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APPROXIMATIONS OF THE JENSEN DIVERGENCE

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ABSTRACT. The Jensen divergence is used to measure the difference between two probability distributions. This divergence has been generalised to allow the comparison of more than two distributions. In this paper, we consider some bounds for generalised Jensen divergence for m-time differentiable functions.

1. INTRODUCTION

In probability theory and statistics, the Jensen divergence is used to measure the difference between two probability distributions. In Burbea and Rao [1], a generalisation of Jensen divergence is considered to allow the comparison of more than two distributions. If Φ is a function defined on an interval I of the real line $\mathbb{R}$, the *(generalised) Jensen divergence* between two elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in I^n (where $n \geq 1$) is given by the following equation, (cf. Burbea and Rao [1])

$$\mathcal{J}_{n,\Phi}(x,y) := \sum_{i=1}^{n} \left[\frac{1}{2}[\Phi(x_i) + \Phi(y_i)] - \Phi\left(\frac{x_i + y_i}{2}\right) \right], \text{ for all } x, y \in I^n \times I^n.$$

Several measures have been proposed to quantify the difference (also known as the *divergence*) between two (or more) probability distributions. We refer to Grosse et. al. [6] for further references.

If $p_1, ..., p_k \geq 0$ $(k \geq 2)$ denote the probability distribution satisfying the usual constraints $\sum_{j=1}^{k} p_j = 1$, then the Jensen divergence between the probability distributions is defined by [1]

$$\mathcal{J}^{p}_{n,\Phi}(y^1, ..., y^k) := \sum_{i=1}^{n} \left[\sum_{j=1}^{k} p_j \Phi(y_i^j) - \Phi\left(\sum_{j=1}^{k} p_j y_i^j \right) \right]$$

for all $(y^1, ..., y^k) \in I^n \times ... \times I^n$ with $y^j = (y_1^j, \ldots, y_n^j)$ for $j = 1, \ldots, k$. In information theory, $\mathcal{J}^p_n$ defines the measure of information on k-input channel for input distribution $p = p_1, ..., p_k$. It also expresses the amount of information supplied by the data for discrimination between these distributions. The divergence $\mathcal{J}_{n,1}$, written as

$$\mathcal{J}_{n,1}(x,y) := \frac{1}{2} \sum_{i=1}^{n} \left[x_i \log x_i + y_i \log y_i - (x_i + y_i) \log\left(\frac{x_i + y_i}{2}\right) \right],$$

is also known as the *Jensen-Shannon divergence*.

Considering the Jensen divergence defined above, we state the following well-known theorem for convex and concave functions.

Theorem 1 (Burbea and Rao [1]). *Let Φ be a C^2 function defined on interval I of real numbers. Then $J_{n,\Phi}$ is convex (concave) on $I^n \times I^n$ if and only if Φ is*

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convex (concave) and $\frac{1}{\Phi''}$ *is concave (convex) on* I. *Further, in this case* $J^p_{n,\Phi}$ *is also convex (concave) on* I^{nk} *for any given probability distribution* p.

Recently, Dragomir, Dragomir and Sherwell [5] obtained several sharp bounds for the Jensen divergence, for different classes of functions. We refer the readers to Section 2 for the detail of these results. In the same spirit, we present bounds for m-time differentiable functions in this paper (cf. Sections 3 and 4). Lastly, we apply these bounds for elementary functions in Section 5.

2. Definitions, notation and previous results

In this section, we provide some definitions and notation that will be used in the paper, and also provide some previous results related to the Jensen divergence. Throughout the paper, we denote p' to be the Hölder conjugate of a real number $1 < p < \infty$, that is, when p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

We use the following notation for Lebesgue integrable functions. Let $a, b, u, v \in \mathbb{R}$ and without loss of generalisation, let us assume that $a \le u \le v < b$. We denote

$$\|g\|_{[u,v],p} := \left(\int_u^v |g(s)|^p \, ds \right)^{1/p} \text{ if } p \ge 1,\ u, v \in [a,b] \text{ and } g \in L_p[a,b];$$

and for $g \in L_\infty[a,b]$ we denote

$$\|g\|_{[u,v],\infty} := \operatorname*{ess\,sup}_{s \in [u,v]} |g(s)|.$$

Definition 2. A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if and only if f is differentiable almost everywhere on $[a,b]$, the derivative f' is Lebesgue integrable on $[a,b]$ and $f(v) - f(u) = \int_u^v f'(t)\, dt$ for any $u, v \in [a,b]$.

Theorem 3 (Dragomir, Dragomir, and Sherwell [5]). *Assume that* $\Phi : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$. *Then we have the bounds*

$$(1) \qquad |\mathcal{J}_{n,\Phi}(x,y)|$$
$$\le \frac{1}{2} \times \begin{cases} \sum_{i=1}^n |y_i - x_i| \, \|\Phi'\|_{[x_i,y_i],\infty}, & \text{if } \Phi' \in L_\infty[a,b]; \\[2ex] \sum_{i=1}^n |y_i - x_i|^{\frac{p-1}{p}} \|\Phi'\|_{[x_i,y_i],p}, & \text{if } \Phi' \in L_p[a,b], p > 1; \\[2ex] \sum_{i=1}^n \|\Phi'\|_{[x_i,y_i],1}, \end{cases}$$
$$\le \frac{1}{2} \times \begin{cases} \|\Phi'\|_{[a,b],\infty} \sum_{i=1}^n |y_i - x_i|, & \text{if } \Phi' \in L_\infty[a,b]; \\[2ex] \|\Phi'\|_{[a,b],p} \sum_{i=1}^n |y_i - x_i|^{\frac{p-1}{p}}, & \text{if } \Phi' \in L_p[a,b], p > 1; \\[2ex] n \|\Phi'\|_{[a,b],1}, \end{cases}$$

for any $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in [a,b]^n$. *The constant* $\frac{1}{4}$ *is best possible in both inequalities.*

For two vectors $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in I^n$ we say that $x \le y$ if for all $i \in \{1, ..., n\}$ we have that $x_i \le y_i$. For $x \le y$, we call the set,

$$[x, y] := \{g = (g_1, ..., g_n) \mid \text{ with } x_i \le g_i \le y_i \text{ for all } i \in \{1, ..., n\}\},$$

the generalised interval generated by x and y.

Theorem 4 (Dragomir, Dragomir, and Sherwell [5]). *Let* $\Phi : I \to \mathbb{R}$ *be a convex function on the interval* I *of real numbers* $\mathbb{R}$.

(i) *If $x,\ y,\ z \in I^n$ are so that $x \leq y \leq z$, then*

$$0 \leq \mathcal{J}_{n,\Phi}(x,y) + \mathcal{J}_{n,\Phi}(y,z) \leq \mathcal{J}_{n,\Phi}(x,z), \tag{2}$$

i.e., $\mathcal{J}_{n,\Phi}$ is super-additive as a functional of the generalized interval;

(ii) *If $x,\ y,\ z, u \in I^n$ are so that $x \leq y \leq z \leq u$, then*

$$0 \leq \mathcal{J}_{n,\Phi}(y,z) \leq \mathcal{J}_{n,\Phi}(x,u), \tag{3}$$

i.e., $\mathcal{J}_{n,\Phi}$ is monotonic nondecreasing as a functional of the generalized interval.

