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ON APPROXIMATION OF THE RIEMANN–STIELTJES INTEGRAL AND APPLICATIONS

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Abstract. Several inequalities of Grüss type for the Stieltjes integral with various type of integrand and integrator are introduced. Some improvements inequalities are proved. Applications to the approximation problem of the Riemann-Stieltjes integral are also pointed out.

1. Introduction

In 2002, Guessab and Schmeisser [19], incorporate the mid-point and the trapezoid inequality together, and they have proved the following companion of Ostrowski's inequality:

Theorem 1. *Assume that the function $f : [a, b] \to \mathbb{R}$ is of r-H–Hölder type, where $r \in (0, 1]$ and $H > 0$ are given, i.e.,*

$$|f(t) - f(s)| \le H |t - s|^r ,$$

for any $t, s \in [a, b]$. Then, for each $x \in [a, \frac{a+b}{2}]$, one has the inequality

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le H \left[\frac{2^{r+1} (x-a)^{r+1} + (a+b-2x)^{r+1}}{2^r (r+1)(b-a)} \right]. \tag{1.1}$$

This inequality is sharp for each admissible x. Equality is obtained if and only if $f = \pm H f_* + c$, with $c \in \mathbb{R}$ and

$$f_*(t) = \begin{cases} (x-t)^r , & a \le t \le x \\ (t-x)^r , & x \le t \le \frac{a+b}{2} \\ f_*(a+b-x) , & \frac{a+b}{2} \le t \le b \end{cases} .$$

In [11], Dragomir has proved the following companion of the Ostrowski inequality for mappings of bounded variation:

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Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right] \cdot \bigvee_a^b (f), \tag{1.2}$$

for any $x \in [a, \frac{a+b}{2}]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $1/4$ is best possible.

Also, Dragomir in [12], has proved some companions of Ostrowski's integral inequality for absolutely continuous mappings. Among others, our interest is incorporated in the following result:

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that $f' \in L_\infty[a,b]$. Then we have the inequality*

$$\begin{aligned}\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \le \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty,\end{aligned} \tag{1.3}$$

for all $x \in [a, \frac{a+b}{2}]$.

For recent results, generalizations and new inequalities concerning the approximation problem of the Riemann–Stieltjes integral under various assumptions, see the recent papers [2]–[18], [20] and the references therein.

By Guessab–Schmeisser functional we mean the functional

$$\mathcal{GS}(f;u) := \int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt,$$

provided that the Stieltjes integral $\int_a^b \frac{f(x)+f(a+b-x)}{2}du(x)$, and the Riemann integral $\int_a^b f(t)\,dt$ exist.

Motivated by Guessab–Schmeisser companion of Ostrowski inequality (1.1), the author of this paper, has established the functional $\mathcal{GS}(f;u)$ in [1], and he has proved the following results in estimating $\mathcal{GS}(f;u)$.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, where r and $H > 0$ are given, and $u : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{H}{r+1}(b-a)^r \cdot \bigvee_a^{\frac{a+b}{2}}(u), \tag{1.4}$$

holds.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, and $u : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, where r and $H, L > 0$ are given. Then the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{LH}{(r+1)(r+2)}(b-a)^{r+1} \tag{1.5}$$

holds.

In this paper we point out several bounds for the functional $\mathcal{GS}(f;u)$ with various type of integrand and integrator. Improvements bounds for $\mathcal{GS}(f;u)$ are proved. Finally, we apply the obtained results to approximate the Riemann-Stieltjes integral

$$\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x)$$

in terms of the Riemann integral $\int_a^b f(t)\,dt$

2. The case of bounded variation integrators

2.1. **The case of bounded variation integrands.**

Theorem 6. *Let $u:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ be continuous and of bounded variation on $[a,b]$. Then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{1}{2}\bigvee_a^b(f)\cdot\bigvee_a^{\frac{a+b}{2}}(u). \tag{2.1}$$

Proof. Using the fact that for a continuous function $p:[a,b]\to\mathbb{R}$ and a function $\nu:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \le \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(\nu). \tag{2.2}$$

As u is of bounded variation on $[a,b]$ and f is continuous, by (2.2) we have

$$\begin{aligned}|\mathcal{GS}(f;u)| &= \left|\int_a^{\frac{a+b}{2}}\left[\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]du(x)\right| \\ &\le \sup_{x\in\left[a,\frac{a+b}{2}\right]}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\cdot\bigvee_a^{\frac{a+b}{2}}(u).\end{aligned}$$

