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SOME INEQUALITIES GENERALIZING KATO'S AND FURUTA'S RESULTS

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Abstract. In this paper we establish a four-operator inequality from which we recapture as particular cases the Furuta's inequality and Kato's inequality as well as we obtain other similar results of interest. Applications for numerical radius inequalities and for functions of operators given by power series are provided as well.

1. Introduction

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$.

If P is a positive selfadjoint operator on H, i.e. $\langle Px,x\rangle \geq 0$ for any $x\in H$, then the following inequality is a generalization of the *Schwarz inequality* in H

$$|\langle Px,y\rangle|^2 \leq \langle Px,x\rangle \langle Py,y\rangle, \tag{1.1}$$

for any $x,y\in H$.

The following inequality is of interest as well, see [14, p. 221].

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2 \leq \|P\| \langle Px,x\rangle \tag{1.2}$$

for any $x\in H$.

In 1952, Kato [15] proved the following celebrated generalization of Schwarz inequality for any operator $T\in\mathcal{B}(H)$:

$$|\langle Tx,y\rangle|^2 \leq \langle (T^*T)^{\alpha} x,x\rangle \left\langle (TT^*)^{1-\alpha} y,y\right\rangle, \tag{K}$$

for any $x,y\in H$, $\alpha\in[0,1]$. Utilizing the modulus notation, i.e. we recall that $|A| := \sqrt{A^*A}$, we can write (K) as follows

$$|\langle Tx,y\rangle|^2 \leq \left\langle |T|^{2\alpha} x,x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y,y\right\rangle \tag{1.3}$$

for any $x,y\in H$, $\alpha\in[0,1]$.

In order to generalize this result, in 1994 Furuta [13] obtained the following result:

$$\left|\left\langle T|T|^{\alpha+\beta-1} x,y\right\rangle\right|^2 \leq \left\langle |T|^{2\alpha} x,x\right\rangle \left\langle |T^*|^{2\beta} y,y\right\rangle \tag{F}$$

for any $x,y\in H$ and $\alpha,\beta\in[0,1]$ with $\alpha+\beta\geq 1$.

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If one analyses the prof from [13], that one realises that the condition $\alpha, \beta \in [0,1]$ is taken only to fit with the result from the *Heinz-Kato inequality*

$$|\langle Tx, y\rangle| \leq \|A^{\alpha}x\| \left\|B^{1-\alpha}y\right\| \tag{HK}$$

for any $x, y \in H$ and $\alpha \in [0,1]$ where A and B are prositive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^{*}y\| \leq \|By\|$ for all $x, y \in H$.

Therefore, we can state the more general result:

Theorem 1 (Furuta Inequality, 1994, [13])**.** *Let $T \in \mathcal{B}(H)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$. Then for any $x, y \in H$ we have the inequality (F).*

If we take $\beta = \alpha$ in Furuta's inequality, then we get

$$\left|\left\langle T\,|T|^{2\alpha-1}\,x, y\right\rangle\right|^{2} \leq \left\langle |T|^{2\alpha}\,x, x\right\rangle \left\langle |T^{*}|^{2\alpha}\,y, y\right\rangle \tag{1.4}$$

for any $x, y \in H$ and $\alpha \geq \frac{1}{2}$. In particular, for $\alpha = 1$ we have

$$|\langle T\,|T|\,x, y\rangle|^{2} \leq \left\langle |T|^{2}\,x, x\right\rangle \left\langle |T^{*}|^{2}\,y, y\right\rangle \tag{1.5}$$

for any $x, y \in H$.

If we take $T = N$ a normal operator, then we get from (F) the following inequality for normal operators

$$\left|\left\langle N\,|N|^{\alpha+\beta-1}\,x, y\right\rangle\right|^{2} \leq \left\langle |N|^{2\alpha}\,x, x\right\rangle \left\langle |N|^{2\beta}\,y, y\right\rangle \tag{1.6}$$

for any $x, y \in H$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$.

This implies the inequalities

$$\left|\left\langle N\,|N|^{2\alpha-1}\,x, y\right\rangle\right|^{2} \leq \left\langle |N|^{2\alpha}\,x, x\right\rangle \left\langle |N|^{2\alpha}\,y, y\right\rangle \tag{1.7}$$

for any $x, y \in H$ and $\alpha \geq \frac{1}{2}$ and

$$|\langle N\,|N|\,x, y\rangle|^{2} \leq \left\langle |N|^{2}\,x, x\right\rangle \left\langle |N|^{2}\,y, y\right\rangle \tag{1.8}$$

for any $x, y \in H$.

Making $y = x$ in (1.7) produces

$$\left|\left\langle N\,|N|^{2\alpha-1}\,x, x\right\rangle\right| \leq \left\langle |N|^{2\alpha}\,x, x\right\rangle$$

for any $x \in H$ and $\alpha \geq \frac{1}{2}$ and, in particular

$$|\langle N\,|N|\,x, x\rangle| \leq \left\langle |N|^{2}\,x, x\right\rangle$$

for any $x \in H$.

For various interesting generalizations, extension of Kato and Furuta inequalities, see the papers [4]-[13], [18]-[22] and [24].

Motivated by the above results, we establish in this paper a simple four-operator inequality from which we show that we can obtain as particular cases the Furuta's inequality (F) and Kato's inequality (K) as well as other similar results of interest. Applications for numerical radius inequalities and for functions of operators given by power series are provided as well.

2. Vector Inequalities

The following result provides a simple however useful extension for four operators of the Schwarz inequality :

Theorem 2. *Let $A,B,C,D\in\mathcal{B}(H)$. Then for $x,y\in H$ we have the inequality*

$$|\langle DCBAx,y\rangle|^2\leq\left\langle A^*|B|^2Ax,x\right\rangle\left\langle D|C^*|^2D^*y,y\right\rangle. \tag{2.1}$$

The equality case holds in (2.1) iff the vectors BAx and C^*D^*y are linearly dependent in H.

Proof. The Schwarz inequality in the Hilbert space H states that for any $u,v\in H$ we have the inequality

$$|\langle u,v\rangle|^2\leq\|u\|^2\|v\|^2 \tag{2.2}$$

with equality if and only if the vectors u and v are linearly dependent in H.

Now, if we take $u=BAx$ and $v=C^*D^*y$ then we have

$$\begin{aligned}\|u\|^2&=\langle BAx,BAx\rangle=\langle B^*BAx,Ax\rangle\\&=\langle A^*B^*BAx,x\rangle=\left\langle A^*|B|^2Ax,x\right\rangle,\end{aligned}$$

$$\begin{aligned}\|v\|^2&=\langle C^*D^*y,C^*D^*y\rangle=\langle CC^*D^*y,D^*y\rangle\\&=\langle DCC^*D^*y,y\rangle=\left\langle D|C^*|^2D^*y,y\right\rangle\end{aligned}$$

and

$$\langle u,v\rangle=\langle BAx,C^*D^*y\rangle=\langle CBAx,D^*y\rangle=\langle DCBAx,y\rangle.$$

Utilising (2.2) we deduce the desired result (2.1). □

Corollary 1. *The Furuta inequality (F) for $\alpha,\beta\geq0$ with $\alpha+\beta\geq1$ is a particular case of (2.1).*

Proof. Let $T=U|T|$ be the polar decomposition of the operator T, where U is partial isometry and the kernel $N(U)=N(|T|)$.