When more information about the derivative of the function Φ is available, then we can state the following result as well

Theorem 5 (Dragomir, Dragomir, and Sherwell [5])**.** *Let $\Phi : [a,b] \to \mathbb{R}$ be a differentiable function on the interval $[a,b]$ of real numbers $\mathbb{R}$.*

(i) *If the derivative Φ' is of bounded variation on $[a,b]$, then*

$$|\mathcal{J}_{n,\Phi}(x,y)| \leq \frac{1}{4}\sum_{i=1}^{n}|y_i - x_i|\left|\bigvee_{x_i}^{y_i}(\Phi')\right| \leq \frac{1}{4}\bigvee_{a}^{b}(\Phi')\sum_{i=1}^{n}|y_i - x_i| \tag{4}$$

for any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$. The constant $\frac{1}{4}$ is best possible in both inequalities (4).

(ii) *If the derivative Φ' is L-Lipschitzian on $[a,b]$ with the constant $L > 0$, then*

$$|\mathcal{J}_{n,\Phi}(x,y)| \leq \frac{1}{8}L\sum_{i=1}^{n}(y_i - x_i)^2 = \frac{1}{2}L\ \mathcal{J}_{n,2}(x,y) \tag{5}$$

for any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$. The constant $\frac{1}{8}$ is best possible in (5).

3. Approximations of Jensen divergence

In this section, we provide some approximations for the following Jensen divergence

$$\mathcal{J}_{1,f}(a,b) = \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)$$

for some classes of function f. We first consider the above Jensen divergence for absolutely continuous functions, and 'weaken' the condition to the case of functions of bounded variation. The results in this section will be used to approximate the Jensen divergence for $\mathcal{J}_{n,\Phi}$ (as defined in Section 1), which we will describe in Section 4.

The following integral identity will be used to obtain an approximation of Jensen divergence. We refer to Cerone, Dragomir and Roumeliotis [2, Lemma 2.1., p. 54].

Lemma 6 (Cerone, Dragomir and Roumeliotis [2])**.** *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(m-1)}$ is absolutely continuous on $[a,b]$ we have the identity*

$$\begin{aligned}\int_a^b f(t)dt &= \sum_{k=0}^{m-1}\left[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\right] \\ &\quad + \frac{(-1)^m}{m!}\int_a^b K_m(x,t)f^{(m)}(t)dt\end{aligned} \tag{6}$$

where the kernel $K_m : [a,b]^2 \to \mathbb{R}$ is given by

$$K_m(x,t) := \begin{cases} (t-a)^m, & \text{if } t \in [a,x] \\ (t-b)^m, & \text{if } t \in (x,b] \end{cases}, \quad x \in [a,b] \tag{7}$$

and m is a natural number, $m \geq 1$.

Corollary 7. *Under the assumptions of Lemma 6, we have the following estimate for the error term in* (6)

$$\left|\int_a^b K_m(x,t)f^{(m)}(t)dt\right|$$
$$(8)\quad \leq \begin{cases} \dfrac{(x-a)^{m+1}+(b-x)^{m+1}}{m+1}\|f^{(m)}\|_{[a,b],\infty}, & \textit{if } f^{(m)}\in L_\infty[a,b];\\ \dfrac{(x-a)^{m+\frac{1}{p}}+(b-x)^{m+\frac{1}{p}}}{(pm+1)^{\frac{1}{p}}}\|f^{(m)}\|_{[a,b],p'}, & \textit{if } f^{(m)}\in L_{p'}[a,b], p>1;\\ [(x-a)^m+(b-x)^m]\|f^{(m)}\|_{[a,b],1}, & \textit{if } f^{(m)}\in L_1[a,b]. \end{cases}$$

Proof. By Hölder's inequality, we have

$$\left|\int_a^b K_m(x,t)f^{(m)}(t)dt\right|$$
$$(9)\qquad \leq \begin{cases} \int_a^b |K_m(x,t)|dt\|f^{(m)}\|_{[a,b],\infty}\\ \left(\int_a^b |K_m(x,t)|^p dt\right)^{1/p}\|f^{(m)}\|_{[a,b],p'}, p>1\\ \sup_{t\in[a,b]}|K_m(x,t)|\|f^{(m)}\|_{[a,b],1}. \end{cases}$$

We evaluate

$$\begin{aligned}\int_a^b |K_m(x,t)|dt &= \int_a^x (t-a)^m dt + \int_x^b (b-t)^m dt\\ &= \left.\frac{(t-a)^{m+1}}{m+1}\right|_a^x + \left.\frac{(b-t)^{m+1}}{m+1}\right|_x^b = \frac{(x-a)^{m+1}+(b-x)^{m+1}}{m+1}\end{aligned}$$

which proves the first part of (8). Now,

$$\begin{aligned}\left(\int_a^b |K_m(x,t)|^p dt\right)^{1/p} &\leq \left(\int_a^x (t-a)^{pm}dt\right)^{1/p} + \left(\int_x^b (b-t)^{pm}dt\right)^{1/p}\\ &= \left[\left.\frac{(t-a)^{pm+1}}{pm+1}\right|_a^x\right]^{1/p} + \left[\left.\frac{(b-t)^{pm+1}}{pm+1}\right|_x^b\right]^{1/p}\\ &= \frac{(x-a)^{m+1/p}+(b-x)^{m+1/p}}{(pm+1)^{1/p}}\end{aligned}$$

which proves the second part of (8). Finally,

$$\begin{aligned}\sup_{t\in[a,b]}|K_m(x,t)| &= \sup_{t\in[a,x]}(t-a)^m + \sup_{t\in(x,b]}(b-t)^m\\ &= (x-a)^m+(b-x)^m\end{aligned}$$

which completes the proof. □

Theorem 8. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(m-1)}$ is absolutely continuous on $[a, b]$. We have the following representation:*

$$
\begin{aligned}
(10) \qquad & \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \\
= \; & \sum_{k=1}^{m-1} \frac{(b-a)^k}{2(k+1)!} \left[\left(\frac{1+(-1)^k}{2^k}\right) f^{(k)}\left(\frac{a+b}{2}\right) - f^{(k)}(a) - (-1)^k f^{(k)}(b) \right] \\
& + \frac{(-1)^m}{2m!(b-a)} \int_a^b C_m(t) f^{(m)}(t)dt,
\end{aligned}
$$

where

$$
C_m(t) := \begin{cases} (t-a)^m - (t-b)^m, & \text{if } t \in [a, (a+b)/2]; \\ (t-b)^m - (t-a)^m, & \text{if } t \in ((a+b)/2, b]. \end{cases}
$$

Proof. By Lemma 6 we have

$$
\begin{aligned}
\int_a^b f(t)dt &= (b-a)f(x) \\
&+ \sum_{k=1}^{m-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right] \qquad (11) \\
&+ \frac{(-1)^m}{m!} \int_a^b K_m(x,t) f^{(m)}(t)dt.
\end{aligned}
$$