Since f is of bounded variation, then using the companion of Ostrowski type inequality (1.2), we may state that

$$\begin{aligned}\sup_{x\in\left[a,\frac{a+b}{2}\right]}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le \sup_{x\in\left[a,\frac{a+b}{2}\right]}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b(f) \\ &\le \frac{1}{2}\bigvee_a^b(f).\end{aligned}$$

It follows that

$$\left|\mathcal{GS}\left(f;u\right)\right| \le \sup_{x\in \ a,\frac{a+b}{2}} \left|\frac{f\left(x\right)+f\left(a+b-x\right)}{2}-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right| \cdot \bigvee_a^{\frac{a+b}{2}}\left(u\right)$$

$$\le \frac{1}{2}\bigvee_a^b\left(f\right)\cdot\bigvee_a^{\frac{a+b}{2}}\left(u\right),$$

and the theorem is proved. □

Remark 1. *We remark that if* $\bigvee_a^{\frac{a+b}{2}}\left(u\right)=\bigvee_{\frac{a+b}{2}}^b\left(u\right)$, *then (2.1) becomes*

$$\left|\mathcal{GS}\left(f;u\right)\right| \le \frac{1}{4}\bigvee_a^b\left(f\right)\cdot\bigvee_a^b\left(u\right) \tag{2.3}$$

Corollary 1. *Let* u *as in Theorem 6.*

(1) *If* $f:[a,b]\to\mathbb{R}$ *be an* L*–Lipschitzian mapping on* $[a,b]$*. Then we have the inequality:*

$$\left|\mathcal{GS}\left(f;u\right)\right| \le \frac{1}{2}L\left(b-a\right)\cdot\bigvee_a^{\frac{a+b}{2}}\left(u\right). \tag{2.4}$$

(2) *If* $f\in C^{(1)}[a,b]$*. Then we have the inequality*

$$\left|\mathcal{GS}\left(f;u\right)\right| \le \frac{1}{2}\bigvee_a^{\frac{a+b}{2}}\left(u\right)\cdot\left\|f'\right\|_{1,[a,b]}. \tag{2.5}$$

(3) *If* $f:[a,b]\to\mathbb{R}$ *be a monotonic mapping. Then we have the inequality*

$$\left|\mathcal{GS}\left(f;u\right)\right| \le \frac{1}{2}\bigvee_a^{\frac{a+b}{2}}\left(u\right)\cdot\left|f\left(b\right)-f\left(a\right)\right|, \tag{2.6}$$

where $\left\|\cdot\right\|_1$ *is the* L_1 *norm, namely* $\left\|f'\right\|_{1,[a,b]}:=\int_a^b\left|f'\left(t\right)\right|dt$.

Corollary 2. *Let* f *as in Theorem 6.*

(1) *If* $u:[a,b]\to\mathbb{R}$ *be an* K*–Lipschitzian mapping on* $[a,b]$*. Then we have the inequality:*

$$\left|\mathcal{GS}\left(f;u\right)\right| \le \frac{1}{4}K\left(b-a\right)\cdot\bigvee_a^b\left(f\right). \tag{2.7}$$

(2) *If* $u\in C^{(1)}[a,b]$*. Then we have the inequality*

$$\left|\mathcal{GS}\left(f;u\right)\right| \le \frac{1}{2}\bigvee_a^b\left(f\right)\cdot\left\|u'\right\|_{1,\ a,\frac{a+b}{2}}. \tag{2.8}$$

(3) *If $u:[a,b]\to\mathbb{R}$ be a monotonic mapping. Then we have the inequality*

$$|\mathcal{GS}(f;u)|\le\frac{1}{2}\bigvee_a^b(f)\cdot\left|u\left(\frac{a+b}{2}\right)-u(a)\right|, \tag{2.9}$$

where $\|\cdot\|_1$ is the L_1 norm, namely $\|u'\|_{1,\ a,\frac{a+b}{2}}:=\int_a^{\frac{a+b}{2}}|u'(t)|\,dt$.