If we take $D=U,C=|T|^\beta,B=1_H$ and $A=|T|^\alpha$ then we have

$$DCBA=U|T|^\beta|T|^\alpha=U|T||T|^{\alpha+\beta-1}=T|T|^{\alpha+\beta-1},$$

$$A^*|B|^2A=|T|^\alpha|T|^\alpha=|T|^{2\alpha}$$

and

$$D|C^*|^2D^*=U|T|^{2\beta}U^*=|T^*|^{2\beta},$$

which by (2.1) imples the desired inequality (F). □

Remark 1. *It is obviuous that Kato's inequality (K) is also a particular case of (2.1).*

The following similar result also holds:

Corollary 2. *For any operator $T\in\mathcal{B}(H)$ and any $\alpha,\beta\geq1$ we have the inequality*

$$\left|\left\langle T|T|^{\beta-1}T|T|^{\alpha-1}x,y\right\rangle\right|^2\leq\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2\beta}y,y\right\rangle, \tag{2.3}$$

for any $x,y\in H$.

Proof. Let $T = U|T|$ be the polar decomposition of the operator T, where U is partial isometry and the kernel $N(U) = N(|T|)$.

If we take $D = U, C = |T|^{\beta}, B = U$ and $A = |T|^{\alpha}$ then we have

$$DCBA = U|T|^{\beta} U|T|^{\alpha} = U|T||T|^{\beta-1} U|T||T|^{\alpha-1} = T|T|^{\beta-1} T|T|^{\alpha-1},$$

$$\begin{aligned} A^* |B|^2 A &= |T|^{\alpha} U^*U|T|^{\alpha} = |T|^{\alpha-1}|T| U^*U|T||T|^{\alpha-1} \\ &= |T|^{\alpha-1} T^*T|T|^{\alpha-1} = |T|^{\alpha-1}|T|^2|T|^{\alpha-1} = |T|^{2\alpha} \end{aligned}$$

and

$$D|C^*|^2 D^* = U|T|^{2\beta} U^* = |T^*|^{2\beta},$$

which by (2.1) implies the desired inequality (2.3). □

Remark 2. *The above inequality (2.3) contains some nice particular inequalities as follows:*

$$\left|\left\langle \left(T|T|^{\alpha-1}\right)^2 x, y\right\rangle\right|^2 \le \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2\alpha} y, y\right\rangle, \tag{2.4}$$

for $\alpha \ge 1$ producing for $\alpha = 1$ the result

$$\left|\left\langle T^2 x, y\right\rangle\right|^2 \le \left\langle |T|^2 x, x\right\rangle \left\langle |T^*|^2 y, y\right\rangle, \tag{2.5}$$

and for $\alpha = 2$ the result

$$\left|\left\langle (T|T|)^2 x, y\right\rangle\right|^2 \le \left\langle |T|^4 x, x\right\rangle \left\langle |T^*|^4 y, y\right\rangle, \tag{2.6}$$

for any $x, y \in H$.

If we take $\alpha = 1$ in (2.3), then we get

$$\left|\left\langle T|T|^{\beta-1} Tx, y\right\rangle\right|^2 \le \left\langle |T|^2 x, x\right\rangle \left\langle |T^*|^{2\beta} y, y\right\rangle, \tag{2.7}$$

for any $\beta \ge 1$ and if we take $\beta = 1$ in (2.3) then we also get

$$\left|\left\langle T^2 |T|^{\alpha-1} x, y\right\rangle\right|^2 \le \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^2 y, y\right\rangle, \tag{2.8}$$

for any $x, y \in H$.

Corollary 3. *For any operator $T \in \mathcal{B}(H)$ and any $\alpha, \beta \ge 0$ with $\alpha + \beta \ge 2$ we have the inequality*

$$\left|\left\langle T^* |T^*|^{\alpha+\beta-2} Tx, y\right\rangle\right|^2 \le \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T|^{2\beta} y, y\right\rangle, \tag{2.9}$$

for any $x, y \in H$.

Proof. Let $T = U|T|$ be the polar decomposition of the operator T, where U is partial isometry and the kernel $N(U) = N(|T|)$. Then $T^* = |T|U^*$.

If we take $D = U$, $C = |T|^{\beta}$, $B = |T|^{\alpha}$and $A = U^*$, then we have

$$DCBA = U|T|^{\beta}|T|^{\alpha} U^* = U|T||T|^{\beta+\alpha-2}|T| U^* = T|T|^{\beta+\alpha-2} T^*,$$

$$A^* |B|^2 A = U|T|^{2\alpha} U^* = |T^*|^{2\alpha}$$

and

$$D|C^*|^2 D^* = U|T|^{2\beta} U^* = |T^*|^{2\beta},$$

which by (2.1) implies the inequality

$$\left|\left\langle T\left|T\right|^{\beta+\alpha-2}T^{*}x,y\right\rangle\right|^{2}\leq\left\langle\left|T^{*}\right|^{2\alpha}x,x\right\rangle\left\langle\left|T^{*}\right|^{2\beta}y,y\right\rangle \tag{2.10}$$

for any $x,y\in H$.

Now, if replace in (2.10) the operator T with T^{*}, then we get the desired result (2.9). □

Remark 3. *We can get from (2.9) the following inequality*

$$\left|\left\langle\left|T\right|^{2}x,y\right\rangle\right|^{2}\leq\left\langle\left|T\right|^{2\alpha}x,x\right\rangle\left\langle\left|T\right|^{2(2-\alpha)}y,y\right\rangle, \tag{2.11}$$

for any $x,y\in H$ and $\alpha\in[0,2]$.

If we make $\beta=\alpha\geq 1$ in (2.9), then we get

$$\left|\left\langle T^{*}\left|T^{*}\right|^{2(\alpha-1)}Tx,y\right\rangle\right|^{2}\leq\left\langle\left|T\right|^{2\alpha}x,x\right\rangle\left\langle\left|T\right|^{2\alpha}y,y\right\rangle, \tag{2.12}$$

for any $x,y\in H$.

Corollary 4. *For any operator $T\in\mathcal{B}(H)$ and any $\gamma,\delta\geq 0$ we have the inequality*

$$\left|\left\langle\left|T\right|^{\gamma}T^{2}\left|T\right|^{\delta}x,y\right\rangle\right|^{2}\leq\left\langle\left|T\right|^{2\delta+2}x,x\right\rangle\left\langle\left|T^{*}\left|T\right|^{\gamma}\right|^{2}y,y\right\rangle, \tag{2.13}$$

for any $x,y\in H$.

Proof. If we take $D=\left|T\right|^{\gamma},C=T,B=T$ and $A=\left|T\right|^{\delta}$ then we have

$$DCBA=\left|T\right|^{\gamma}T^{2}\left|T\right|^{\delta},$$

$$A^{*}\left|B\right|^{2}A=\left|T\right|^{\delta}\left|T\right|^{2}\left|T\right|^{\delta}=\left|T\right|^{2\delta+2}$$

and

$$\begin{aligned}D\left|C^{*}\right|^{2}D^{*}&=\left|T\right|^{\gamma}\left|T^{*}\right|^{2}\left|T\right|^{\gamma}=\left|T\right|^{\gamma}TT^{*}\left|T\right|^{\gamma}\\&=\left|T\right|^{\gamma}T\left(\left|T\right|^{\gamma}T\right)^{*}=\left|\left(\left|T\right|^{\gamma}T\right)^{*}\right|^{2}=\left|T^{*}\left|T\right|^{\gamma}\right|^{2},\end{aligned}$$

which by (2.1) implies the desired inequality (2.13). □

Remark 4. *The particular case $\gamma=\delta=1$ provides the inequality*

$$\left|\left\langle\left|T\right|T^{2}\left|T\right|x,y\right\rangle\right|^{2}\leq\left\langle\left|T\right|^{4}x,x\right\rangle\left\langle\left|T^{*}\left|T\right|\right|^{2}y,y\right\rangle, \tag{2.14}$$

for any $x,y\in H$.