Choose $x = a$ in (11) to obtain

$$
\begin{aligned}
\int_a^b f(t)dt &= (b-a)f(a) + \sum_{k=1}^{m-1} \left[\frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \right] \qquad (12) \\
&+ \frac{(-1)^m}{m!} \int_a^b (t-b)^m f^{(m)}(t)dt;
\end{aligned}
$$

and choose $x = b$ in (11) to obtain

$$
\begin{aligned}
\int_a^b f(t)dt &= (b-a)f(b) + \sum_{k=1}^{m-1} \left[\frac{(-1)^k (b-a)^{k+1}}{(k+1)!} f^{(k)}(b) \right] \qquad (13) \\
&+ \frac{(-1)^m}{m!} \int_a^b (t-a)^m f^{(m)}(t)dt.
\end{aligned}
$$

Adding (12) and (13), and divide the sum by 2, we obtain

$$
\begin{aligned}
\int_a^b f(t)dt &= (b-a)\frac{f(a)+f(b)}{2} \\
&+ \frac{1}{2} \sum_{k=1}^{m-1} \left[\frac{(b-a)^{k+1}}{(k+1)!} [f^{(k)}(a) + (-1)^k f^{(k)}(b)] \right] \qquad (14) \\
&+ \frac{(-1)^m}{2m!} \int_a^b \left[(t-a)^m + (t-b)^m \right] f^{(m)}(t)dt.
\end{aligned}
$$

We also have the following by choosing $x = (a+b)/2$ in (11)

$$
\begin{aligned}
\int_a^b f(t)dt &= (b-a)f\left(\frac{a+b}{2}\right) \\
&\quad + \sum_{k=1}^{m-1} \frac{(\frac{b-a}{2})^{k+1} + (-1)^k(\frac{b-a}{2})^{k+1}}{(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \\
&\quad + \frac{(-1)^m}{m!}\int_a^b M_m(t) f^{(m)}(t)dt \\
&= (b-a)f\left(\frac{a+b}{2}\right) \\
&\quad + \sum_{k=1}^{m-1} \left(\frac{[1+(-1)^k]\ (b-a)^{k+1}}{2^{k+1}(k+1)!}\right) f^{(k)}\left(\frac{a+b}{2}\right) \\
&\quad + \frac{(-1)^m}{m!}\int_a^b M_m(t) f^{(m)}(t)dt,
\end{aligned} \tag{15}
$$

where

$$
M_m(t) := \begin{cases} (t-a)^m, & \text{if } t \in [a,(a+b)/2] \\ (t-b)^m, & \text{if } t \in ((a+b)/2, b]. \end{cases}
$$

Equating (14) and (15) yields

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \\
&= \frac{1}{b-a}\left[-\sum_{k=1}^{m-1}\left[\frac{(b-a)^{k+1}}{2(k+1)!}[f^{(k)}(a) + (-1)^k f^{(k)}(b)]\right]\right. \\
&\quad + \sum_{k=1}^{m-1}\left(\frac{[1+(-1)^k]\ (b-a)^{k+1}}{2^{k+1}(k+1)!}\right) f^{(k)}\left(\frac{a+b}{2}\right) \\
&\quad \left. - \frac{(-1)^m}{2m!}\int_a^b [(t-a)^m + (t-b)^m] f^{(m)}(t)dt + \frac{(-1)^m}{m!}\int_a^b M_m(t) f^{(m)}(t)dt \right] \\
&= \frac{1}{b-a}\left[\sum_{k=1}^{m-1}\frac{(b-a)^{k+1}}{2(k+1)!}\left[\left(\frac{1+(-1)^k}{2^k}\right) f^{(k)}\left(\frac{a+b}{2}\right) - f^{(k)}(a) - (-1)^k f^{(k)}(b)\right]\right. \\
&\quad + \frac{(-1)^m}{2m!}\int_a^{\frac{a+b}{2}} ((t-a)^m - (t-b)^m) f^{(m)}(t)dt \\
&\quad \left. + \frac{(-1)^m}{2m!}\int_{\frac{a+b}{2}}^b ((t-b)^m - (t-a)^m) f^{(m)}(t)dt \right] \\
&= \sum_{k=1}^{m-1}\frac{(b-a)^k}{2(k+1)!}\left[\left(\frac{1+(-1)^k}{2^k}\right) f^{(k)}\left(\frac{a+b}{2}\right) - f^{(k)}(a) - (-1)^k f^{(k)}(b)\right] \\
&\quad + \frac{(-1)^m}{2m!(b-a)}\int_a^b C_m(t) f^{(m)}(t)dt,
\end{aligned}
$$

as required. □

Corollary 9. *Under the assumptions of Theorem 8, we have the following estimate for the error term in* (10)*:*

$$\left|\int_a^b C_m(x,t) f^{(m)}(t)dt\right| \leq \begin{cases} \frac{2(b-a)^{m+1}}{m+1}\|f^{(m)}\|_{[a,b],\infty}, & \text{if } f^{(m)} \in L_\infty[a,b] \\ \frac{2(b-a)^{m+1/p}}{(pm+1)^{1/p}}\|f^{(m)}\|_{[a,b],p'}, & \text{if } f^{(m)} \in L_{p'}[a,b] \\ 2(b-a)^m\|f^{(m)}\|_{[a,b],1}, & \text{if } f^{(m)} \in L_1[a,b] \end{cases}.$$

Proof. Similarly to the proof of Corollary 7, we use Hölder's inequality to estimate the error term (cf. (9)). So, we want to quantify:

$$\begin{aligned}\int_a^b |C_m(t)|dt &= \int_a^b |(t-a)^m - (t-b)^m|dt \\ &\leq \int_a^b (t-a)^m dt + \int_a^b (b-t)^m dt \\ &= \left.\frac{(t-a)^{m+1}}{m+1}\right|_a^b - \left.\frac{(b-t)^{m+1}}{m+1}\right|_a^b = \frac{2(b-a)^{m+1}}{m+1}.\end{aligned}$$

We also quantify

$$\begin{aligned}\left(\int_a^b |C_m(t)|^p dt\right)^{1/p} &= \left(\int_a^b |(t-a)^m - (t-b)^m|^p dt\right)^{1/p} \\ &\leq \left(\int_a^b (t-a)^{pm} dt\right)^{1/p} + \left(\int_a^b (b-t)^{pm} dt\right)^{1/p} \\ &= \left[\left.\frac{(t-a)^{pm+1}}{pm+1}\right|_a^b\right]^{1/p} + \left[-\left.\frac{(b-t)^{pm+1}}{pm+1}\right|_a^b\right]^{1/p} \\ &= 2\frac{(b-a)^{m+1/p}}{(pm+1)^{1/p}}.\end{aligned}$$

