|h\right|\right) = \alpha F\left(\left|f\right|\right) \vee\, \beta F\left(\left|h\right|\right).$$

2. F *is non-negatively continuous from below, i.e. if* $(f_n) \subset \mathcal{A}^{\Phi}$ *is a non-negative increasing sequence, then*

$$\lim_{n\to\infty} F\left(f_n\right) = F\left(\lim_{n\to\infty} f_n\right).$$

3. *There is some constant* $C > 0$ *for which*

$$F\left(\left|f\right|\right) \leq C \left\|f\right\|_{\mathcal{A}^{\Phi}}, \text{ whenever } f \in \mathcal{A}^{\Phi}.$$

The set $\widetilde{\mathcal{A}^{\Phi}}$ will thus be referred to as the "*laud*" *space* of $\mathcal{A}^{\Phi}$, in contrast with the "dual" space of L^{Φ} in Measure Theory.

The counterpart of Proposition IX-2-2 in the appendix of [10] can be stated as follows.

Theorem 2.2. *The following assertions hold.*

1. *The mapping* $\left\|\cdot\right\|_{\mathcal{A}^{\Phi}} : \mathcal{A}^{\Phi} \to [0,\, \infty)$ *defined by (2.2) is a norm.*

2. $\mathcal{A}^{\Phi} \subset \mathcal{A}^{1}$, *i.e. there exist some constant* $\delta > 0$ *such that*

$$\delta \left\|f\right\|_{\mathcal{A}^{1}} \leq \left\|f\right\|_{\mathcal{A}^{\Phi}},$$

whenever $f \in \mathcal{A}^{\Phi}$.

3. $\mathcal{A}^{\Phi}$ *is a Banach space, i.e. every Cauchy sequence in* $\mathcal{A}^{\Phi}$ *converges to a measurable function in* $\mathcal{A}^{\Phi}$*-norm.*

4. *If* $f \in \mathcal{A}^{\Phi}$ *and* $h \in \mathcal{A}^{\Psi}$, *then*

$$\left\|fh\right\|_{\mathcal{A}^{1}} \leq 2 \left\|f\right\|_{\mathcal{A}^{\Phi}} \cdot \left\|h\right\|_{\mathcal{A}^{\Psi}},$$

which shall be referred to as the Optimal Hölder Inequality.

5. *Given any* $h \in \mathcal{A}^{\Psi}$, *the mapping* $F_h \circ |\cdot| : \mathcal{A}^{\Phi} \to [0,\, \infty)$ *defined by*

$$F_h\left(\left|f\right|\right) = \bigwedge_{\Omega} \left|fh\right| dp,$$

belongs to the laud space of $\mathcal{A}^{\Phi}$. *Moreover, letting* $\mathfrak{M}$ *stand for the set of all measurable functions defined on* $(\Omega,\, \mathcal{F})$, *the quantity*

$$\left\|h\right\|^{*}_{\mathcal{A}^{\Phi}} := \sup_{f\in\mathcal{A}^{\Phi}\setminus\{0\}} \frac{F_h\left(\left|f\right|\right)}{\left\|f\right\|_{\mathcal{A}^{\Phi}}} = \sup\left\{ F_h\left(\left|f\right|\right) : f \in \mathfrak{M},\quad \bigwedge_{\Omega} \Phi\left(\left|f\right|\right) dp \leq 1 \right\} \tag{2.3}$$

defines a norm on the space $\mathcal{A}^{\Psi}$ *which is equivalent to the norm* $\left\|\cdot\right\|_{\mathcal{A}^{\Psi}}$, *more precisely*

$$\lambda \left\|h\right\|_{\mathcal{A}^{\Psi}} \leq \left\|h\right\|^{*}_{\mathcal{A}^{\Phi}} \leq 2 \left\|h\right\|_{\mathcal{A}^{\Psi}},$$

for some constant $\lambda \in (0,\, 2]$ *and all* $h \in \mathcal{A}^{\Psi}$.