We also have

Corollary 5. *For any operator $T\in\mathcal{B}(H)$ and any $\gamma,\delta\geq 0$ we have the inequalities*

$$\left|\left\langle\left|T\right|^{\gamma+\delta+2}x,y\right\rangle\right|^{2}\leq\left\langle\left|T\right|^{2\delta+2}x,x\right\rangle\left\langle\left|T\right|^{2\gamma+2}y,y\right\rangle \tag{2.15}$$

and

$$\left|\left\langle\left|T\right|^{\gamma}\left|T^{*}\right|^{2}\left|T\right|^{\delta}x,y\right\rangle\right|^{2}\leq\left\langle\left|T^{*}\left|T\right|^{\delta}\right|^{2}x,x\right\rangle\left\langle\left|T^{*}\left|T\right|^{\gamma}\right|^{2}y,y\right\rangle \tag{2.16}$$

for any $x,y\in H$.

Proof. If we take $D=|T|^{\gamma}, C=T^{*}, B=T$ and $A=|T|^{\delta}$ then we have

$$DCBA=|T|^{\gamma}T^{*}T|T|^{\delta}=|T|^{\gamma}|T|^{2}|T|^{\delta}=|T|^{\gamma+\delta+2},$$
$$A^{*}|B|^{2}A=|T|^{\delta}|T|^{2}|T|^{\delta}=|T|^{2\delta+2}$$

and

$$D|C^{*}|^{2}D^{*}=|T|^{\gamma}|T|^{2}|T|^{\gamma}=|T|^{2\gamma+2}$$

which by (2.1) implies the desired inequality (2.15).

The dual choice $D=|T|^{\gamma}, C=T, B=T^{*}$ and $A=|T|^{\delta}$ gives

$$DCBA=|T|^{\gamma}|T^{*}|^{2}|T|^{\delta},$$
$$A^{*}|B|^{2}A=|T|^{\delta}|T^{*}|^{2}|T|^{\delta}=\left|T^{*}|T|^{\delta}\right|^{2}$$

and

$$D|C^{*}|^{2}D^{*}=|T|^{\gamma}|T^{*}|^{2}|T|^{\gamma}=|T^{*}|T|^{\gamma}|^{2},$$

which by (2.1) produces (2.16). □

Remark 5. *If we take $\delta=\gamma$ in (2.16), then we get*

$$\left|\left\langle |T|^{\gamma}|T^{*}|^{2}|T|^{\gamma}x,y\right\rangle\right|^{2}\leq\left\langle |T^{*}|T|^{\gamma}|^{2}x,x\right\rangle\left\langle |T^{*}|T|^{\gamma}|^{2}y,y\right\rangle \tag{2.17}$$

for any $x,y\in H$.

The following corollary also holds

Corollary 6. *For any operator $T\in\mathcal{B}(H)$ and any $\beta\geq 0$ we have the inequalities*

$$\left|\left\langle T|T^{*}|^{\beta}Tx,y\right\rangle\right|^{2}\leq\left\langle |T|^{2}x,x\right\rangle\left\langle T|T^{*}|^{2\beta}T^{*}y,y\right\rangle \tag{2.18}$$

and

$$\left|\left\langle T|T|^{\beta}Tx,y\right\rangle\right|^{2}\leq\left\langle |T|^{2}x,x\right\rangle\left\langle T|T|^{2\beta}T^{*}y,y\right\rangle \tag{2.19}$$

for any $x,y\in H$.

Proof. Let $T=U|T|$ be the polar decomposition of the operator T, where U is partial isometry and the kernel $N(U)=N(|T|)$.

If we take $D=U, C=|T||T^{*}|^{\beta}, B=U$ and $A=|T|$ then we have

$$DCBA=U|T||T^{*}|^{\beta}U|T|=T|T^{*}|^{\beta}T,$$
$$A^{*}|B|^{2}A=|T|U^{*}U|T|=T^{*}T=|T|^{2}$$

and

$$\begin{aligned}D|C^{*}|^{2}D^{*}&=UCC^{*}U^{*}=U|T||T^{*}|^{\beta}|T^{*}|^{\beta}|T|U^{*}\\&=T|T^{*}|^{2\beta}T^{*},\end{aligned}$$

which by (2.1) produces (2.18).

Now, if we take $D=U, C=|T|^{\beta+1}, B=U$ and $A=|T|$ then we have

$$DCBA=U|T|^{\beta+1}U|T|=T|T|^{\beta}T,$$
$$A^{*}|B|^{2}A=|T|^{2}$$

and

$$D|C^{*}|^{2}D^{*}=UCC^{*}U^{*}=U|T|^{\beta+1}|T|^{\beta+1}U^{*}=T|T|^{2\beta}T^{*}.$$

□

Remark 6. *The case $\beta = 1$ produces from the inequalities (2.18) and (2.19) the simple results*

$$|\langle T |T^*| Tx, y\rangle|^2 \le \left\langle |T|^2 x, x \right\rangle \left\langle T^2 (T^*)^2 y, y \right\rangle \tag{2.20}$$

and

$$|\langle T |T| Tx, y\rangle|^2 \le \left\langle |T|^2 x, x \right\rangle \left\langle |T^*|^4 y, y \right\rangle \tag{2.21}$$

for any $x, y \in H$.

3. Norm and Numerical Radius Inequalities

We can state the following corollary of Furuta's inequality for the numerical radius w of an operator $V \in \mathcal{B}(H)$, namely $w(V) = \sup_{\|x\|=1} |\langle Vx, x\rangle|$, which satisfies the following basic inequalities

$$\frac{1}{2} \|V\| \le w(V) \le \|V\|. \tag{3.1}$$

Theorem 3. *Let $A, B, C, D \in \mathcal{B}(H)$. Then we have*

$$\|DCBA\|^2 \le \left\| A^* |B|^2 A \right\| \left\| D |C^*|^2 D^* \right\| \tag{3.2}$$

and for any $r \ge 1$

$$w^r (DCBA) \le \frac{1}{2} \left\| \left(A^* |B|^2 A \right)^r + \left(D |C^*|^2 D^* \right)^r \right\|. \tag{3.3}$$

Proof. Taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ in (2.1) we have

$$\begin{aligned}
\|DCBA\|^2 &= \sup_{\|x\|=\|y\|=1} |\langle DCBAx, y\rangle|^2 \\
&\le \sup_{\|x\|=\|y\|=1} \left[\left\langle A^* |B|^2 Ax, x \right\rangle \left\langle D |C^*|^2 D^* y, y \right\rangle \right] \\
&= \sup_{\|x\|=1} \left\langle A^* |B|^2 Ax, x \right\rangle \sup_{\|y\|=1} \left\langle D |C^*|^2 D^* y, y \right\rangle \\
&= \left\| A^* |B|^2 A \right\| \left\| D |C^*|^2 D^* \right\|
\end{aligned}$$

and the inequality (3.2) is proved.