And finally,

$$\begin{aligned}\sup_{t\in[a,b]} |C_m(t)| &= \sup |(t-a)^m - (t-b)^m| \\ &\leq \sup_{t\in[a,b]} (t-a)^m + \sup_{t\in[a,b]} (b-t)^m = 2(b-a)^m\end{aligned}$$

which completes the proof. □

Remark 10. For the case of $m = 1$, we have the following

$$\begin{aligned}\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) &= -\frac{1}{2(b-a)}\int_a^b C_1(t)f'(t)dt \\ &= \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} (a-b)f'(t)dt + \int_{\frac{a+b}{2}}^b (b-a)f'(t)dt\right] \\ &= -\frac{1}{2}\int_a^{\frac{a+b}{2}} f'(t)dt + \frac{1}{2}\int_{\frac{a+b}{2}}^b f'(t)dt\end{aligned}$$

and thus

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right| &= \left|-\frac{1}{2}\int_a^{\frac{a+b}{2}} f'(t)dt+\frac{1}{2}\int_{\frac{a+b}{2}}^b f'(t)dt\right| \\ &\leq \frac{1}{2}\left|\int_a^{\frac{a+b}{2}} f'(t)dt\right|+\frac{1}{2}\left|\int_{\frac{a+b}{2}}^b f'(t)dt\right| \\ &\leq \frac{1}{2}\int_a^{\frac{a+b}{2}} |f'(t)|dt+\frac{1}{2}\int_{\frac{a+b}{2}}^b |f'(t)|dt \\ &= \frac{1}{2}\int_a^b |f'(t)|dt=\frac{1}{2}\|f'\|_{L^1[a,b]}.\end{aligned}$$

This recaptures the last case in Theorem 3 for $n=1$.

Theorem 11. *Let $f:[a,b]\to\mathbb{R}$ be a function whose m^{th} derivatives $f^{(m)}$ are of locally bounded variation on $[a,b]$. Then,*

$$\begin{aligned}\frac{f(a)+f(b)}{2}-\left(\frac{a+b}{2}\right) &= \sum_{k=1}^{m}\left[\frac{(-1)^k+1}{2^{k+1}k!}\right](b-a)^k f^{(k)}\left(\frac{a+b}{2}\right) \\ &+ \frac{1}{2m!}(-1)^{m+1}\int_a^{\frac{a+b}{2}}(s-a)^m\ d(f^{(m)}(s)) \\ &+ \frac{1}{2m!}\int_{\frac{a+b}{2}}^b (b-s)^m\ d(f^{(m)}(s)) \qquad (16)\end{aligned}$$

Proof. We utilise the following Taylor's representation for *m-time* differentiable functions $f:[a,b]\to\mathbb{R}$ whose m^{th} derivatives $f^{(m)}$ are of locally bounded variation on $[a,b]$ (see [8]).

$$f(t)=\sum_{k=0}^{m}\frac{1}{k!}(t-c)^k f^{(k)}(c)+\frac{1}{m!}\int_c^t (t-s)^m\ d(f^{(m)}(s)) \qquad (17)$$

where t and c are in $[a,b]$ and the integral in the remainder is taken in the Riemann-Stieltjes sence.
If we choose in (17), $c=\frac{a+b}{2}$ and $t=a$, then we get,

$$\begin{aligned}f(a) &= \sum_{k=0}^{m}\frac{1}{k!}\left(\frac{a-b}{2}\right)^k f^{(k)}\left(\frac{a+b}{2}\right)+\frac{1}{m!}\int_{\frac{a+b}{2}}^a (a-s)^m\ d(f^{(m)}(s)) \\ &= \sum_{k=0}^{m}\frac{(-1)^k}{2^k k!}(b-a)^k f^{(k)}\left(\frac{a+b}{2}\right) \\ &\quad +\frac{(-1)^{m+1}}{m!}\int_a^{\frac{a+b}{2}}(s-a)^m\ d(f^{(m)}(s)) \qquad (18) \\ &= f\left(\frac{a+b}{2}\right)+\sum_{k=1}^{m}\frac{(-1)^k}{2^k k!}(b-a)^k f^{(k)}\left(\frac{a+b}{2}\right) \\ &\quad +\frac{(-1)^{m+1}}{m!}\int_a^{\frac{a+b}{2}}(s-a)^m\ d(f^{(m)}(s)).\end{aligned}$$

If we choose in (17), $c = \frac{a+b}{2}$ and $t = b$, then we also get,

$$
\begin{aligned}
f(b) &= \sum_{k=0}^{m} \frac{1}{2^k k!}(b-a)^k f^{(k)}\left(\frac{a+b}{2}\right) + \frac{1}{m!}\int_{\frac{a+b}{2}}^{b} (b-s)^m \; d(f^{(m)}(s)) \\
&= f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{m} \frac{1}{2^k k!}(b-a)^k f^{(k)}\left(\frac{a+b}{2}\right) \\
&\quad + \frac{1}{m!}\int_{\frac{a+b}{2}}^{b} (b-s)^m \; d(f^{(m)}(s)).
\end{aligned} \tag{19}
$$

If we add the equality (18) with (19) and divide the sum by 2, then we get,

$$
\begin{aligned}
\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right) &+ \sum_{k=1}^{m} \left[\frac{(-1)^k+1}{2^{k+1}k!}\right](b-a)^k f^{(k)}\left(\frac{a+b}{2}\right) \\
&+ \frac{1}{2m!}(-1)^{m+1}\int_{a}^{\frac{a+b}{2}} (s-a)^m \; d(f^{(m)}(s)) \\
&+ \frac{1}{2m!}\int_{\frac{a+b}{2}}^{b} (b-s)^m \; d(f^{(m)}(s))
\end{aligned}
$$

which completes the proof. □

Corollary 12. *Under the assumptions of Theorem 11, we have the following estimate for the error term in* (16)

$$
\begin{aligned}
&\frac{1}{2m!}\left|(-1)^{m+1}\int_{a}^{\frac{a+b}{2}} (s-a)^m \; d(f^{(m)}(s)) + \int_{\frac{a+b}{2}}^{b} (b-s)^m \; d(f^{(m)}(s))\right| \\
\leq\; &\frac{(b-a)^m}{2^{m+1}m!}\bigvee_{a}^{b}(f^{(m)}).
\end{aligned} \tag{20}
$$

Proof. Note that for any continuous function $p : [\alpha, \beta] \to \mathbb{R}$ and $v : [\alpha, \beta] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_{\alpha}^{\beta} p(t)dv(t)$ exists and

$$
\left|\int_{\alpha}^{\beta} p(t)dv(t)\right| \leq \max_{t\in[\alpha,\beta]} |p(t)| \bigvee_{\alpha}^{\beta}(v). \tag{21}
$$