6. *If* $F \circ |\cdot| : \mathcal{A}^{\Phi} \to [0,\, \infty)$ *is a mapping belonging to* $\widetilde{\mathcal{A}^{\Phi}}$, *then there is an* $h \in \mathcal{A}^{\Psi}$ *with* $\left\|h\right\|_{\mathcal{A}^{\Psi}} \leq C$ *(the constant* C *being as in Definition 2.3) such that for all* $f \in \mathcal{A}^{\Phi}$,

$$F\left(\left|f\right|\right) = \bigwedge_{\Omega} \left|fh\right| dp.$$

3. Proofs

We prepare the ground for the proof of Theorem 2.1.

Let $J \subset \mathbb{N}$ be an index set. Then the weighted supremum of a sequence $(b_n)_{n\in J} \subset [0, \infty)$ is defined by $\sup_{n\in J} b_n\alpha_n$, where $(\alpha_n)_{n\in J} \subset [0, 1]$ is a prescribed sequence with 0 as its unique limit point if the index set is infinite (in symbol $|J| = \infty$).

Remark 3.1. *For all $d \in \mathbb{R}$, $c \in (0, \infty)$ and $(b_n)_{n\in J} \subset [0, \infty)$, where J is an index set, then*

$$\sup_{n\in J} (d + cb_n) = d + c \sup_{n\in J} b_n.$$

Remark 3.1 is obvious.

Lemma 3.1. *Let $J \subset \mathbb{N}$ be an index set and $\Phi : [0, \infty) \to [0, \infty)$ be any function. Consider two sequences $(b_n)_{n\in J} \subset [0, \infty)$ and $(\alpha_n)_{n\in J} \subset [0, 1]$ possessing 0 as its unique limit point if $|J| = \infty$. Then*

$$\Phi\left(\sup_{n\in J} b_n\alpha_n\right) \le \sup_{n\in J} \Phi(b_n)\,\alpha_n$$

provided that Φ is a convex Young function. Furthermore, the inequality is reversed if Φ is a concave Young function.

Proof. We shall prove the lemma for convex Young functions only, since the other case can similarly be done. In fact, without loss of generality assume that $x_0 := \sup_{n\in J} b_n\alpha_n < \infty$. By applying the definition and the tangent property of convex functions together with Remark 3.1, we observe that

$$\sup_{n\in J} \Phi(b_n)\,\alpha_n \ge \sup_{n\in J} \Phi(b_n\alpha_n) \ge \sup_{n\in J} (d + cb_n\alpha_n) = d + c \sup_{n\in J} b_n\alpha_n,$$

where $d \in \mathbb{R}$ and $c \in (0, \infty)$ are such that $\Phi(x_0) = d + cx_0$ and $d + cx \le \Phi(x)$ for all $x \in [0, \infty)$.

This leads to the desire result. □

Corollary 3.1. *Let $J \subset \mathbb{N}$ be an index set, $(x_n)_{n\in J} \subset [0, \infty)$ and $(y_n)_{n\in J} \subset [0, \infty)$ be any sequences. Suppose that the numbers $\beta, \gamma \in (1, \infty)$ satisfy the inequality $\beta < \gamma$. Consider a sequence $(\alpha_n)_{n\in J} \subset [0, 1]$ with 0 as its unique limit point if $|J| = \infty$. Then*

$$\left(\sup_{n\in J} x_n^\beta\alpha_n\right)^{\beta^{-1}} \le \left(\sup_{n\in J} x_n^\gamma\alpha_n\right)^{\gamma^{-1}},$$

provided that $\sup_{n\in J} x_n^\gamma\alpha_n < \infty$.