By taking $x = y$ in (2.1) and utilising the increasing monotonicity of the power means for two positive numbers, we have for any $r \ge 1$ that

$$\begin{aligned}
|\langle DCBAx, x\rangle| &\le \left[\left\langle A^* |B|^2 Ax, x \right\rangle \left\langle D |C^*|^2 D^* x, x \right\rangle \right]^{1/2} \\
&\le \frac{\left\langle A^* |B|^2 Ax, x \right\rangle + \left\langle D |C^*|^2 D^* x, x \right\rangle}{2} \\
&\le \left[\frac{\left\langle A^* |B|^2 Ax, x \right\rangle^r + \left\langle D |C^*|^2 D^* x, x \right\rangle^r}{2} \right]^{1/r}
\end{aligned} \tag{3.4}$$

for any $x \in H$.

Now, utilising Hölder-McCarthy inequality $\langle Px,x\rangle^r \le \langle P^r x,x\rangle, x\in H, \|x\|=1$ that holds for any positive operator P and any power $r\ge 1$ we have

$$\begin{aligned}\frac{\left\langle A^*|B|^2Ax,x\right\rangle^r+\left\langle D|C^*|^2D^*x,x\right\rangle^r}{2} \\ \le \frac{\left\langle \left(A^*|B|^2A\right)^r x,x\right\rangle+\left\langle\left(D|C^*|^2D^*\right)^r x,x\right\rangle}{2} \\ = \left\langle \frac{\left(A^*|B|^2A\right)^r+\left(D|C^*|^2D^*\right)^r}{2}x,x\right\rangle\end{aligned} \tag{3.5}$$

for any $x\in H$, $\|x\|=1$.

By making use of (3.4) and (3.5) we get the inequality of interest

$$|\langle DCBAx,x\rangle|^r \le \left\langle \frac{\left(A^*|B|^2A\right)^r+\left(D|C^*|^2D^*\right)^r}{2}x,x\right\rangle \tag{3.6}$$

for any $x\in H$, $\|x\|=1$.

Finally, by taking the supremum over $x\in H$, $\|x\|=1$ in (3.6) we deduce the desired result (3.3). □

The above theorem has a number of particular cases for one operator that are of interest:

Corollary 7. *1. Let $T\in\mathcal{B}(H), r\ge 1$ and $\alpha,\beta\ge 0$ with $\alpha+\beta\ge 1$. Then we have*

$$w^r\left(T|T|^{\alpha+\beta-1}\right)\le \frac{1}{2}\left\||T|^{2\alpha r}+|T^*|^{2\beta r}\right\|. \tag{3.7}$$

In particular, we also have

$$w^r\left(T|T|^{2\alpha-1}\right)\le \frac{1}{2}\left\||T|^{2\alpha r}+|T^*|^{2\alpha r}\right\|, \tag{3.8}$$

for any $\alpha\ge\frac{1}{2}$ and

$$w^r\left(T|T|\right)\le \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\|. \tag{3.9}$$

2. For any operator $T\in\mathcal{B}(H), r\ge 1$ and any $\alpha,\beta\ge 1$ we have the inequality

$$w^r\left(T|T|^{\beta-1}T|T|^{\alpha-1}\right)\le \frac{1}{2}\left\||T|^{2\alpha r}+|T^*|^{2\beta r}\right\|. \tag{3.10}$$

In particular, we also have

$$w^r\left(\left(T|T|^{\alpha-1}\right)^2\right)\le \frac{1}{2}\left\||T|^{2\alpha r}+|T^*|^{2\alpha r}\right\|, \tag{3.11}$$

for any $\alpha\ge 1$ which provides the result

$$w^r\left(T^2\right)\le \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\|. \tag{3.12}$$

3.For any operator $T\in\mathcal{B}(H), r\ge 1$ and any $\beta\ge 0$ we have the inequalities

$$w^r\left(T|T^*|^{\beta}T\right)\le \frac{1}{2}\left\||T|^{2r}+\left[T|T^*|^{2\beta}T^*\right]^r\right\| \tag{3.13}$$

and

$$w^r\left(T|T|^{\beta}T\right)\leq\frac{1}{2}\left\||T|^{2r}+\left[T|T^*|^{2\beta}T^*\right]^r\right\|. \tag{3.14}$$

In particular, we have

$$w^r\left(T|T^*|T\right)\leq\frac{1}{2}\left\||T|^{2r}+\left[T^2\left(T^*\right)^2\right]^r\right\| \tag{3.15}$$

and

$$w^r\left(T|T|T\right)\leq\frac{1}{2}\left\||T|^{2r}+|T^*|^{4r}\right\|. \tag{3.16}$$

4. Inequalities for Power Series

Now, by the help of power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n\geq 0$, then $f_A=f$.

In the recent paper [3], by utilising the Furuta's inequality (F) for $\alpha,\beta\geq 0$ with $\alpha+\beta\geq 1$ we established the following result for power series:

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and be $g(z)=\sum_{n=0}^{\infty}b_nz^n$ be two functions defined by power series with real coefficients and both of them convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If T is a bounded linear operator on the Hilbert space H and $z,u\in\mathbb{C}$ with the property that*

$$|z|^2,|u|^2,\|T\|^2<R, \tag{4.1}$$

then we have the inequality

$$\begin{aligned}&\left|\left\langle Tf\left(z|T|\right)g\left(u|T|\right)x,y\right\rangle\right|^2\\&\leq f_A\left(|z|^2\right)g_A\left(|u|^2\right)\left\langle f_A\left(|T|^2\right)x,x\right\rangle\left\langle|T^*|^2g_A\left(|T^*|^2\right)y,y\right\rangle\end{aligned} \tag{4.2}$$

for any $x,y\in H$.

By utilising Corollary 2 we have the following similar result:

Theorem 5. *With the assumptions in Theorem 4, we have the inequality*

$$\begin{aligned}&\left|\left\langle Tf\left(z|T|\right)Tg\left(u|T|\right)x,y\right\rangle\right|^2\\&\leq f_A\left(|z|^2\right)g_A\left(|u|^2\right)\left\langle|T|^2f_A\left(|T|^2\right)x,x\right\rangle\left\langle|T^*|^2g_A\left(|T^*|^2\right)y,y\right\rangle\end{aligned} \tag{4.3}$$

for any $x,y\in H$.

Proof. From (2.3) we have for any natural numbers $n\geq 1$ and $m\geq 1$ the following power inequality

$$\left|\left\langle T|T|^{n-1}T|T|^{m-1}x,y\right\rangle\right|^2\leq\left\langle|T|^{2n}x,x\right\rangle\left\langle|T^*|^{2m}y,y\right\rangle, \tag{4.4}$$

for any $x,y\in H$.