Remark 2. *In Corollary 1, we have the following cases:*

(1) *If f is L–Lipschitzian mapping on $[a,b]$ and*

(a) *u is K–Lipschitzian mapping on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{4}KL(b-a)^2. \tag{2.10}$$

(b) *$u\in C^{(1)}[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{2}L(b-a)\cdot\|u'\|_{1,\ a,\frac{a+b}{2}}. \tag{2.11}$$

(c) *u is monotonic on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{2}L(b-a)\cdot\left|u\left(\frac{a+b}{2}\right)-u(a)\right|. \tag{2.12}$$

(2) *If $f\in C^{(1)}[a,b]$, and*

(a) *u is K–Lipschitzian mapping on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{4}K(b-a)\|f'\|_{1,[a,b]}. \tag{2.13}$$

(b) *$u\in C^{(1)}[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{2}\|f'\|_{1,[a,b]}\|u'\|_{1,\ a,\frac{a+b}{2}}. \tag{2.14}$$

(c) *u is monotonic on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{2}\|f'\|_{1,[a,b]}\cdot\left|u\left(\frac{a+b}{2}\right)-u(a)\right|. \tag{2.15}$$

(3) *If f is monotonic on $[a,b]$, and*

(a) *u is K–Lipschitzian mapping on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{4}K(b-a)|f(b)-f(a)|. \tag{2.16}$$

(b) *$u\in C^{(1)}[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\le\frac{1}{2}|f(b)-f(a)|\,\|u'\|_{1,\ a,\frac{a+b}{2}}. \tag{2.17}$$

(c) *u is monotonic on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{1}{2}|f(b)-f(a)| \cdot \left|u\left(\frac{a+b}{2}\right)-u(a)\right|. \tag{2.18}$$

Remark 3. *In Corollary 2, we have the following cases:*

(1) *If u is K–Lipschitzian mapping on $[a,b]$ and*
   (a) *f is L–Lipschitzian mapping on $[a,b]$, then the inequality (2.10) holds.*
   (b) *$f \in C^{(1)}[a,b]$, then the inequality (2.13) holds.*
   (c) *f is monotonic on $[a,b]$, then the inequality (2.16) holds.*

(2) *If $u \in C^{(1)}[a,b]$, and*
   (a) *f is L–Lipschitzian mapping on $[a,b]$, then the inequality (2.11) holds.*
   (b) *$f \in C^{(1)}[a,b]$, then the inequality (2.14) holds.*
   (c) *f is monotonic on $[a,b]$, then the inequality (2.17) holds.*

(3) *If u is monotonic on $[a,b]$, and*
   (a) *f is L–Lipschitzian mapping on $[a,b]$, then the inequality (2.12) holds.*
   (b) *$f \in C^{(1)}[a,b]$, then the inequality (2.15) holds.*
   (c) *f is monotonic on $[a,b]$, then the inequality (2.18) holds.*

2.2. **The case of r-H–Hölder type integrands.**

Theorem 7. *Let $u:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is of r-H–Hölder type mapping on $[a,b]$. Then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{H}{2^r(r+1)}(b-a)^r \cdot \bigvee_a^{\frac{a+b}{2}}(u). \tag{2.19}$$

Proof. As u is of bounded variation on $[a,b]$ and f is of r-H–Hölder type on $[a,b]$, by (2.2) we have

$$\begin{aligned}
|\mathcal{GS}(f;u)| &= \left|\int_a^{\frac{a+b}{2}}\left[\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]du(x)\right| \\
&\le \sup_{x\in\left[a,\frac{a+b}{2}\right]}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \cdot \bigvee_a^{\frac{a+b}{2}}(u).
\end{aligned}$$

Using the companion of Ostrowski type inequality (1.1), we may state that

$$\begin{aligned}
&\sup_{x\in\ a,\frac{a+b}{2}} \left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le H \sup_{x\in\ a,\frac{a+b}{2}} \left[\frac{2^{r+1}(x-a)^{r+1}+(a+b-2x)^{r+1}}{2^r(r+1)(b-a)} \right] \\
&\le H\frac{(b-a)^r}{2^r(r+1)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\mathcal{GS}(f;u)| &\le \sup_{x\in\ a,\frac{a+b}{2}} \left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \cdot \bigvee_a^{\frac{a+b}{2}}(u) \\
&\le H\frac{(b-a)^r}{2^r(r+1)} \cdot \bigvee_a^{\frac{a+b}{2}}(u),
\end{aligned}$$

and the theorem is proved. □

Remark 4. *We note that, the inequality (2.19) improves the inequality (1.4) by the constant* $\frac{1}{2^r}$*, and therefore, (2.19) is better than (1.4).*

Corollary 3. *Let* $u:[a,b]\to\mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$ *and* $f:[a,b]\to\mathbb{R}$ *is of* L*–Lipschitzian type mapping on* $[a,b]$*. Then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{1}{4}L(b-a)\cdot \bigvee_a^{\frac{a+b}{2}}(u). \tag{2.20}$$