Proof. Apply Lemma 3.1 for the convex Young function $\Phi(t) = t^{\beta^{-1}\gamma}$ to obtain

$$\left(\sup_{n\in J} x_n^\beta\alpha_n\right)^{\beta^{-1}\gamma} \le \sup_{n\in J} \left(x_n^\beta\right)^{\beta^{-1}\gamma} \alpha_n,$$

which yields the desired result. □

The Proof of Theorem 2.1. We note that the proof follows from the conjunction of both Proposition 2.1 in [3] and the above Lemma 3.1. □

Corollary 3.2. *Let the numbers* $\beta, \gamma \in (1, \infty)$ *satisfy the inequality* $\beta < \gamma$, f *be a non-negative measurable function and* p *an optimal measure. Then*

$$\left(\int_{\Omega} f^{\beta} dp \right)^{\beta^{-1}} \leq \left(\int_{\Omega} f^{\gamma} dp \right)^{\gamma^{-1}}.$$

Before tackling the proof of Theorem 2.2 (which goes down the line of the proof given in [10] for Proposition IX-2-2), some essential results need to be checked.

Lemma 3.2. *Let* y *be a bounded measurable function and consider the quasi-optimal measure* $q_y : \mathcal{F} \to [0, \infty)$,

$$q_y(A) = \int_A |y| \, dp.$$

Then $dq_y = |y| \, dp$ p*-a.e. Moreover,*

$$|y| = \max \left\{ \frac{q_y(H)}{p(H)} \cdot \chi_H : H \in \mathcal{H}(p), \, q_y(H) > 0 \right\}$$

on $\bigcup \mathcal{H}(p)$.

Proof. Clearly, $q_y \ll p$, i.e. q_y is absolutely continuous with respect to p. Then by Lemma 2.3 (in [2]), the system

$$\mathcal{H}^*(p) = \{ H \in \mathcal{H}(p) : q_h(H) > 0 \}$$

is a q_y-generating system. As we know, the Optimal Radon-Nikodym Theorem (cf. [2]) says that there is a unique measurable function $h \geq 0$ for which

$$q_y(A) = \int_A h dp,$$

for all $A \in \mathcal{F}$ with $q_y(A) > 0$, where

$$h = \max \left\{ \frac{q_y(H)}{p(H)} \cdot \chi_H : H \in \mathcal{H}^*(p) \right\}.$$

From the unicity it ensues that $|y| = h$, p- almost everywhere. ☐

Remark 3.2. *Given any convex Young function* Φ, *for every* $f \in \mathcal{A}^{\Phi}$ *we have*

$$\|f\|_{\mathcal{A}^{\Phi}} \leq \max \left\{ 1; \int_{\Omega} \Phi(|f|) \, dp \right\}.$$

Proof. In fact, by the convexity

$$\int_{\Omega} \Phi \left(\frac{|f|}{\max \left\{ 1; \int_{\Omega} \Phi(|f|) \, dp \right\}} \right) dp \leq \frac{\int_{\Omega} \Phi(|f|) \, dp}{\max \left\{ 1; \int_{\Omega} \Phi(|f|) \, dp \right\}} \leq 1,$$

provided that the optimal average of $\Phi(|f|)$ is finite. Finally we just point out that the case, when the optimal average of $\Phi(|f|)$ is infinite, is obvious. ☐

Remark 3.3. *For every measurable function f we have that $\|f\|_{\mathcal{A}^\Phi} \le 1$ if and only if*

$$\int_\Omega \Phi(|f|)\, dp \le 1.$$

Remark 3.4. *For any convex Young function Ψ and any measurable simple function of the form $h = b\chi_A$ where $A \in \mathcal{F}$ with $p(A) > 0$ we have*

$$\|h\|_{\mathcal{A}^\Psi} = \frac{|b|}{\Psi^{-1}\left(\frac{1}{p(A)}\right)}.$$

Remarks 3.3 and 3.4 can be easily checked, so we shall omit their proofs.

The Proof of Theorem 2.2.