If we multiply this inequality with the positive quantities $|a_{n-1}|\,|z|^{n-1}$ and $|b_{m-1}|\,|u|^{m-1}$, use the triangle inequality and the Cauchy-Bunyakowsky-Schwarz discrete inequality, we have successively:

$$\begin{aligned}(4.5)\qquad &\left|\sum_{n=1}^{k}\sum_{m=1}^{l} a_{n-1}z^{n-1}b_{m-1}u^{m-1}\left\langle T\,|T|^{n-1}\,T\,|T|^{m-1}\,x,y\right\rangle\right| \\ &\leq \sum_{n=1}^{k}\sum_{m=1}^{l} |a_{n-1}|\,|z|^{n-1}\,|b_{m-1}|\,|u|^{m-1}\left|\left\langle T\,|T|^{n-1}\,T\,|T|^{m-1}\,x,y\right\rangle\right| \\ &\leq \sum_{n=1}^{k} |a_{n-1}|\,|z|^{n-1}\left\langle |T|^{2n}\,x,x\right\rangle^{1/2}\sum_{m=1}^{l} |b_{m-1}|\,|u|^{m-1}\left\langle |T^{*}|^{2m}\,y,y\right\rangle^{1/2} \\ &\leq \left(\sum_{n=1}^{k} |a_{n-1}|\,|z|^{2(n-1)}\right)^{1/2}\left\langle \sum_{n=1}^{k} |a_{n-1}|\,|T|^{2n}\,x,x\right\rangle^{1/2} \\ &\times \left(\sum_{m=1}^{l} |b_{m-1}|\,|u|^{2(m-1)}\right)^{1/2}\left\langle \sum_{m=1}^{l} |b_{m-1}|\,|T^{*}|^{2m}\,y,y\right\rangle^{1/2}\end{aligned}$$

for any $x,y\in H$ and $k\geq 0, l\geq 1$.

Observe also that

$$\begin{aligned}(4.6)\qquad &\sum_{n=1}^{k}\sum_{m=1}^{l} a_{n-1}z^{n-1}b_{m-1}u^{m-1}\left\langle T\,|T|^{n-1}\,T\,|T|^{m-1}\,x,y\right\rangle \\ &= \left\langle T\left(\sum_{n=1}^{k} a_{n-1}z^{n-1}\,|T|^{n-1}\right)T\left(\sum_{m=1}^{l} b_{m-1}u^{m-1}\,|T|^{m-1}\right)x,y\right\rangle\end{aligned}$$

for any $x,y\in H$ and $k\geq 0, l\geq 1$.

Making use of (4.5) and (4.6) we get

$$\begin{aligned}(4.7)\qquad &\left|\left\langle T\left(\sum_{n=1}^{k} a_{n-1}z^{n-1}\,|T|^{n-1}\right)T\left(\sum_{m=1}^{l} b_{m-1}u^{m-1}\,|T|^{m-1}\right)x,y\right\rangle\right| \\ &\leq \left(\sum_{n=1}^{k} |a_{n-1}|\,|z|^{2(n-1)}\right)^{1/2}\left\langle |T|^{2}\sum_{n=1}^{k} |a_{n-1}|\,|T|^{2(n-1)}\,x,x\right\rangle^{1/2} \\ &\times \left(\sum_{m=1}^{l} |b_{m-1}|\,|u|^{2(m-1)}\right)^{1/2}\left\langle |T^{*}|^{2}\sum_{m=1}^{l} |b_{m-1}|\,|T^{*}|^{2(m-1)}\,y,y\right\rangle^{1/2}\end{aligned}$$

for any $x,y\in H$ and $k\geq 0, l\geq 1$.

Due to the assumption (4.1) in theorem, we have that the series whose soms are involved in the inequality (4.7) are convergent and then, by taking the limit over $k\to\infty$ and $l\to\infty$ in (4.7), we deduce the desired result (4.3). □

Corollary 8. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If T is a bounded linear operator on the Hilbert space H and $z\in\mathbb{C}$ with the property that*

$$(4.8)\qquad |z|^{2}, \|T\|^{2} < R,$$

then we have the inequality

$$\left|\left\langle [Tf(z|T|)]^2 x, y\right\rangle\right| \le f_A\left(|z|^2\right)\left\langle |T|^2 f_A\left(|T|^2\right)x,x\right\rangle^{1/2}\left\langle |T^*|^2 f_A\left(|T^*|^2\right)y,y\right\rangle^{1/2} \tag{4.9}$$

for any $x, y \in H$.

On making use of a similar argument and the inequality from Corollary 3, we can state the following result as well:

Theorem 6. *With the assumptions in Theorem 4 we have the inequality*

$$\left|\langle T^* f(z|T^*|)\, g(u|T^*|)\, Tx, y\rangle\right|^2 \le f_A\left(|z|^2\right) g_A\left(|u|^2\right)\left\langle |T|^2 f_A\left(|T|^2\right)x,x\right\rangle\left\langle |T^*|^2 g_A\left(|T^*|^2\right)y,y\right\rangle \tag{4.10}$$

for any $x, y \in H$.

In particular we have:

Corollary 9. *With the assumptions of Corollary 8 we have*

$$\left|\left\langle T^* f^2(z|T^*|)\, Tx, y\right\rangle\right| \le f_A\left(|z|^2\right)\left\langle |T|^2 f_A\left(|T|^2\right)x,x\right\rangle^{1/2}\left\langle |T^*|^2 f_A\left(|T^*|^2\right)y,y\right\rangle^{1/2} \tag{4.11}$$

for any $x, y \in H$.

Another result for one power series that was obtained in [3] is incorporated in:

Theorem 7. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If T is a bounded linear operator on the Hilbert space H with the property that $\|T\|^2 < R$, then we have the inequality*

$$\left|\left\langle T|T|\, f\left(|T|^2\right)x,y\right\rangle\right|^2 \le \left\langle |T|^2 f_A\left(|T|^2\right)x,x\right\rangle\left\langle |T^*|^2 f_A\left(|T^*|^2\right)y,y\right\rangle \tag{4.12}$$

for any $x, y \in H$.

We are able now, by utilising the inequality (2.12) to complement this result as follows:

Theorem 8. *With the assumptions in Theorem 7, we have the inequality*

$$\left|\left\langle T^* f\left(|T^*|^2\right)Tx,y\right\rangle\right|^2 \le \left\langle |T|^2 f_A\left(|T|^2\right)x,x\right\rangle\left\langle |T|^2 f_A\left(|T|^2\right)y,y\right\rangle, \tag{4.13}$$

for any $x, y \in H$.

Proof. If we write the inequality (2.12) for a natural number $n \ge 1$ we have

$$\left|\left\langle T^*|T^*|^{2(n-1)}Tx,y\right\rangle\right| \le \left\langle |T|^{2n}x,x\right\rangle^{1/2}\left\langle |T|^{2n}y,y\right\rangle^{1/2}, \tag{4.14}$$

for any $x, y \in H$.