Using (21) we have the following

$$
\begin{aligned}
&\left|\frac{1}{2m!}(-1)^{m+1}\int_{a}^{\frac{a+b}{2}} (s-a)^m \; d(f^{(m)}(s)) + \frac{1}{2m!}\int_{\frac{a+b}{2}}^{b} (b-s)^m \; d(f^{(m)}(s))\right| \\
\leq\; &\frac{1}{2m!}\left[\max_{t\in[a,(a+b)/2]}(s-a)^m \bigvee_{a}^{\frac{a+b}{2}}(f^{(m)}) + \max_{t\in[(a+b)/2,b]}(b-s)^m \bigvee_{\frac{a+b}{2}}^{b}(f^{(m)})\right] \\
=\; &\frac{1}{2m!}\left[\frac{(b-a)^m}{2^m}\bigvee_{a}^{\frac{a+b}{2}}(f^{(m)}) + \frac{(b-a)^m}{2^m}\bigvee_{\frac{a+b}{2}}^{b}(f^{(m)})\right] \\
=\; &\frac{1}{2m!}\left[\frac{(b-a)^m}{2^m}\bigvee_{a}^{b}(f^{(m)})\right] = \frac{(b-a)^m}{2^{m+1}m!}\bigvee_{a}^{b}(f^{(m)})
\end{aligned}
$$

which completes the proof. □

Theorem 13 (Dragomir [4])**.** *Let $f:[a,b]\to\mathbb{R}$ be a function whose m^{th} derivatives $f^{(m)}$ are of locally bounded variation on $[a,b]$.*

$$f\left(\frac{a+b}{2}\right) = \frac{f(a)+f(b)}{2} + \sum_{k=1}^{m}\frac{(b-a)^k}{2^{k+1}k!}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right] + \int_a^b M_n(t)d\left(f^{(m)}(t)\right), \tag{22}$$

where

$$M_n(t)=\frac{1}{2n!}\times\begin{cases}\left(\frac{a+b}{2}-t\right)^n, & \text{if } t\in[a,\frac{a+b}{2}]\\ (-1)^n\left(t-\frac{a+b}{2}\right)^n, & \text{if } t\in(\frac{a+b}{2},b]\end{cases} \tag{23}$$

We refer to [4, Corollary 2] for the proof of this theorem.

Remark 14. By utilising (21) we have the following bound for the error term in (22)

$$\left|\int_a^b M_m(t)d\left(f^{(m)}(t)\right)\right| \le \frac{(b-a)^m}{2^{m+1}m!}\bigvee_a^b(f^{(m)}). \tag{24}$$

We refer to [4, Corollary 3] for the proof.

4. Approximations of generalised Jensen divergence

We consider a function $\Phi: I\to\mathbb{R}$ that is *m-time* differentiable ($m\ge 1$) and the derivative $\Phi^{(m-1)}$ is locally absolutely continuous on I, this means that it is absolutely continuous on any closed subinterval [a,b] of I. For $k=1,2,\dots,m$, we define

$$\begin{aligned}
P_{n,\Phi,k}(x,y) &:= \sum_{i=1}^{n}(y_i-x_i)^k\Phi^{(k)}\left(\frac{x_i+y_i}{2}\right)\\
Q_{n,\Phi,k}(x,y) &:= \sum_{i=1}^{n}(y_i-x_i)^k\left[\Phi^{(k)}(x_i)+(-1)^k\Phi^{(k)}(y_i)\right]\\
E_{n,\Phi,m}(x,y) &:= \frac{(-1)^m}{2m!}\sum_{i=1}^{n}\left[\frac{1}{y_i-x_i}\int_{x_i}^{\frac{x_i+y_i}{2}}\left[(t-x_i)^m-(t-y_i)^m\right]\Phi^{(m)}dt\right.\\
&\quad\left.+\frac{1}{y_i-x_i}\int_{\frac{x_i+y_i}{2}}^{y_i}\left[(t-y_i)^m-(t-x_1)^m\right]\Phi^{(m)}dt\right]
\end{aligned}$$

where $x=(x_1,...,x_n),\ y=(y_1,...,y_n)\ \in I^n$ and the integral above is taken in the sense of a Riemann-Stieltjes. The following representation for the $\mathcal{J}$-divergence can be stated.

Theorem 15. *Let $\Phi: I\to\mathbb{R}$ be a function on I such that the derivative $\Phi^{(m-1)}$ be absolutely continuous on I. Then, Then,*

$$\mathcal{J}_{n,\Phi}(x,y) := \sum_{k=1}^{m-1}\frac{1}{2(k+1)!}\left[\frac{(-1)^k+1}{2^k}P_{n,\Phi,k}(x,y)-Q_{n,\Phi,k}(x,y)\right]+E_{n,\Phi,k}(x,y) \tag{25}$$

for any vector $x,y\in I^n$.

Proof. We employ the result of Theorem 8 for $f = \Phi$, $a = x_i$ and $b = y_i$, $i \in \{1, ..., n\}$ and sum over i from 1 to n, then we deduce the desired representation; and the proof is completed. □

Corollary 16. *Under the assumptions of Theorem 15, we have the following estimate:*

$$|E_{n,\Phi,m}(x,y)| \leq \frac{1}{2m!} \times \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{2|y_i - x_i|^m}{m+1} \max_{i\in\{1,\ldots,n\}} \|\Phi^{(m)}\|_{[x_i,y_i],\infty}, \\ \displaystyle\sum_{i=1}^{n} \frac{2|y_i - x_i|^{m+1/p-1}}{(pm+1)^{1/p}} \max_{i\in\{1,\ldots,n\}} \|\Phi^{(m)}\|_{[x_i,y_i],p'}, \quad p > 1, \\ \displaystyle\sum_{i=1}^{n} 2|y_i - x_i|^{m-1} \max_{i\in\{1,\ldots,n\}} \|\Phi^{(m)}\|_{[x_i,y_i],1}. \end{cases}$$

The proof follows by Corollary 9.

We consider now, a function $\Phi : I \to \mathbb{R}$ that is *m-time* differentiable ($m \geq 1$) and the m^{th} derivative $\Phi^{(m)}$ is of locally bounded variation on I, this means that it is of bounded variation on any closed subinterval [a,b] of I. For $k = 1, 2, ..., m$, we recall

$$P_{n,\Phi,k}(x,y) := \sum_{i=1}^{n} (y_i - x_i)^k \Phi^{(k)}\left(\frac{x_i + y_i}{2}\right)$$

and define

$$\begin{aligned} R_{n,\Phi,m}(x,y) \quad &:= \quad \frac{1}{2m!} \sum_{i=1}^{n} \left[(-1)^{m+1} \int_{x_i}^{\frac{x_i+y_i}{2}} (t - x_i)^m d(\Phi^{(m)}(t)) \right. \\ &\quad \left. + \int_{\frac{x_i+y_i}{2}}^{y_i} (t - x_i)^m d(\Phi^{(m)}(t)) \right] \end{aligned}$$

where $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in I^n$ and the integral above is taken in the sense of a Riemann-Stieltjes.

The following representation for the $\mathcal{J}$-divergence can be stated.

Theorem 17. *Let $\Phi : I \to \mathbb{R}$ be a m-time differentiable function on I and the m^{th} derivative $\Phi^{(m)}$ be of locally bounded variation on I. Then,*

$$\mathcal{J}_{n,\Phi}(x,y) := \sum_{k=1}^{m} \left[\frac{(-1)^k + 1}{2^{k+1} k!} \right] P_{n,\Phi,k}(x,y) + R_{n,\Phi,m}(x,y) \tag{26}$$

for any vector $x, y \in I^n$.