PART 1. Let f, h be any measurable functions. It is trivial that $\|f\|_{\mathcal{A}^\Phi} \ge 0$. We want to prove that if $\|f\|_{\mathcal{A}^\Phi} = 0$, then $p(|f| \ne 0) = 0$. In fact, suppose that $\|f\|_{\mathcal{A}^\Phi} = 0$ but $p(0 < |f| \le \infty) = p(|f| \ne 0) > 0$. Then by Remark 1.1 a non-empty subset J_0 of the index set J exists such that $p(H_n \cap (0 < |f| \le \infty)) > 0$, whenever $n \in J_0$ and $p(H_n \cap (0 < |f| \le \infty)) = 0$ otherwise, where J is the index set of the generating system $\mathcal{H}(p) = \{H_n : n \in J\}$. Note that $\|f\|_{\mathcal{A}^\Phi} = \inf S$, where

$$S = \left\{ \delta > 0 : \int_\Omega \Phi\left(\frac{|f|}{\delta}\right) dp \le 1 \right\}.$$

From the assumption and the definition of the infimum there is a sequence $(\delta_k) \subset S$ such that $0 < \delta_k < \frac{1}{k}$ for all $k \in \mathbb{N}$. By applying the Optimal Jensen Inequality we can observe that

$$1 \ge \int_\Omega \Phi\left(\frac{|f|}{\delta_k}\right) dp \ge \Phi\left(\int_\Omega \frac{|f|}{\delta_k} dp\right).$$

Hence

$$\delta_k \Phi^{-1}(1) \ge \int_\Omega |f|\, dp,$$

which implies, via Proposition 2.1 in [3], that

$$\sup_{n \in J_0} \int_{H_n \cap (0 < |f| \le \infty)} |f|\, dp = \int_\Omega |f|\, dp = 0. \tag{3.1}$$

Clearly, $p(|f| = \infty) = 0$, otherwise the left hand side of (3.1) would assume the value ∞, a contradiction. Then necessarily, $p(H_n \cap (0 < |f| < \infty)) = 0$ for every $n \in J_0$, which is impossible because of the assumption. By this absurdity we have thus proved that if $\|f\|_{\mathcal{A}^\Phi} = 0$, then $f = 0$, p-a.e. Note that its converse is obvious. We show the triangle inequality in the next step. In fact, via the monotonicity and the convexity, we observe that

$$\Phi\left(\frac{|f+h|}{\|f\|_{\mathcal{A}^\Phi}+\|h\|_{\mathcal{A}^\Phi}}\right) \leq \Phi\left(\frac{|f|+|h|}{\|f\|_{\mathcal{A}^\Phi}+\|h\|_{\mathcal{A}^\Phi}}\right) \leq$$

$$\leq \frac{\|f\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi}+\|h\|_{\mathcal{A}^\Phi}}\Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) + \frac{\|h\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi}+\|h\|_{\mathcal{A}^\Phi}}\Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right).$$

Hence

$$\bigcurlywedge_{\Omega} \Phi\left(\frac{|f+h|}{\|f\|_{\mathcal{A}^\Phi}+\|h\|_{\mathcal{A}^\Phi}}\right) \leq \frac{\|f\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi}+\|h\|_{\mathcal{A}^\Phi}} \bigcurlywedge_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp +$$

$$+ \frac{\|h\|_{\mathcal{A}^\Phi}}{\|f\|_{\mathcal{A}^\Phi}+\|h\|_{\mathcal{A}^\Phi}} \bigcurlywedge_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right) dp \leq 1,$$

since

$$\bigcurlywedge_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp \leq 1 \quad \text{and} \quad \bigcurlywedge_{\Omega} \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Phi}}\right) dp \leq 1.$$

Consequently,

$$\|f+h\|_{\mathcal{A}^\Phi} \leq \|f\|_{\mathcal{A}^\Phi} + \|h\|_{\mathcal{A}^\Phi}.$$

We leave to the reader the verification of the homogeneity axiom.

