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# ON THE TORRICELLIAN POINTS IN NORMED LINEAR SPACES AND APPLICATIONS

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Abstract. We prove the existence of Torricellian points for an arbitrary set of $n$ points in a given normed space. In particular, we prove the existence of Torricellian points in reflexive normed spaces, non-expansive subspaces and evidently, inner product spaces. A case for collinear points is given and is utilised to characterise strict convexity. It is shown that the set of Torriceliian points contains a unique point when the space is strictly convex. Evidently, the Torricelian point is unique in inner product spaces.

*Key words and Phrases*: Fermat point, Torricellian point, characterisation of strictly convex spaces

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## Introduction

In 1643, Fermat raised the following problem [1]:

> "Given three distinct points in the plane, find the (unique) point having the minimal sum of distances to these three points."

This problem was first solved by Torricelli, which was published by his pupil Viviani in 1659. This point is often referred to as Fermat point, Fermat-Torricelli point, or Torricellian point. In this text, we will refer to it as Torricellian point. If all angles of the triangle are less than $\frac{2\pi}{3}$, then the Torricellian point is the interior point from which each side subtends of an angle of $\frac{2\pi}{3}$. If one of the angles is greater than $\frac{2\pi}{3}$ then the Torricellian point lies at the obtuse angled vertex. The Torricellian point is the solution to the (Euclidean) Steiner tree problem:

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> "Given $n$ points in the plane, the goal is to connect them by lines of minimum total length in such a way that any two points may be interconnected by line segments either directly or via other points and line segments"

for the case of $n = 3$.

In Dalla [1], a generalisation of Torricellian point of a set of $n$ points in $\mathbb{R}^d$ is considered. It is noted that for the case of collinear points (i.e. when these points all lie on a single straight line), the Torricellian point is the median of these points (see also Simons [2]). In the solution of the original problem, the Torricellian point is the isogonal point (median of the triangle on a plane). The same also holds true for the case of 4 points in $\mathbb{R}^3$ [1]. Dalla also proved that the isogonality property fails for any simplex of dimension higher than 3 [1].

In Simons [2], a generalisation of Torricellian point for an arbitrary set of $n$ given points is considered, when these points lie in one, two and three dimensional spaces. When all $n$ points lie in one dimensional space (collinear), the Torricellian point $F$ coincides with the centre point of the set $\{P_1, \ldots, P_n\}$ when $n$ is odd, i.e. $F = P_l$, where $l = \frac{(n+1)}{2}$; and $F$ is any point lying between $P_l$ and $P_{l+1}$ where $l = \frac{n}{2}$ when $n$ is even (cf. Dalla [1]).

Furthermore, the general Torricellian point in 2-D has application as the optimum point to site a central terminal for supply utilities such as gas, electricity or water to a set of users in order to minimise the total length of piping, wire etc., required for the distribution [2, p. 60–61]. Another application of the Torricellian point is the characterizations of isosceles tetrahedra with respect to their vertex set [3].

Torricelli's problem can be defined in any metric spaces, but if we would like to have nice properties in characterising Torricellian points, we have to restrict ourselves to the case of normed linear spaces. The set of Torricellian points can be utilised to characterise strictly convex normed spaces and inner product spaces.

Locating the Torricellian point in normed spaces has been associated with characterisations of inner product spaces. Benítez, Fernández, and Soriano [4] consider a generalisation of Torricellian point in a normed space of dimension greater than 3. It is shown that the set of Torricellian points intersects the convex hull of the three vertices if and only if the normed space is an inner product space.

In Martini, Swanepoel and Weiss [5], a generalisation of the Torricelli's problem in Minkowski spaces is considered. It is shown that the Torricelli point is unique if and only if the Minkowski space is strictly convex. The case of collinear points is also considered; and their results generalised the result of Simons [2] as mentioned above.