Now, if we multiply the inequality (4.15) by $|a_{n-1}| \geq 0$, sum over n from 1 to k, utilize the generalized triangle inequality and the Cauchy-Bunyakowsky-Schwarz discrete inequality, we have successively:

$$\begin{aligned}
&\left|\left\langle T^{*}\sum_{n=1}^{k} a_{n-1}\left|T^{*}\right|^{2(n-1)} Tx, y\right\rangle\right| \\
&\leq \sum_{n=1}^{k}\left|a_{n-1}\right|\left|\left\langle T^{*}\left|T^{*}\right|^{2(n-1)} Tx, y\right\rangle\right| \\
&\leq \sum_{n=1}^{k}\left|a_{n-1}\right|\left\langle |T|^{2n} x, x\right\rangle^{1/2}\left\langle |T|^{2n} y, y\right\rangle^{1/2} \\
&\leq \left\langle \sum_{n=1}^{k}\left|a_{n-1}\right| |T|^{2n} x, x\right\rangle^{1/2}\left\langle \sum_{n=1}^{k}\left|a_{n-1}\right| |T|^{2n} y, y\right\rangle^{1/2} \\
&= \left\langle |T|^{2}\sum_{n=1}^{k}\left|a_{n-1}\right| |T|^{2(n-1)} x, x\right\rangle^{1/2}\left\langle |T|^{2}\sum_{n=1}^{k}\left|a_{n-1}\right| |T|^{2(n-1)} y, y\right\rangle^{1/2}
\end{aligned} \tag{4.15}$$

for any $x, y \in H$.

Since all the series whose partial sums are involved in the inequality (4.15) are convergent, then by letting $k \to \infty$ in (4.15) we deduce the desired result (4.13). □

We give here some examples of operator inequalities for some fundamental functions expressed by power series.

Example 1. *1) For any operator $T \in \mathcal{B}(H)$ with $\|T\| < 1$ and any $z \in \mathbb{C}$ with $|z| < 1$ we have the inequalities*

$$\begin{aligned}
&\left|\left\langle\left[T\left(1_H \pm z|T|\right)^{-1}\right]^{2} x, y\right\rangle\right| \\
&\leq \frac{\left\langle |T|^{2}\left(1_H-|T|^{2}\right)^{-1} x, x\right\rangle^{1/2}\left\langle \left|T^{*}\right|^{2}\left(1_H-\left|T^{*}\right|^{2}\right)^{-1} y, y\right\rangle^{1/2}}{1-|z|^{2}}
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&\left|\left\langle\left[T \ln\left(1_H \pm z|T|\right)^{-1}\right]^{2} x, y\right\rangle\right| \\
&\leq \frac{\left\langle |T|^{2}\ln\left(1_H-|T|^{2}\right)^{-1} x, x\right\rangle^{1/2}\left\langle \left|T^{*}\right|^{2}\ln\left(1_H-\left|T^{*}\right|^{2}\right)^{-1} y, y\right\rangle^{1/2}}{1-|z|^{2}}
\end{aligned} \tag{4.17}$$

for any $x, y \in H$.

2) For any operator $T \in \mathcal{B}(H)$ and any $z \in \mathbb{C}$ we have the inequalities

$$\begin{aligned}
&\left|\left\langle\left[T \sin\left(z|T|\right)\right]^{2} x, y\right\rangle\right|, \left|\left\langle\left[T \sinh\left(z|T|\right)\right]^{2} x, y\right\rangle\right| \\
&\leq \sinh\left(|z|^{2}\right)\left\langle |T|^{2}\sinh\left(|T|^{2}\right) x, x\right\rangle^{1/2}\left\langle \left|T^{*}\right|^{2}\sinh\left(\left|T^{*}\right|^{2}\right) y, y\right\rangle^{1/2},
\end{aligned} \tag{4.18}$$

$$\left|\left\langle\left[T\cos\left(z\left|T\right|\right)\right]^{2}x,y\right\rangle\right|,\left|\left\langle\left[T\cosh\left(z\left|T\right|\right)\right]^{2}x,y\right\rangle\right| \leq \cosh\left(\left|z\right|^{2}\right)\left\langle\left|T\right|^{2}\sinh\left(\left|T\right|^{2}\right)x,x\right\rangle^{1/2}\left\langle\left|T^{*}\right|^{2}\cosh\left(\left|T^{*}\right|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.19}$$

and

$$\left|\left\langle\left[T\exp\left(z\left|T\right|\right)\right]^{2}x,y\right\rangle\right| \leq \exp\left(\left|z\right|^{2}\right)\left\langle\left|T\right|^{2}\exp\left(\left|T\right|^{2}\right)x,x\right\rangle^{1/2}\left\langle\left|T^{*}\right|^{2}\exp\left(\left|T^{*}\right|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.20}$$

for any $x,y\in H$.

The proof follows from the inequality (4.9).

Example 2. *For any operator* $T\in\mathcal{B}(H)$ *with* $\|T\|<1$ *and any* $z\in\mathbb{C}$ *with* $|z|<1$ *we have the inequalities*

$$\left|\left\langle T^{*}\left(1_{H}\pm z\left|T^{*}\right|\right)^{-2}Tx,y\right\rangle\right| \leq \frac{\left\langle\left|T\right|^{2}\left(1_{H}-\left|T\right|^{2}\right)^{-1}x,x\right\rangle^{1/2}\left\langle\left|T^{*}\right|^{2}\left(1_{H}-\left|T^{*}\right|^{2}\right)^{-1}y,y\right\rangle^{1/2}}{1-\left|z\right|^{2}} \tag{4.21}$$

and

$$\left|\left\langle T^{*}\left[\ln\left(1_{H}\pm z\left|T^{*}\right|\right)^{-1}\right]^{2}Tx,y\right\rangle\right| \leq \ln\left(1-\left|z\right|^{2}\right)^{-1} \times\left\langle\left|T\right|^{2}\ln\left(1_{H}-\left|T\right|^{2}\right)^{-1}x,x\right\rangle^{1/2}\left\langle\left|T^{*}\right|^{2}\ln\left(1_{H}-\left|T^{*}\right|^{2}\right)^{-1}y,y\right\rangle^{1/2} \tag{4.22}$$

for any $x,y\in H$.

2) For any operator $T\in\mathcal{B}(H)$ *and any* $z\in\mathbb{C}$ *we have the inequalities*

$$\left|\left\langle T^{*}\sin^{2}\left(z\left|T^{*}\right|\right)Tx,y\right\rangle\right|,\left|\left\langle T^{*}\sinh^{2}\left(z\left|T^{*}\right|\right)Tx,y\right\rangle\right| \leq \sinh\left(\left|z\right|^{2}\right)\left\langle\left|T\right|^{2}\sinh\left(\left|T\right|^{2}\right)x,x\right\rangle^{1/2}\left\langle\left|T^{*}\right|^{2}\sinh\left(\left|T^{*}\right|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.23}$$

$$\left|\left\langle T^{*}\cos^{2}\left(z\left|T^{*}\right|\right)Tx,y\right\rangle\right|,\left|\left\langle T^{*}\cosh^{2}\left(z\left|T^{*}\right|\right)Tx,y\right\rangle\right| \leq \cosh\left(\left|z\right|^{2}\right)\left\langle\left|T\right|^{2}\sinh\left(\left|T\right|^{2}\right)x,x\right\rangle^{1/2}\left\langle\left|T^{*}\right|^{2}\cosh\left(\left|T^{*}\right|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.24}$$

and

$$\left|\left\langle T^{*}\exp\left(2z\left|T^{*}\right|\right)Tx,y\right\rangle\right| \leq \exp\left(\left|z\right|^{2}\right)\left\langle\left|T\right|^{2}\exp\left(\left|T\right|^{2}\right)x,x\right\rangle^{1/2}\left\langle\left|T^{*}\right|^{2}\exp\left(\left|T^{*}\right|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.25}$$

for any $x,y\in H$.

The proof follows by the inequality (4.11).