Proof. We employ the result of Theorem 11 for $f = \Phi$, $a = x_i$ and $b = y_i$, $i \in \{1, ..., n\}$ and sum over i from 1 to n, then we deduce the desired representation (26); and the proof is completed. □

Corollary 18. *Under the assumptions of Theorem 17, we have the error estimate:*

$$\begin{aligned} |R_{n,\Phi,m}(x,y)| &\leq \frac{1}{2^{m+1}m!} \sum_{i=1}^{n} |y_i - x_i|^m \left| \bigvee_{x_i}^{y_i} (\Phi^{(m)}) \right| \\ &\leq \frac{1}{2^{m+1}m!} \max_{i\in\{1,\ldots n\}} \left| \bigvee_{x_i}^{y_i} (\Phi^{(m)}) \right| \sum_{i=1}^{n} |y_i - x_i|^m, \end{aligned} \tag{27}$$

for any $x, y \in I^m$.

In particular if $x, y \in [a,b]^n \subset I^n$, then we have the simpler bound:

$$|R_{n,\Phi,m}(x,y)| \le \frac{1}{2^{m+1}m!}\bigvee_a^b(\Phi^{(m)})\Big(\sum_{i=1}^n |y_i - x_i|^m\Big). \tag{28}$$

The proof follows by Corollary 12.

Corollary 19. *Under the assumptions of Theorem 17 and if the m^{th} derivative $\Phi^{(m)}$ is Lipschitzian with the constant $L_m \ge 0$ on I, then we have the error estimate:*

$$|R_{n,\Phi,m}(x,y)| \le \frac{L_m}{2^{m+1}(m+1)!}\sum_{i=1}^n |y_i - x_i|^{m+1}, \tag{29}$$

for any $x, y \in I^n$.

Proof. It is well known that if $p : [c,d] \to \mathbb{R}$ is a Riemann integrable function and $v : [c,d] \to \mathbb{R}$ is Lipschitzian with constant L, then the Riemann-Stieltjes integral $\int_c^d p(t)\ dv(t)$ exists and,

$$\left|\int_c^d p(t)\ dv(t)\right| \le L\int_c^d |p(t)|\ dt.$$

Therefore,

$$\begin{aligned}
&|R_{n,\Phi,m}(x,y)| \\
\le\ & \frac{1}{2m!}\left\{\sum_{i=1}^n\left[\left|\int_{x_i}^{\frac{x_i+y_i}{2}}(t-x_i)^m\ d(\Phi^{(m)}(t))\right| + \left|\int_{\frac{x_i+y_i}{2}}^{y_i}(y_i-t)^m\ d(\Phi^{(m)}(t))\right|\right]\right\} \\
\le\ & \frac{L_m}{2m!}\left\{\sum_{i=1}^n\left[\left|\int_{x_i}^{\frac{x_i+y_i}{2}}(t-x_i)^m\ dt\right| + \left|\int_{\frac{x_i+y_i}{2}}^{y_i}(y_i-t)^m\ dt\right|\right]\right\} \\
=\ & \frac{L_m}{2m!}\sum_{i=1}^n\left[\frac{|y_i-x_i|^{m+1}}{(m+1)2^{m+1}} + \frac{|y_i-x_i|^{m+1}}{(m+1)2^{m+1}}\right] \\
=\ & \frac{L_m}{2^{m+1}(m+1)!}\sum_{i=1}^n |y_i-x_i|^{m+1}, \quad \text{for any } x,y \in I^n.
\end{aligned}$$

This proves the desired inequality (29). ∎

For $k = 1, 2, \dots, m$, we now define

$$\begin{aligned}
Y_{n,\Phi,k}(x,y) \ :=\ & -\frac{1}{2m!}\sum_{i=1}^n \frac{1}{y_i - x_i}\int_{x_i}^{\frac{x_i+y_i}{2}}\left(\frac{x_i+y_i}{2} - t\right)\ d\big(\Phi^{(m)}(t)\big) \\
& -\frac{1}{2m!}\sum_{i=1}^n \frac{1}{y_i - x_i}\int_{\frac{x_i+y_i}{2}}^{y_i}(-1)^n\left(t - \frac{x_i+y_i}{2}\right)\ d\big(\Phi^{(m)}(t)\big)
\end{aligned}$$

where $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n) \in I^n$ and the integral above is taken in the sense of a Riemann-Stieltjes. The following representation for the $\mathcal{J}$-divergence can be stated.

Theorem 20. *Let $\Phi : I \to \mathbb{R}$ be a m-time differentiable function on I and the m^{th} derivative $\Phi^{(m)}$ be of locally bounded variation on I. Then,*

$$\mathcal{J}_{n,\Phi}(x,y) := -\sum_{i=1}^n\sum_{k=1}^m \frac{(y_i-x_i)^k}{2^{k+1}k!}\left[\Phi^{(k)}(x_i) + (-1)^k\Phi^{(k)}(y_i)\right] + Y_{n,\Phi,m}(x,y), \tag{30}$$

for any vector $x, y \in I^n$.

The proof follows by Theorem 13.

Corollary 21. *We have the following error estimates*

$$|Y_{n,\Phi,m}(x,y)| \leq \sum_{i=1}^{n} \frac{|y_i - x_i|^m}{2^{m+1}m!} \max_{i\{1,\ldots,n\}} \{\bigvee_{x_i}^{y_i}(\Phi^{(m)})\}.$$

The proof follows by Remark 14.

5. Application to some elementary functions

In this section, we consider the approximation of Jensen divergence for some elementary functions.

(1) First, we consider the exponential function, i.e. $\Phi(t) = e^t$. We have, from Theorem 15

$$\mathcal{J}_{n,e^t}(x,y) \approx \sum_{i=1}^{n}\sum_{k=1}^{m} \frac{(y_i - x_i)^k}{2(k+1)!}\left[\frac{1+(-1)^k}{2^k} e^{\frac{x_i+y_i}{2}} - e^{x_i} - (-1)^k e^{y_i}\right]$$

with the remainder $E_{n,e^t,m}(x,y)$ satisfies the bound,

$$|E_{n,e^t,m}| \leq \frac{1}{2m!} \times \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{2|y_i - x_i|^m}{m+1} \max_{i\in\{1,\ldots,n\}} e^{y_i}, \\ \displaystyle\sum_{i=1}^{n} \frac{2|y_i - x_i|^{m+1/p-1}}{(pm+1)^{1/p}} \max_{i\in\{1,\ldots,n\}} \left(\frac{e^{p'y_i} - e^{p'x_i}}{p'}\right)^{1/p'}, & p > 1, \\ \displaystyle\sum_{i=1}^{n} 2|y_i - x_i|^{m-1} \max_{i\in\{1,\ldots,n\}} \left(e^{y_i} - e^{x_i}\right). \end{cases}$$