Dragomir et al. in [6], considered and solved Torricelli's problem in a real inner product space of dimension greater than 1. A generalisation to $n$ arbitrary points in a real inner product space is considered in Dragomir and Comănescu [7].

The aim of this paper is to generalise these results to a more general settings of normed linear spaces (cf. Section 1). In particular (cf. Section 2), we consider the existence of Torricellian points on reflexive normed spaces, non-expansive subspaces, and as a corollary, we consider the existence in inner product spaces which recaptures the results of Dragomir and Comănescu [7]. In Section 3, we consider the case of collinear points in any normed spaces; and in particular, strictly convex spaces and inner product spaces, which recaptures the results of Simons [2]. Lastly, we show that for the case of non-collinear points, the Torricellian point is unique in strictly convex spaces and inner product spaces. This generalises the results of Martini, Swanepoel and Weiss [5].

## 1. Generalisation of Torricellian points

We start this section by pointing out the main properties of norm derivatives that will be used in the sequel. For more details connected with these norm derivatives, we refer to Dragomir [8] where further references are given.

Let $(X, \|\cdot\|)$ be a real normed linear space and consider the semi-inner products $\langle\cdot,\cdot\rangle_i$, $\langle\cdot,\cdot\rangle_s$ given by:

$$\langle x, y\rangle_i = \lim_{t\to 0-} \frac{\|y+tx\|^2 - \|y\|^2}{2t};$$
$$\langle x, y\rangle_s = \lim_{t\to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t};$$

which are well defined for all $x, y$ in $X$. For the sake of completeness we list some usual properties of these semi-inner products:

(i) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;

(ii) $\langle -x,y\rangle_s = \langle x,-y\rangle_s = -\langle x,y\rangle_i$ if $x,y\in X$;

(iii) $\langle \alpha x,\beta y\rangle_p = \alpha\beta\langle x,y\rangle_p$ for all $x,y\in X$ and $\alpha,\beta\geq 0$;

(iv) $\langle \alpha x+y,x\rangle_p = \alpha\langle x,x\rangle_p + \langle y,x\rangle_p$ if $x,y\in X$, $\alpha\in\mathbb{R}$;

(v) $\langle x+y,z\rangle_p \leq \|x\|\|z\| + \langle y,z\rangle_p$ for all $x,y,z\in X$;

(vi) the space $(X,\|\cdot\|)$ is smooth, i.e., the norm $\|\cdot\|$ is Gâteaux differentiable on $X\setminus\{0\}$ iff $\langle x,y\rangle_i = \langle x,y\rangle_s$ for all $x,y\in X$; or iff $\langle\cdot,\cdot\rangle_p$ is linear in the first variable;

where $p=s$ or $p=i$. For some applications of these norm derivatives in the theory of representation for the continuous linear functionals or in best approximation theory, we refer to papers [9], [10], [11], [12], and [13] where further references are given.

Let $(X,\|\cdot\|)$ be a real normed linear space, $n\geq 1$ a natural number and $\mathcal{A}=\{a_1,\ldots,a_n\}$ a set of $n$ distinct points in $X$. Throughout the paper we denote $\overline{1,n}=1,\ldots,n$. The functional $T=T_{\mathcal{A}}:X\to[0,\infty)$, $T(x):=\sum_{i=1}^{n}\|x-a_i\|$ will be called the **Torricellian functional** associated with the family of points $\mathcal{A}$. The main properties of this mapping can be summarised in the following. We refer to Dragomir and Comănescu [7, Proposition 1] for the detailed proof.

**Proposition 1.1.** *With the above assumptions, we have:*

*(i)* $T$ *is nonlinear on* $X$;

*(ii)* $T$ *is continuous on* $X$ *in the norm topology;*

*(iii)* $T$ *is nonnegative and*

$$\lim_{\|x\|\to\infty} T(x) = \infty;$$

*(iv)* $T$ *is convex on* $X$.