Part 2. We prove that $\delta_1 \|f\|_{\mathcal{A}^1} \leq \|f\|_{\mathcal{A}^\Phi}$ for some constant $\delta_1 > 0$ and all $f \in \mathcal{A}^\Phi$. In fact, let $u_0 \in (0, \infty)$ such that $\varphi(u_0) > 0$ and $u_0 + (\varphi(u_0))^{-1} \geq 1$. Making use of the inequality here below (proved in [10] on page 198)

$$\Phi(x) \geq (x-u_0)^+ \varphi(u_0), \; x \in [0, \infty),$$

we have

$$1 \geq \bigcurlywedge_{\Omega} \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}}\right) dp \geq \varphi(u_0) \bigcurlywedge_{\Omega} \left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} - u_0\right)^+ dp$$

and hence by Remark 3.1,

$$u_0 + \frac{1}{\varphi(u_0)} \geq \bigcurlywedge_{\Omega} \left[u_0 + \left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} - u_0\right)^+\right] dp \geq \bigcurlywedge_{\Omega} \frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} dp.$$

Whence, $\|f\|_{\mathcal{A}^1} \leq \left(u_0 + \frac{1}{\varphi(u_0)}\right) \|f\|_{\mathcal{A}^\Phi}$.

Part 3. Let $(f_n) \subset \mathcal{A}^\Phi$ be any Cauchy sequence. Then we can extract from it a subsequence (f_{n_k}) such that

$$\sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{A}^\Phi} < \infty$$

and hence by Part 2,

$$\sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{\mathcal{A}^1} < \infty.$$

Since $\mathcal{A}^1$ is a Banach space, the limit $\lim\limits_{k\to\infty} f_{n_k} = f$ exists almost everywhere. Clearly, for every $k \in \mathbb{N}$,

$$f_{n_k} = f_{n_1} + \sum_{j=1}^{k-1} \left(f_{n_{j+1}} - f_{n_j} \right),$$

Write

$$S_{n_k} = |f_{n_1}| + \sum_{j=1}^{k-1} \left| f_{n_{j+1}} - f_{n_j} \right|, \quad k \in \mathbb{N}.$$

Obviously,

$$\|S_{n_k}\|_{\mathcal{A}^\Phi} \le \|f_{n_1}\|_{\mathcal{A}^\Phi} + \sum_{j=1}^{k-1} \left\| f_{n_{j+1}} - f_{n_j} \right\|_{\mathcal{A}^\Phi}, \quad k \in \mathbb{N}.$$

Since (S_{n_k}) is an increasing sequence it ensues that

$$\|f\|_{\mathcal{A}^\Phi} \le \liminf_{k\to\infty} \|S_{n_k}\|_{\mathcal{A}^\Phi} \le \|f_{n_1}\|_{\mathcal{A}^\Phi} + \sum_{j=1}^{\infty} \left\| f_{n_{j+1}} - f_{n_j} \right\|_{\mathcal{A}^\Phi} < \infty.$$

Hence $f \in \mathcal{A}^\Phi$. Note that

$$\|f - f_{n_k}\|_{\mathcal{A}^\Phi} \le \sum_{j=k+1}^{\infty} \left\| f_{n_{j+1}} - f_{n_j} \right\|_{\mathcal{A}^\Phi}$$

which yields

$$\lim_{k\to\infty} \|f - f_{n_k}\|_{\mathcal{A}^\Phi} = 0.$$

By the triangle inequality we have

$$\|f - f_n\|_{\mathcal{A}^\Phi} \le \|f - f_{n_k}\|_{\mathcal{A}^\Phi} + \|f_n - f_{n_k}\|_{\mathcal{A}^\Phi} \to 0,$$

as $k \to \infty$ and $n \to \infty$.

PART 4. Let $f \in \mathcal{A}^\Phi$ and $h \in \mathcal{A}^\Psi$ be arbitrary such that $\|f\|_{\mathcal{A}^\Phi} > 0$ and $\|h\|_{\mathcal{A}^\Psi} > 0$. Then by applying the fundamental inequality (1.1) to $u = \dfrac{|f|}{\|f\|_{\mathcal{A}^\Phi}}$ and $v = \dfrac{|h|}{\|h\|_{\mathcal{A}^\Psi}}$ yields

$$\int_\Omega |fh|\, dp \le \|f\|_{\mathcal{A}^\Phi} \cdot \|h\|_{\mathcal{A}^\Psi} \left(\int_\Omega \Phi\left(\frac{|f|}{\|f\|_{\mathcal{A}^\Phi}} \right) dp + \int_\Omega \Phi\left(\frac{|h|}{\|h\|_{\mathcal{A}^\Psi}} \right) dp \right) \le$$
$$\le 2\, \|f\|_{\mathcal{A}^\Phi} \cdot \|h\|_{\mathcal{A}^\Psi}.$$