Example 3. *For any operator $T\in\mathcal{B}(H)$ with $\|T\|<1$ and any $z\in\mathbb{C}$ with $|z|<1$ we have the inequalities*

$$\left|\left\langle T^{*}\left(1_{H}\pm|T^{*}|^{2}\right)^{-1}Tx,y\right\rangle\right| \leq\left\langle |T|^{2}\left(1_{H}-|T|^{2}\right)^{-1}x,x\right\rangle^{1/2}\left\langle |T|^{2}\left(1_{H}-|T|^{2}\right)^{-1}y,y\right\rangle^{1/2} \tag{4.26}$$

and

$$\left|\left\langle T^{*}\ln\left(1_{H}\pm|T^{*}|^{2}\right)^{-1}Tx,y\right\rangle\right| \leq\left\langle |T|^{2}\ln\left(1_{H}-|T|^{2}\right)^{-1}x,x\right\rangle^{1/2}\left\langle |T|^{2}\ln\left(1_{H}-|T|^{2}\right)^{-1}y,y\right\rangle^{1/2} \tag{4.27}$$

for any $x,y\in H$.

2) For any operator $T\in\mathcal{B}(H)$ and any $z\in\mathbb{C}$ we have the inequalities

$$\left|\left\langle T^{*}\sin\left(|T^{*}|^{2}\right)Tx,y\right\rangle\right|,\left|\left\langle T^{*}\sinh\left(|T^{*}|^{2}\right)Tx,y\right\rangle\right| \leq\left\langle |T|^{2}\sinh\left(|T|^{2}\right)x,x\right\rangle^{1/2}\left\langle |T|^{2}\sinh\left(|T|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.28}$$

$$\left|\left\langle T^{*}\cos\left(|T^{*}|^{2}\right)Tx,y\right\rangle\right|,\left|\left\langle T^{*}\cosh\left(|T^{*}|^{2}\right)Tx,y\right\rangle\right| \leq\left\langle |T|^{2}\sinh\left(|T|^{2}\right)x,x\right\rangle^{1/2}\left\langle |T|^{2}\cosh\left(|T|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.29}$$

and

$$\left|\left\langle T^{*}\exp\left(|T|^{2}\right)Tx,y\right\rangle\right| \leq\left\langle |T|^{2}\exp\left(|T|^{2}\right)x,x\right\rangle^{1/2}\left\langle |T|^{2}\exp\left(|T|^{2}\right)y,y\right\rangle^{1/2}, \tag{4.30}$$

for any $x,y\in H$.

The proof follows by the inequality (4.13).

5. More Results for Normal Operators

In the recent paper [3], by utlising Furuta's inequality (F) for $\alpha,\beta\geq 0$ with $\alpha+\beta\geq 1$ we established the following result for normal operators:

Theorem 9. *Let $f(z)=\sum_{n=0}^{\infty}a_{n}z^{n}$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If N is a normal operator on the Hilbert space H and $\alpha,\beta\geq 0$ with $\alpha+\beta\geq 1$ with the property that $\|N\|^{2\alpha},\|N\|^{2\beta}<R$, then we have the inequality*

$$\left|\left\langle f\left(N|N|^{\alpha+\beta-1}\right)x,y\right\rangle\right|^{2}\leq\left\langle f_{A}\left(|N|^{2\alpha}\right)x,x\right\rangle\left\langle f_{A}\left(|N|^{2\beta}\right)y,y\right\rangle \tag{5.1}$$

for any $x,y\in H$.

We can provide here some companion results as follows:

Theorem 10. *Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If N is a normal operator on the Hilbert space H and $\alpha,\beta\geq 1$, with the property that $\|N\|^{2\alpha},\|N\|^{2\beta}<R$, then we have the inequality*

$$\left|\left\langle f\left(N^2|N|^{\alpha+\beta-2}\right)x,y\right\rangle\right|^2\leq\left\langle f_A\left(|N|^{2\alpha}\right)x,x\right\rangle\left\langle f_A\left(|N|^{2\beta}\right)y,y\right\rangle, \tag{5.2}$$

for any $x,y\in H$.

Proof. Utilising the inequality (2.3) for the normal operator operator $T=N^n$ with n a natural number, we have

$$\left|\left\langle N^n|N^n|^{\beta-1}N^n|N^n|^{\alpha-1}x,y\right\rangle\right|^2\leq\left\langle|N^n|^{2\alpha}x,x\right\rangle\left\langle|N^n|^{2\beta}y,y\right\rangle, \tag{5.3}$$

for any $x,y\in H$.

Utilising the spectral representation for Borel functions of normal operators on Hilbert spaces, see for instance [1, p. 67], we have for any $\alpha,\beta\geq 1$ and for any $n\in\mathbb{N}$ that

$$\begin{aligned}N^n|N^n|^{\beta-1}N^n|N^n|^{\alpha-1}&=\int_{\sigma(N)}z^n|z^n|^{\beta-1}z^n|z^n|^{\alpha-1}dP(z)\\&=\int_{\sigma(N)}\left[z^2|z|^{\alpha+\beta-2}\right]^n dP(z)\\&=\left[N^2|N|^{\alpha+\beta-2}\right]^n,\end{aligned}$$

where P is the spectral measure associated to the operator N and $\sigma(N)$ is its spectrum.

Similarly,

$$|N^n|^{2\alpha}=\left(|N|^{2\alpha}\right)^n \text{ and } |N^n|^{2\beta}=\left(|N|^{2\beta}\right)^n$$

Therefore, the inequality (5.3) can be written as

$$\left|\left\langle\left[N^2|N|^{\alpha+\beta-2}\right]^n x,y\right\rangle\right|\leq\left\langle\left(|N|^{2\alpha}\right)^n x,x\right\rangle^{1/2}\left\langle\left(|N|^{2\beta}\right)^n y,y\right\rangle^{1/2}, \tag{5.4}$$

for any $\alpha,\beta\geq 1$ and for any $n\in\mathbb{N}$, for any $x,y\in H$.

If we multiply the inequality (5.4) by $|a_n|\geq 0$, sum over n from 0 to $k\geq 1$ and utilize the Cauchy-Bunyakowsky-Schwarz discrete inequality, we have successively

$$\begin{aligned}&\left|\left\langle\sum_{n=0}^{k}a_n\left[N^2|N|^{\alpha+\beta-2}\right]^n x,y\right\rangle\right|\\&\leq\sum_{n=0}^{k}|a_n|\left|\left\langle\left[N^2|N|^{\alpha+\beta-2}\right]^n x,y\right\rangle\right|\\&\leq\sum_{n=0}^{k}|a_n|\left\langle\left[|N|^{2\alpha}\right]^n x,x\right\rangle^{1/2}\left\langle\left[|N|^{2\beta}\right]^n y,y\right\rangle^{1/2}\\&\leq\left\langle\sum_{n=0}^{k}|a_n|\left[|N|^{2\alpha}\right]^n x,x\right\rangle^{1/2}\left\langle\sum_{n=0}^{k}|a_n|\left[|N|^{2\beta}\right]^n y,y\right\rangle^{1/2}\end{aligned} \tag{5.5}$$

for any $x,y\in H$ and for any $k\geq 1$.