Theorem 17 gives us

$$\mathcal{J}_{n,e^t}(x,y) \approx \sum_{i=1}^{n}\sum_{k=1}^{m} \frac{(-1)^k + 1}{2^{k+1}k!}(y_i - x_i)^k e^{\frac{x_i+y_i}{2}}$$

where the remainder, $R_{n,e^t,m}(x,y)$ satisfies the bound,

$$\begin{aligned} |R_{n,e^t,m}(x,y)| &\leq \frac{1}{2^{m+1}m!} \max_{i\in\{1,\ldots n\}} \left(e^{y_i} - e^{x_i}\right) \sum_{i=1}^{n} |y_i - x_i|^m \\ &\leq \frac{1}{2^{m+1}m!} \max_{i\in\{1,\ldots n\}} \left(e^{y_i} - e^{x_i}\right) \left(\sum_{i=1}^{n} |y_i - x_i|\right)^m. \end{aligned}$$

Finally, Theorem 20 gives us

$$\mathcal{J}_{n,e^t}(x,y) \approx -\sum_{i=1}^{n}\sum_{k=1}^{m} \frac{(y_i - x_i)^k}{2^{k+1}k!}\left[e^{x_i} + (-1)^k e^{y_i}\right]$$

with the remainder $Y_{n,e^t,m}(x,y)$ satisfies the bound,

$$|Y_{n,e^t,m}(x,y)| \leq \frac{1}{2^{m+1}m!} \max_{i\in\{1,\ldots n\}} \left(e^{y_i} - e^{x_i}\right) \sum_{i=1}^{n} |y_i - x_i|^m.$$

(2) We now consider the function $\Phi(t) = t^p$, where $p > m$. We have, from Theorem 15

$$\begin{aligned} \mathcal{J}_{n,t^p}(x,y) \approx{}& \sum_{i=1}^{n}\sum_{k=1}^{m} \frac{(y_i - x_i)^k}{2(k+1)!}\left[\frac{1+(-1)^k}{2^k}\frac{p!}{(p-k)!}\left(\frac{x_i+y_i}{2}\right)^{p-k}\right. \\ &\left. - \frac{p!}{(p-k)!}{x_i}^{p-k} - \frac{p!(-1)^k}{(p-k)!}{y_i}^{p-k}\right] \end{aligned}$$

with the remainder $E_{n,e^t,m}(x,y)$ satisfies the bound,

$$|E_{n,t^p,m}| \leq \frac{p!}{2m!(p-m)!} \times \begin{cases} \sum_{i=1}^{n} \frac{2|y_i - x_i|^m}{m+1} \max_{i \in \{1,\ldots,n\}} \|t^{p-m}\|_{[x_i,y_i],\infty}, \\ \sum_{i=1}^{n} \frac{2|y_i - x_i|^{m+1/p-1}}{(pm+1)^{1/p}} \max_{i \in \{1,\ldots,n\}} \|t^{p-m}\|_{[x_i,y_i],p'}, \quad p>1, \\ \sum_{i=1}^{n} 2|y_i - x_i|^{m-1} \max_{i \in \{1,\ldots,n\}} \|t^{p-m}\|_{[x_i,y_i],1}. \end{cases}$$

Theorem 17 gives us

$$\mathcal{J}_{n,t^p}(x,y) \approx \sum_{i=1}^{n}\sum_{k=1}^{m} \frac{(-1)^k+1}{2^{k+1}k!}(x_i-y_i)^k \frac{p!}{(p-k)!}\left(\frac{x_i+y_i}{2}\right)^{p-k}$$

where the remainder, $R_{n,t^p,m}(x,y)$ satisfies the bound,

$$|R_{n,e^t,m}(x,y)| \leq \frac{p!}{2^{m+1}m!(p-m-1)!} \max_{i \in \{1,\ldots n\}} \|t^{p-m-1}\|_{[x_i,y+i,1]} \sum_{i=1}^{n} |y_i - x_i|^m.$$

Finally, Theorem 20 gives us

$$\mathcal{J}_{n,e^t}(x,y) \approx -\sum_{i=1}^{n}\sum_{k=1}^{m} \frac{(y_i-x_i)^k}{2^{k+1}k!}\frac{p!}{(p-k)!}\left[{x_i}^{p-k} + (-1)^k {y_i}^{p-k}\right]$$

with the remainder $Y_{n,e^t,m}(x,y)$ satisfies the bound,

$$|Y_{n,e^t,m}(x,y)| \leq \frac{p!}{2^{m+1}m!(p-m-1)!} \max_{i \in \{1,\ldots n\}} \|t^{p-m-1}\|_{[x_i,y+i,1]} \sum_{i=1}^{n} |y_i - x_i|^m.$$

(3) We consider the function $\Phi(t) = -\log(t)$, where $t \geq 1$. We have, from Theorem 15

$$\mathcal{J}_{n,t^p}(x,y) \approx \sum_{i=1}^{n}\sum_{k=1}^{m} \frac{(y_i-x_i)^k}{2(k+1)!}\left[\frac{1+(-1)^k}{2^k}\frac{(-1)^{k-1}(k-1)!}{\left(\frac{x_i+y_i}{2}\right)^k} - \frac{(-1)^{k-1}(k-1)!}{x_i^k} - \frac{(-1)^{2k-1}(k-1)!}{y_i^k}\right]$$

with the remainder $E_{n,e^t,m}(x,y)$ satisfies the bound,

$$|E_{n,t^p,m}| \leq \frac{(m-1)!}{2m!} \times \begin{cases} \sum_{i=1}^{n} \frac{2|y_i - x_i|^m}{m+1} \max_{i \in \{1,\ldots,n\}} \|t^{-m}\|_{[x_i,y_i],\infty}, \\ \sum_{i=1}^{n} \frac{2|y_i - x_i|^{m+1/p-1}}{(pm+1)^{1/p}} \max_{i \in \{1,\ldots,n\}} \|t^{-m}\|_{[x_i,y_i],p'}, \quad p>1, \\ \sum_{i=1}^{n} 2|y_i - x_i|^{m-1} \max_{i \in \{1,\ldots,n\}} \|t^{-m}\|_{[x_i,y_i],1}. \end{cases}$$

Theorem 17 gives us

$$\mathcal{J}_{n,t^p}(x,y) \quad \approx \quad \sum_{i=1}^{n}\sum_{k=1}^{m}\frac{(-1)^k+1}{2^{k+1}k}(x_i-y_i)^k\,\frac{(-1)^{k-1}}{\left(\frac{x_i+y_i}{2}\right)^k}$$

where the remainder, $R_{n,t^p,m}\left(x,y\right)$ satisfies the bound,

$$|R_{n,e^t,m}(x,y)| \leq \frac{1}{2^{m+1}}\max_{i\in\{1,\ldots n\}}\|t^{-m-1}\|_{[x_i,y+i,1]}\sum_{i=1}^{n}|y_i-x_i|^m.$$

Finally, Theorem 20 gives us

$$\mathcal{J}_{n,e^t}(x,y) \approx -\sum_{i=1}^{n}\sum_{k=1}^{m}\frac{(y_i-x_i)^k}{2^{k+1}k}(-1)^{k-1}\left[{x_i}^{-k}+(-1)^k{y_i}^{-k}\right]$$

with the remainder $Y_{n,e^t,m}\left(x,y\right)$ satisfies the bound,

$$|Y_{n,e^t,m}(x,y)| \leq \frac{1}{2^{m+1}}\max_{i\in\{1,\ldots n\}}\|t^{-m-1}\|_{[x_i,y+i,1]}\sum_{i=1}^{n}|y_i-x_i|^m.$$

In the following figures, choose $n = 20$, $m = 2,4,6$ for $\Phi(t) = -\log(t)$, $I = [10,20]$. We observe that the approximation in Theorem 17 converges faster than the other two, whilst the approximation in Theorem 15 is the slowest.