Now, let us consider the Gâteaux directional derivatives for a functional $T$:

$$(V_+T)(x_0)\cdot x := \lim_{t\to 0+}\frac{T(x_0+tx)-T(x_0)}{t}$$

and

$$(V_-T)(x_0)\cdot x := \lim_{t\to 0-}\frac{T(x_0+tx)-T(x_0)}{t}$$

where $x_0,x\in X$. The following proposition also holds.

**Proposition 1.2.** *With the above assumption we have:*

$$(1)\qquad (V_{\pm}T)(x_0)\cdot x = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\langle x, x_0 - a_i\rangle_{s(i)}}{\|x_0 - a_i\|}, & \text{if } x_0 \notin \mathcal{A}; \\ \displaystyle\sum_{i=1}^{n} \frac{\langle x, a_j - a_i\rangle_{s(i)}}{\|a_j - a_i\|} \pm \|x\|, & \text{if } x_0 = a_j \text{ for } j \in \{1,\dots,n\} \end{cases}$$

*for all* $x \in X$.

*Proof.* Suppose that $x_0 \notin \mathcal{A}$. Then we have:

$$\begin{aligned} (V_{\pm}T)(x_0)\cdot x &:= \sum_{i=1}^{n} \lim_{t\to 0\pm} \frac{\|x_0 - a_i + tx\| - \|x_0 - a_i\|}{t} \\ &= \sum_{i=1}^{n} \lim_{t\to 0\pm} \left[ \frac{\|x_0 - a_i + tx\|^2 - \|x_0 - a_i\|^2}{2t} \cdot \frac{2}{\|x_0 - a_i + tx\| + \|x_0 - a_i\|} \right] \\ &= \sum_{i=1}^{n} \frac{\langle x, x_0 - a_i\rangle_{s(i)}}{\|x_0 - a_i\|} \end{aligned}$$

for all $x \in X$.

Let $x_0 = a_j$ with a fixed $j \in \{1,\dots,n\}$. Then

$$\begin{aligned} (V_{\pm}T)(x_0)\cdot x &:= \sum_{i=1 i\neq j}^{n} \lim_{t\to 0\pm} \frac{\|a_j - a_i + tx\| - \|a_j - a_i\|}{t} + \lim_{t\to 0\pm} \frac{|t|\,\|x\|}{t} \\ &= \sum_{i=1 i\neq j}^{n} \frac{\langle x, a_j - a_i\rangle_{s(i)}}{\|a_j - a_i\|} \pm \|x\| \end{aligned}$$

and this completes the proof. □

**Corollary 1.3.** *Let* $(X, \|\cdot\|)$ *be a smooth normed space and denote* $[\cdot,\cdot] := \langle\cdot,\cdot\rangle_s = \langle\cdot,\cdot\rangle_i$ *the semi-inner product which generates the norm* $\|\cdot\|$. *Then* $T$ *is Gâteaux differentiable on* $X\backslash\mathcal{A}$ *and*

$$(VT)(x_0)\cdot x = \sum_{i=1}^{n} \frac{[x, x_0 - a_i]}{\|x_0 - a_i\|}, \quad \text{if } x_0 \notin \mathcal{A}$$

*for all* $x \in X$.

**Remark 1.4.** Corollary 1.3 also holds for any inner product space $(X, \langle\cdot,\cdot\rangle)$ as every inner product space is smooth. This recaptures the results in Dragomir and Comănescu [7, Proposition 3], that $T$ is Gâteaux differentiable on $X \backslash \mathcal{A}$ and

$$(VT)(x_0) \cdot x = \sum_{i=1}^{n} \frac{\langle x, x_0 - a_i \rangle}{\|x_0 - a_i\|}, \quad \text{for all } x_0 \notin \mathcal{A}$$

We refer to [7, Proposition 2] for the proof of the following proposition.