Since $\|N\|^{2\alpha}, \|N\|^{2\beta} < R$ then $\left\| N^2 |N|^{\alpha+\beta-2} \right\| < R$ and the series

$$\sum_{n=0}^{\infty} |a_n| \left[|N|^{2\alpha} \right]^n, \sum_{n=0}^{\infty} |a_n| \left[|N|^{2\beta} \right]^n$$

and

$$\sum_{n=0}^{\infty} a_n \left[N^2 |N|^{\alpha+\beta-2} \right]^n$$

are convergent in the Banach algebra $B(H)$.

Taking the limit over $k \to \infty$ in the inequality (5.5) we deduce the desired result from (5.2). □

Corollary 10. *With the assumptions of Theorem 10, we have the inequality*

$$\left\| f\left(N^2 |N|^{\alpha+\beta-2} \right) \right\|^2 \le \left\| f_A\left(|N|^{2\alpha} \right) \right\| \left\| f_A\left(|N|^{2\beta} \right) \right\|. \tag{5.6}$$

Remark 7. *If we take $\beta = 2 - \alpha$ with $\alpha \in [1, 2]$ in (5.2), then we get the following result*

$$\left| \left\langle f\left(N^2\right) x, y \right\rangle \right|^2 \le \left\langle f_A\left(|N|^{2\alpha} \right) x, x \right\rangle \left\langle f_A\left(|N|^{2(2-\alpha)} \right) y, y \right\rangle \tag{5.7}$$

for any $x, y \in H$ and $\|N\|^{2\alpha}, \|N\|^{2(2-\alpha)} < R$.

Example 4. *1. Let N be a normal operator with $\|N\| < 1$. If $\alpha, \beta \ge 1$, then we have the inequalities*

$$\begin{aligned} &\left| \left\langle \left(1_H \pm N^2 |N|^{\alpha+\beta-2} \right)^{-1} x, y \right\rangle \right|^2 \\ &\le \left\langle \left(1_H - |N|^{2\alpha} \right)^{-1} x, x \right\rangle \left\langle \left(1_H - |N|^{2\beta} \right)^{-1} y, y \right\rangle, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &\left| \left\langle \ln\left(1_H \pm N^2 |N|^{\alpha+\beta-2} \right)^{-1} x, y \right\rangle \right|^2 \\ &\le \left\langle \ln\left(1_H - |N|^{2\alpha} \right)^{-1} x, x \right\rangle \left\langle \ln\left(1_H - |N|^{2\beta} \right)^{-1} y, y \right\rangle, \end{aligned} \tag{5.9}$$

for any $x, y \in H$.

2. Let N be a normal operator and $\alpha, \beta \ge 1$, then we have the inequalities

$$\begin{aligned} &\left| \left\langle \sin\left(N^2 |N|^{\alpha+\beta-2} \right) x, y \right\rangle \right|^2, \left| \left\langle \sinh\left(N^2 |N|^{\alpha+\beta-2} \right) x, y \right\rangle \right|^2 \\ &\le \left\langle \sinh\left(|N|^{2\alpha} \right) x, x \right\rangle \left\langle \sinh\left(|N|^{2\beta} \right) y, y \right\rangle, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\left| \left\langle \cos\left(N^2 |N|^{\alpha+\beta-2} \right) x, y \right\rangle \right|^2, \left| \left\langle \cosh\left(N^2 |N|^{\alpha+\beta-2} \right) x, y \right\rangle \right|^2 \\ &\le \left\langle \cosh\left(|N|^{2\alpha} \right) x, x \right\rangle \left\langle \cosh\left(|N|^{2\beta} \right) y, y \right\rangle, \end{aligned} \tag{5.11}$$

and

$$\left| \left\langle \exp\left(N^2 |N|^{\alpha+\beta-2} \right) x, y \right\rangle \right|^2 \le \left\langle \exp\left(|N|^{2\alpha} \right) x, x \right\rangle \left\langle \exp\left(|N|^{2\beta} \right) y, y \right\rangle, \tag{5.12}$$

for any $x, y \in H$.

Remark 8. *We remark that the choice $\beta = 2 - \alpha$ with $\alpha \in [1, 2]$ produces some simpler inequalities in (5.8)-(5.12). The details are omitted.*

References

[1] W. Arveson, *A Short Course on Spectral Theory*, 2002, Springer-Verlag Inc., New York.

[2] S.S. Dragomir, The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications. *J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 52, 22 pp.

[3] S.S. Dragomir, Some inequalities of Furuta's type for functions of operators defined by power series, Preprint, *RGMIA Res. Rep. Coll.* 15(2012), Article 17, pp. 12. [Online http://rgmia.org/v15.php]

[4] M. Fujii, C.-S. Lin and R. Nakamoto, Alternative extensions of Heinz-Kato-Furuta inequality. *Sci. Math.* **2** (1999), no. 2, 215–221.

[5] M. Fujii and T. Furuta, Löwner-Heinz, Cordes and Heinz-Kato inequalities. *Math. Japon.* **38** (1993), no. 1, 73–78.

[6] M. Fujii, E. Kamei, C. Kotari and H. Yamada, Furuta's determinant type generalizations of Heinz-Kato inequality. *Math. Japon.* **40** (1994), no. 2, 259–267

[7] M. Fujii, Y.O. Kim, and Y. Seo, Further extensions of Wielandt type Heinz-Kato-Furuta inequalities via Furuta inequality. *Arch. Inequal. Appl.* **1** (2003), no. 2, 275–283

[8] M. Fujii, Y.O. Kim and M. Tominaga, Extensions of the Heinz-Kato-Furuta inequality by using operator monotone functions. *Far East J. Math. Sci.* (FJMS) **6** (2002), no. 3, 225–238

[9] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **128** (2000), no. 1, 223–228.

[10] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. II. *J. Inequal. Appl.* **3** (1999), no. 3, 293–302,

[11] T. Furuta, Equivalence relations among Reid, Löwner-Heinz and Heinz-Kato inequalities, and extensions of these inequalities. *Integral Equations Operator Theory* **29** (1997), no. 1, 1–9.

[12] T. Furuta, Determinant type generalizations of Heinz-Kato theorem via Furuta inequality. *Proc. Amer. Math. Soc.* **120** (1994), no. 1, 223–231.

[13] T. Furuta, An extension of the Heinz-Kato theorem. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 785–787.

[14] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[15] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.* **125**(1952), 208-212.

[16] F. Kittaneh, Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.

[17] F. Kittaneh, Norm inequalities for fractional powers of positive operators. *Lett. Math. Phys.* **27** (1993), no. 4, 279–285.

[18] C.-S. Lin, On Heinz-Kato-Furuta inequality with best bounds. J. *Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **15** (2008), no. 1, 93–101.

[19] C.-S. Lin, On chaotic order and generalized Heinz-Kato-Furuta-type inequality. *Int. Math. Forum* **2** (2007), no. 37-40, 1849–1858,

[20] C.-S. Lin, On inequalities of Heinz and Kato, and Furuta for linear operators. *Math. Japon.* **50** (1999), no. 3, 463–468.

[21] C.-S. Lin, On Heinz-Kato type characterizations of the Furuta inequality. II. *Math. Inequal. Appl.* **2** (1999), no. 2, 283–287.

[22] C.A. McCarthy, c_p. *Israel J. Math.*, **5**(1967), 249-271.

[23] G. Popescu, Unitary invariants in multivariable operator theory. *Mem. Amer. Math. Soc.* **200** (2009), no. 941, vi+91 pp.

[24] M. Uchiyama, Further extension of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **127** (1999), no. 10, 2899–2904.

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