Corollary 4. *Let* $f:[a,b]\to\mathbb{R}$ *is of* r*-*H*–Hölder type mapping on* $[a,b]$*.*

(1) *If* u *is* K*-Lipschitzian on* $[a,b]$*, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{HK}{2^{r+1}(r+1)}(b-a)^{r+1}. \tag{2.21}$$

(2) *If* $u\in C^{(1)}[a,b]$*, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{H}{2^r(r+1)}(b-a)^r\cdot \|u'\|_{1,\ a,\frac{a+b}{2}}. \tag{2.22}$$

(3) *If* u *is monotonic on* $[a,b]$*, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{H}{2^r(r+1)}(b-a)^r\cdot \left| u\left(\frac{a+b}{2}\right) - u(a) \right|. \tag{2.23}$$

Therefore, we may deduce the following result:

Corollary 5. *Let $f:[a,b]\to\mathbb{R}$ be L–Lipschitzian mapping on $[a,b]$.*

(1) *If u is K-Lipschitzian on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\leq\frac{1}{8}KL(b-a)^2. \tag{2.24}$$

(2) *If $u\in C^{(1)}[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\leq\frac{1}{4}L(b-a)\cdot\|u'\|_{1,\ a,\frac{a+b}{2}}. \tag{2.25}$$

(3) *If u is monotonic on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)|\leq\frac{1}{4}L(b-a)\cdot\left|u\left(\frac{a+b}{2}\right)-u(a)\right|. \tag{2.26}$$

2.3. **The case of absolutely continuous integrands.**

Theorem 8. *Let $u:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$|\mathcal{GS}(f;u)|\leq\frac{1}{4}(b-a)\|f'\|_{\infty,[a,b]}\cdot\bigvee_a^{\frac{a+b}{2}}(u). \tag{2.27}$$

Proof. As u is of bounded variation on $[a,b]$ and f is continuous, by (2.2) we have

$$|\mathcal{GS}(f;u)|=\left|\int_a^{\frac{a+b}{2}}\left[\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]du(x)\right|$$
$$\leq\sup_{x\in\left[a,\frac{a+b}{2}\right]}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\cdot\bigvee_a^{\frac{a+b}{2}}(u).$$

Since f is absolutely continuous on $[a,b]$, then using the companion of Ostrowski type inequality (1.3), we may state that

$$\sup_{x\in\left[a,\frac{a+b}{2}\right]}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq\sup_{x\in\left[a,\frac{a+b}{2}\right]}\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_{\infty,[a,b]}$$
$$\leq\frac{1}{4}(b-a)\|f'\|_{\infty,[a,b]}.$$

It follows that

$$|\mathcal{GS}(f;u)|\leq\sup_{x\in\left[a,\frac{a+b}{2}\right]}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\cdot\bigvee_a^{\frac{a+b}{2}}(u)$$
$$\leq\frac{1}{4}(b-a)\|f'\|_{\infty,[a,b]}\cdot\bigvee_a^{\frac{a+b}{2}}(u),$$

and the theorem is proved. □

Corollary 6. *Let f be as in Theorem 8.*

(1) *If u is K-Lipschitzian on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq \frac{1}{8}K(b-a)^2 \|f'\|_{\infty,[a,b]}. \tag{2.28}$$

(2) *If $u \in C^{(1)}[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq \frac{1}{4}(b-a)\|f'\|_{\infty,[a,b]} \cdot \|u'\|_{1,\ a,\frac{a+b}{2}}. \tag{2.29}$$

(3) *If u is monotonic on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq \frac{1}{4}(b-a)\|f'\|_{\infty,[a,b]} \cdot \left|u\left(\frac{a+b}{2}\right) - u(a)\right|. \tag{2.30}$$

3. The case of Lipschitzian integrators

3.1. The case of bounded variation integrands.

Theorem 9. *Let $u:[a,b]\to\mathbb{R}$ be an K-Lipschitzian on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq \frac{3}{16}K(b-a)\bigvee_a^b(f). \tag{3.1}$$

Proof. It is well-known that for a Riemann integrable function $p:[a,b]\to\mathbb{R}$ and L–Lipschitzian function $\nu:[a,b]\to\mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \leq L\int_a^b |p(t)|\,dt. \tag{3.2}$$

Therefore, as u is K-Lipschitzian on $[a,b]$, by (3.2) we have

$$\begin{aligned}|\mathcal{GS}(f;u)| &= \left|\int_a^{\frac{a+b}{2}}\left[\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right]du(x)\right| \\ &\leq K\int_a^{\frac{a+b}{2}}\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right|dx\end{aligned}$$