**Proposition 1.5.** *Let $(X, \|\cdot\|)$ be a strictly convex normed linear space. If $\mathcal{A} = \{a_1, \ldots, a_n\}$ with $n \geq 3$ is a set of non-collinear points in $X$, then $T$ is strictly convex on $X$.*

Now we formally define the Torricellian points for a given set of points.

**Definition 1.6.** Let $(X, \|\cdot\|)$ be a real normed linear space, $n \geq 1$ a natural number and $\{a_1, \ldots, a_n\}$ a set of distinct elements in $X$. The point $x_0 \in X$ will be called a **Torricellian point** related to the set $\{a_1, \ldots, a_n\}$ if it minimises the Torricellian functional $T$, i.e.,

$$T(x_0) = \inf_{x \in X} T(x) \tag{2}$$

or, equivalently,

$$\sum_{i=1}^{n} \|x_0 - a_i\| \leq \sum_{i=1}^{n} \|x - a_i\| \quad \text{for all } \ x \in X. \tag{3}$$

The set of all Torricellian points associated with the family $\{a_1, \ldots, a_n\}$ will be denoted by $\mathcal{T}_X\{a_1, \ldots, a_n\}$.

The following proposition gives the main properties of $\mathcal{T}_X\{a_1, \ldots, a_n\}$ for any $\{a_1, \ldots, a_n\} \subset X$.

**Proposition 1.7.** *With the above assumptions, we have that $\mathcal{T}_X\{a_1, \ldots, a_n\}$ is a convex, closed and bounded subset of the normed linear space $X$.*

*Proof.* Assume that $\mathcal{T}_X\{a_1, \ldots, a_n\}$ is non empty, as it is a vacuous proof when the set is empty. If $x_1, x_2 \in \mathcal{T}_X\{a_1, \ldots, a_n\}$ and $\lambda \in [0,1]$, then

$$\begin{aligned} T(\lambda x_1 + (1-\lambda) x_2) &\leq \lambda T(x_1) + (1-\lambda) T(x_2) \\ &\leq \lambda T(x) + (1-\lambda) T(x) = T(x) \end{aligned}$$

for all $x \in X$, which shows that $[x_1, x_2] \subset \mathcal{T}_X\{a_1, \ldots, a_n\}$, i.e., the set $\mathcal{T}_X\{a_1, \ldots, a_n\}$ is convex.

Now, let $x_0 \in \mathcal{T}_X\{a_1,\ldots,a_n\}$ and put $\alpha_0 = T(x_0)$. Then it is clear that $\mathcal{T}_X\{a_1,\ldots,a_n\} = T^{-1}(\{\alpha_0\})$ which is closed as the mapping is continuous in the norm topology of $X$ and $\{\alpha_0\}$ is closed in $\mathbb{R}$.

Let $x_0 \in \mathcal{T}_X\{a_1,\ldots,a_n\}$ and put $\alpha = T(x_0) = \inf_{x\in X} T(x)$. Then

$$\begin{aligned}\alpha = T(x_0) &= \sum_{i=1}^{n} \|x_0 - a_i\| \\ &\geq \sum_{i=1}^{n} \left|\|x_0\| - \|a_i\|\right| \geq n\|x_0\| - \sum_{i=1}^{n}\|a_i\|\end{aligned}$$

which gives us

$$\|x_0\| \leq \frac{\alpha + \sum_{i=1}^{n}\|a_i\|}{n}$$

which proves the boundedness of the set $\mathcal{T}_X\{a_1,\ldots,a_n\}$. □

We will introduce the following definition.

**Definition 1.8.** The Torricellian point $x_0 \in \mathcal{T}_X\{a_1,\ldots,a_n\}$ will be called **segmentally inferior** relating to the set $\mathcal{T}_X\{a_1,\ldots,a_n\}$ if there exists $x_1, x_2 \in \mathcal{T}_X\{a_1,\ldots,a_n\}$ so that $x_0 \neq x_1, x_2$ and $x_0 \in [x_1, x_2]$, i.e., $x_0 \in (x_1, x_2)$.