PART 5. To show that $\|\cdot\|^*_{\mathcal{A}^\Phi}$ is a norm we shall only verify the biconditional $\|h\|^*_{\mathcal{A}^\Phi} = 0$ if and only if $h = 0$, p-a.e. because the two other norm axioms can be easily checked. To this end we need to prove first that $\|h\|^*_{\mathcal{A}^\Phi} = 0$ implies $h = 0$, p-a.e. In fact, suppose (by the contrapositive) that there is some $H \in \mathcal{H}(p)$ for which the inequality $p(H \cap (|h| > 0)) > 0$ holds. Write $A := H \cap (|h| > 0)$. Consider the measurable function $f_\delta = \delta \chi_A$ with $\delta > 0$ such that

$$\int_\Omega \Psi(f_\delta)\, dp = \Psi(\delta)\, p(A) \le 1.$$

This can be done, because Ψ is a convex Young function. Then

$$\|h\|^*_{\mathcal{A}^\Phi} \geq \int_\Omega |h|\, f_\delta dp > 0.$$

Therefore, $\|h\|^*_{\mathcal{A}^\Phi} = 0$ implies $h = 0$, p-a.e. We point out that the converse is straightforward.

By applying the Optimal Hölder Inequality, we observe from (2.3) that

$$\|h\|^*_{\mathcal{A}^\Phi} = \sup_{\left\{f\in\mathfrak{M}:\ \|f\|_{\mathcal{A}^\Phi}\leq 1\right\}} \int_\Omega |fh|\, dp \leq 2\, \|h\|_{\mathcal{A}^\Psi}\,.$$

Next, we shall show the inequality $\lambda\, \|h\|_{\mathcal{A}^\Psi} \leq \|h\|^*_{\mathcal{A}^\Phi}$ for some constant $\lambda \in (0,\, 2]$ and all $h \in \mathcal{A}^\Psi$. In fact, assume the contrary, i.e. for every constant $\lambda \in (0,\, 2]$ we can find an $h \in \mathcal{A}^\Psi$ for which $\lambda\, \|h\|_{\mathcal{A}^\Psi} > \|h\|^*_{\mathcal{A}^\Phi}$. Now, choose $f_0 = \dfrac{\|h\|_{\mathcal{A}^\Psi}}{\rho p\,(H)}\chi_H$, where $H \in \mathcal{H}\,(p)\,,\ \rho > 0$ and $p\,(H \cap (|h| = \rho)) = p\,(H)$. Then $f_0 \in \mathcal{A}^\Phi$, via Remark 3.4. Consequently,

$$\lambda\, \|h\|_{\mathcal{A}^\Psi} > \|h\|^*_{\mathcal{A}^\Phi} = \sup_{\left\{f\in\mathfrak{M}:\ \|f\|_{\mathcal{A}^\Phi}\leq 1\right\}} \int_\Omega |fh|\, dp \geq \int_\Omega |f_0|\, |h|\, dp = \|h\|_{\mathcal{A}^\Psi}$$

so that $\lambda > 1$ for all $\lambda \in (0,\, 2]$. Letting $\lambda \to 0$ would entail $0 > 1$ which is absurd, indeed. Therefore, the inequality $\lambda\, \|h\|_{\mathcal{A}^\Psi} \leq \|h\|^*_{\mathcal{A}^\Phi}$ fulfils for some constant $\lambda \in (0,\, 2]$ and all $h \in \mathcal{A}^\Psi$.