Since f is of bounded variation, then using the companion of Ostrowski type inequality (1.2), we may state that

$$\begin{aligned}\int_a^{\frac{a+b}{2}}\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right|dx &\leq \bigvee_a^b(f)\cdot\int_a^{\frac{a+b}{2}}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]dx \\ &\leq \frac{3}{16}(b-a)\bigvee_a^b(f).\end{aligned}$$

It follows that

$$\begin{aligned}|\mathcal{GS}(f;u)| &\leq K\int_a^{\frac{a+b}{2}}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|dx\\ &\leq \frac{3}{16}K(b-a)\bigvee_a^b(f),\end{aligned}$$

and the theorem is proved. □

Corollary 7. *Let u be as in Theorem 9.*

(1) *If f is L-Lipschitzian on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq \frac{3}{16}KL(b-a)^2. \tag{3.3}$$

(2) *If $f \in C^{(1)}[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq \frac{3}{16}K(b-a)\|f'\|_{1,[a,b]}. \tag{3.4}$$

(3) *If f is monotonic on $[a,b]$, then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq \frac{3}{16}K(b-a)\cdot|f(b)-f(a)|. \tag{3.5}$$

3.2. The case of r-H–Hölder type integrands.

Theorem 10. *Let $u:[a,b]\to\mathbb{R}$ be an K-Lipschitzian on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is of r-H–Hölder type mapping on $[a,b]$. Then we have the inequality:*

$$|\mathcal{GS}(f;u)| \leq KH\frac{(b-a)^{r+1}}{2^r(r+1)(r+2)}. \tag{3.6}$$

Proof. As u is K-Lipschitzian on $[a,b]$ and f is continuous, by (3.2) we have

$$|\mathcal{GS}(f;u)| \leq K\int_a^{\frac{a+b}{2}}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|dx$$

Using the companion of Ostrowski type inequality (1.1), we may state that

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|dx\\ &\leq H\int_a^{\frac{a+b}{2}}\left[\frac{2^{r+1}(x-a)^{r+1}+(a+b-2x)^{r+1}}{2^r(r+1)(b-a)}\right]dx\\ &\leq H\frac{(b-a)^{r+1}}{2^r(r+1)(r+2)}.\end{aligned}$$

It follows that

$$|\mathcal{GS}(f;u)| \le K\int_a^{\frac{a+b}{2}} \left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| dx$$
$$\le KH\frac{(b-a)^{r+1}}{2^r(r+1)(r+2)},$$

and the theorem is proved. □

Remark 5. *We note that, the inequality (3.6) improves the inequality (1.5) by the constant $\frac{1}{2^r}$, and therefore (3.6) is better than the inequality (1.5).*

Corollary 8. *Let u as in Theorem 10, and $f : [a,b] \to \mathbb{R}$ is of L–Lipschitzian type mapping on $[a,b]$. Then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{1}{12}KL(b-a)^2. \tag{3.7}$$

3.3. **The case of absolutely continuous integrands.**

Theorem 11. *Let $u : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$|\mathcal{GS}(f;u)| \le \frac{1}{12}K(b-a)^2\|f'\|_{\infty,[a,b]}. \tag{3.8}$$

Proof. As u is K-Lipschitzian on $[a,b]$ and f is continuous, by (3.2) we have

$$|\mathcal{GS}(f;u)| \le K\int_a^{\frac{a+b}{2}} \left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| dx$$

Using the companion of Ostrowski type inequality (1.3), we may state that

$$\int_a^{\frac{a+b}{2}} \left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| dx$$
$$\le (b-a)\|f'\|_{\infty,[a,b]}\cdot\int_a^{\frac{a+b}{2}}\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]dx$$
$$\le \frac{1}{12}(b-a)^2\|f'\|_{\infty,[a,b]}.$$

It follows that

$$|\mathcal{GS}(f;u)| \le K\int_a^{\frac{a+b}{2}} \left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| dx$$
$$\le \frac{1}{12}K(b-a)^2\|f'\|_{\infty,[a,b]},$$

and the theorem is proved. □

4. A Numerical quadrature formula for the Riemann–Stieltjes integral

In this section, we use the results in the previous sections to approximate the Riemann–Stieltjes integral $\int_a^{\frac{a+b}{2}} \left[\frac{f(x)+f(a+b-x)}{2}\right] du(x)$, in terms of the Riemann integral $\int_a^b f(t)\,dt$.