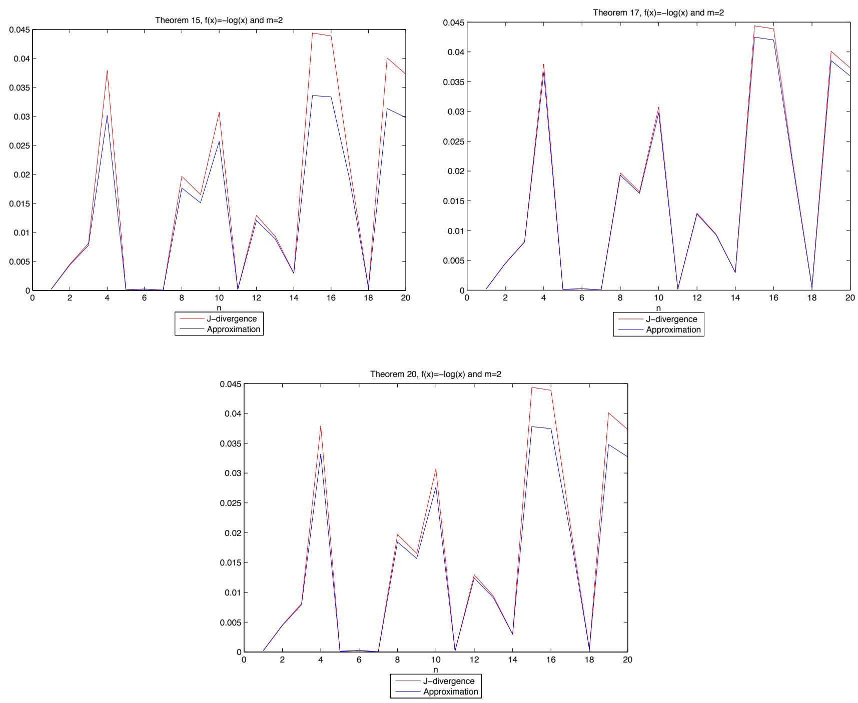

FIGURE 1: JENSEN DIVERGENCE AND ITS APPROXIMATION ($m = 2$)

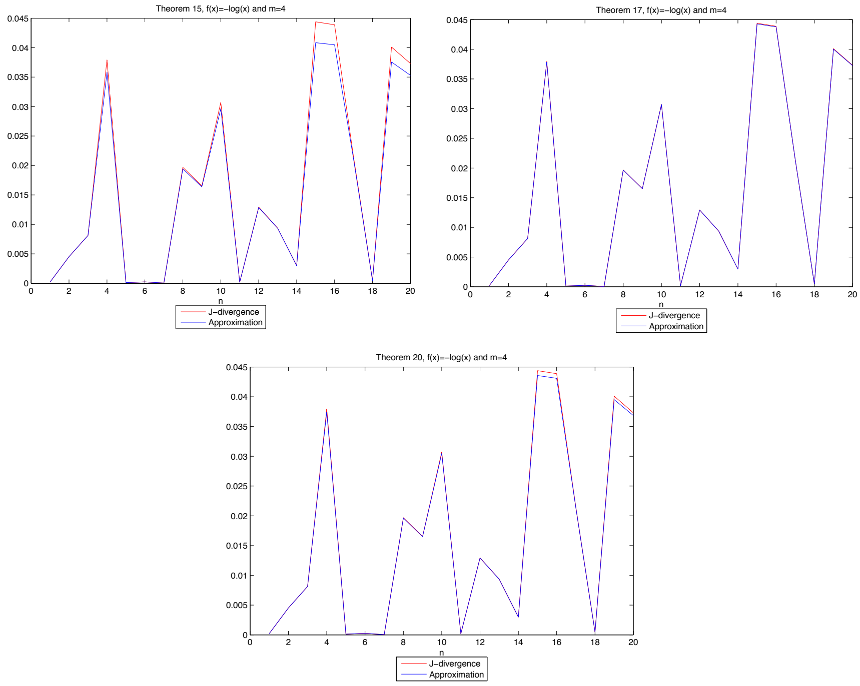

FIGURE 2: JENSEN DIVERGENCE AND ITS APPROXIMATION $(m = 4)$

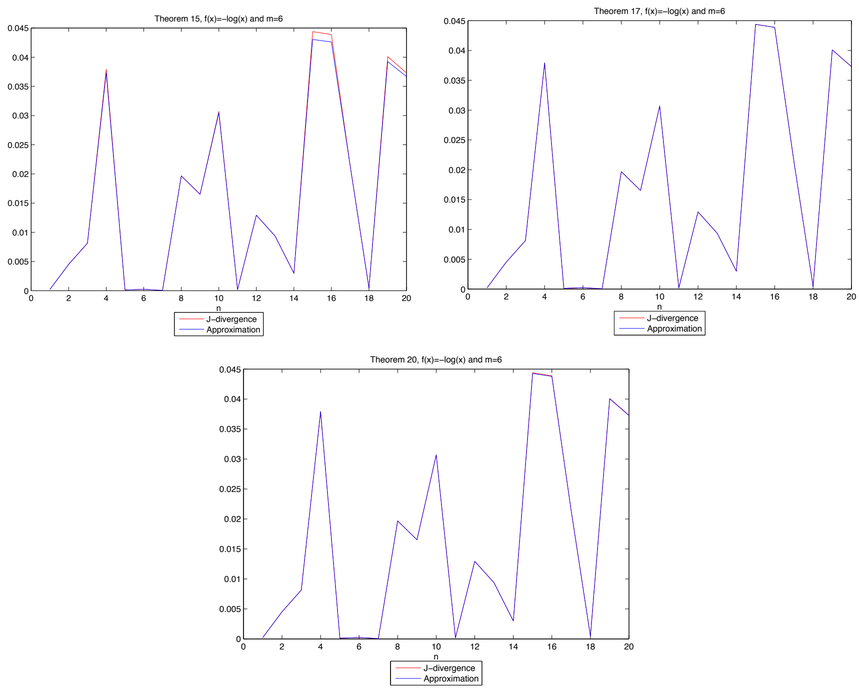

FIGURE 3: JENSEN DIVERGENCE AND ITS APPROXIMATION $(m = 6)$

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