Part 6. Let $F \circ |\cdot| \in \widetilde{\mathcal{A}^\Phi}$. Define the function $q : \mathcal{F} \to [0,\, \infty)$ by $q\,(A) = F\,(\chi_A)$. Via the assumption for every $A \in \mathcal{F}$,

$$q\,(A) \leq C\, \|\chi_A\|_{\mathcal{A}^\Phi}\,.$$

Consider the continuous function

$$\eta\,(t) = \begin{cases} \frac{1}{\Phi^{-1}\left(\frac{1}{t}\right)} & \text{whenever } t > 0 \\ 0 & \text{if } \ t = 0. \end{cases}$$

A simple calculus shows that

$$\int_\Omega \Phi\left(\frac{\chi_A}{\eta\,(p\,(A))}\right) dp = \Phi\left(\frac{1}{\eta\,(p\,(A))}\right) p\,(A) = 1.$$

Hence $q\,(A) \leq C\eta\,(p\,(A))$, whenever $A \in \mathcal{F}$. Consequently, $q \ll p$, i.e. q is absolutely continuous with respect to p. Then by Theorem 2.4 of [2],

$$h = \max\left\{\frac{q\,(H)}{p\,(H)} \cdot \chi_H : H \in \mathcal{H}\,(p)\,,\ q\,(H) > 0\right\}$$

is the unique measurable function such that $dq = h \cdot dp$ almost everywhere. Consequently, for every measurable simple function

$$s = \sum_{i=1}^n b_i \chi_{B_i} = \bigvee_{i=1}^n b_i \chi_{B_i}$$

we have

$$\bigvee_{i=1}^{n} |b_i| F(\chi_{B_i}) = \bigvee_{i=1}^{n} F(|b_i| \chi_{B_i}) = F\left(\bigvee_{i=1}^{n} |b_i| \chi_{B_i}\right) = \int_{\Omega} h \bigvee_{i=1}^{n} |b_i| \chi_{B_i} dp = \int_{\Omega} h |s| \, dp = F(|s|).$$

Next, we show that $\|h\|_{\mathcal{A}^{\Psi}} \leq 2C$. To this end, let (s_n) be a sequence of non-negative measurable simple functions tending increasingly to h. Then by the Young equality (1.2) one can observe that

$$\Psi\left(\frac{s_n}{2C}\right) + \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) = \frac{s_n}{2C}\psi\left(\frac{s_n}{2C}\right).$$

On the one hand,

$$\int_{\Omega} \left[\Psi\left(\frac{s_n}{2C}\right) + \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)\right] dp \geq \int_{\Omega} \max\left\{\Psi\left(\frac{s_n}{2C}\right); \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right)\right\} dp = \max\left\{\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp; \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\}.$$

On the other hand we observe via Remark 3.2 that

$$\int_{\Omega} \frac{s_n}{2C}\psi\left(\frac{s_n}{2C}\right) dp \leq \frac{1}{2C}\int_{\Omega} h\psi\left(\frac{s_n}{2C}\right) dp = \frac{1}{2C}F\left(\psi\left(\frac{s_n}{2C}\right)\right) \leq \frac{1}{2}\left\|\psi\left(\frac{s_n}{2C}\right)\right\|_{\mathcal{A}^{\Phi}} \leq \frac{1}{2}\max\left\{1; \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\}.$$

Consequently,

$$\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp + \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp \leq 2\max\left\{\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp; \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\} \leq \max\left\{1; \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp\right\} \leq 1 + \int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp.$$

This implies that

$$\int_{\Omega} \Psi\left(\frac{s_n}{2C}\right) dp \leq 1, \ n \in \mathbb{N}, \tag{3.2}$$

since

$$\int_{\Omega} \Phi\left(\psi\left(\frac{s_n}{2C}\right)\right) dp < \infty.$$

Finally, letting $n \to \infty$ in (3.2), the Optimal Monotone Convergence Theorem (cf. [1], Theorem 3.1/i) thus implies that

$$\bigvee_{\Omega} \Psi\left(\frac{h}{2C}\right) dp \leq 1.$$

Therefore, $h \in \mathcal{A}^{\Psi}$.

□

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