Theorem 12. *Let f, u be as in Theorem 6 and consider*

$$I_h := \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\},$$

be a partition of $[a, b]$. Denote $h_i = x_{i+1} - x_i$, $i = 1, 2, \cdots n-1$. Then we have

$$\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) = A_n(f,u,I_h) + R_n(f,u,I_h) \tag{4.1}$$

where,

$$A_n(f,u,I_h) = \sum_{i=0}^{n-1} \frac{u\left(\frac{x_{i+1}+x_i}{2}\right) - u(x_i)}{h_i} \times \int_{x_i}^{\frac{x_{i+1}+x_i}{2}} f(t)\,dt \tag{4.2}$$

and the Remainder $R_n(f,u,I_h)$ satisfies the estimation

$$|R_n(f,u,I_h)| \le \frac{1}{2} \cdot \max_{i=\overline{0,n-1}} \left\{ \bigvee_{x_i}^{x_{i+1}} (f) \right\} \cdot \bigvee_a^{\frac{a+b}{2}} (u). \tag{4.3}$$

Proof. Applying Theorem 6 on the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \cdots n-1$, we get

$$\left| \int_{x_i}^{\frac{x_{i+1}+x_i}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - \frac{u\left(\frac{x_{i+1}+x_i}{2}\right) - u(x_i)}{h_i} \int_{x_i}^{\frac{x_{i+1}+x_i}{2}} f(t)\,dt \right| \le \frac{1}{2} \bigvee_{x_i}^{x_{i+1}} (f) \cdot \bigvee_{x_i}^{\frac{x_{i+1}+x_i}{2}} (u).$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned} \left| \int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - A_n(f,u,I_h) \right| &\le \frac{1}{2} \cdot \sum_{i=0}^{n-1} \left[\bigvee_{x_i}^{x_{i+1}} (f) \cdot \bigvee_{x_i}^{\frac{x_{i+1}+x_i}{2}} (u) \right] \\ &= \frac{1}{2} \cdot \max_{i=\overline{0,n-1}} \left\{ \bigvee_{x_i}^{x_{i+1}} (f) \right\} \cdot \sum_{i=0}^{n-1} \bigvee_{x_i}^{\frac{x_{i+1}+x_i}{2}} (u) \\ &= \frac{1}{2} \cdot \max_{i=\overline{0,n-1}} \left\{ \bigvee_{x_i}^{x_{i+1}} (f) \right\} \cdot \bigvee_a^{\frac{a+b}{2}} (u), \end{aligned}$$

and the theorem is proved. □

Theorem 13. *Let f, u be as in Theorem 10. Let I_h as above. Then we have*

$$\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) = A_n(f,u,I_h) + R_n(f,u,I_h) \tag{4.4}$$

where, $A_n(f,u,I_h)$ is defined in (4.2) and the Remainder $R_n(f,u,I_h)$ satisfies the estimation

$$|R_n(f,u,I_h)| \le \frac{KH}{2^r(r+1)(r+2)} \cdot [\nu(h)]^r \cdot (b-a) \tag{4.5}$$

where, $\nu(h) = \max_{i=\overline{0,n-1}} \{h_i\}$.

Proof. Applying Theorem 10 on the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \cdots n-1$, we get

$$\left| \int_{x_i}^{\frac{x_{i+1}+x_i}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - \frac{u\left(\frac{x_{i+1}+x_i}{2}\right) - u(x_i)}{h_i} \int_{x_i}^{\frac{x_{i+1}+x_i}{2}} f(t)\,dt \right| \le \frac{KH}{2^r(r+1)(r+2)} \cdot h_i^{r+1}.$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned} \left| \int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - A_n(f,u,I_h) \right| &\le \frac{KH}{2^r(r+1)(r+2)} \cdot \sum_{i=0}^{n-1} h_i^{r+1} \\ &\le \frac{KH}{2^r(r+1)(r+2)} \cdot \left[\max_{i=\overline{0,n-1}} \{h_i\} \right]^r \cdot \sum_{i=0}^{n-1} h_i \\ &\le \frac{KH}{2^r(r+1)(r+2)} \cdot [\nu(h)]^r \cdot (b-a), \end{aligned}$$

and the theorem is proved. □

Remark 6. *In order to approximate the Riemann–Stieltjes integral (4.1), one may state several interesting error estimations for the remainder $R_n(f,u,I_h)$ under various assumptions using the inequalities in the sections 2 and 3. We shall omit the details.*

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