**Proposition 1.9.** *If the point $x_0 \in \mathcal{T}_X\{a_1,\ldots,a_n\}$ is segmentally inferior for the set $\mathcal{T}_X\{a_1,\ldots,a_n\}$, then $T$ is Gâteaux differentiable on $x_0$ in the direction $x_2 - x_1$ and*

$$(VT)(x_0)\cdot(x_2 - x_1) = 0.$$

*Proof.* Let $x_0$ be a segmentally inferior point of $\mathcal{T}_X\{a_1,\ldots,a_n\}$. Then there exists $x_1, x_2$, $x_1 \neq x_2$, so that $x_0 \in (x_1, x_2)$, i.e., $x_0 = sx_1 + (1-s)x_2$ with $s \in (0,1)$. If $x_0 = sx_1 + (1-s)x_2$ with $s \in (0,1)$, then, $x_0 + t(x_2 - x_1) \in [x_1, x_2]$ if and only if $t \in (1, 1-s)$. For $t \in (1-s, s)$ we have $T(x_0 + t(x_2 - x_1)) = T(x_0)$ and hence

$$(VT)(x_0)\cdot(x_2 - x_1) = \lim_{t\to 0}\frac{T(x_0 + t(x_2 - x_1)) - T(x_0)}{t} = 0,$$

which completes the proof. □

**Corollary 1.10.** *The points $a_j$ with $j = \overline{1,n}$ cannot be segmentally inferior for the set $\mathcal{T}_X\{a_1,\ldots,a_n\}$.*

*Proof.* Let us assume that $a_j$ is segmentally inferior for the set $\mathcal{T}_X\{a_1,\ldots,a_n\}$, $j\in\{1,\ldots,n\}$. Then there exists $x_1,x_2\in\mathcal{T}_X\{a_1,\ldots,a_n\}$ such that $a_j\neq x_1,x_2$ and $a_j\in(x_1,x_2)$. By the above proposition we have:

$$(V_+T)(a_j)(x_2-x_1)=(V_-T)(a_j)(x_2-x_1)=0.$$

By Proposition 1.2 we have

$$0=(V_+T)(a_j)(x_2-x_1)=\sum_{i=1 i\neq j}^{n}\frac{\langle x_2-x_1,a_j-a_i\rangle_s}{\|a_j-a_i\|}+\|x_2-x_1\|$$

and

$$0=(V_-T)(a_j)(x_2-x_1)=\sum_{i=1 i\neq j}^{n}\frac{\langle x_2-x_1,a_j-a_i\rangle_i}{\|a_j-a_i\|}-\|x_2-x_1\|$$

from where we get

$$\sum_{i=1 i\neq j}^{n}\frac{\langle x_2-x_1,a_j-a_i\rangle_s}{\|a_j-a_i\|}-\sum_{i=1 i\neq j}^{n}\frac{\langle x_2-x_1,a_j-a_i\rangle_i}{\|a_j-a_i\|}=-2\|x_2-x_1\|<0. \tag{4}$$

Since $\langle x_2-x_1,a_j-a_i\rangle_s\geq\langle x_2-x_1,a_j-a_i\rangle_i$ we have

$$\sum_{i=1 i\neq j}^{n}\left(\frac{\langle x_2-x_1,a_j-a_i\rangle_s-\langle x_2-x_1,a_j-a_i\rangle_i}{\|a_j-a_i\|}\right)\geq 0,$$

which contradicts (4) and thus proves the corollary. □

We will give now a definition regarding the eigenvalue of two sets of $n$ distinct points in a normed linear space.

**Definition 1.11.** Let $X$ be a normed linear space and the subsets of distinct points $\mathcal{A}=\{a_1,\ldots,a_n\}$ and $\mathcal{B}=\{b_1,\ldots,b_n\}$. The subsets $\mathcal{A}$ and $\mathcal{B}$ are *isometrically equivalent* if there exists a distance preserving bijective function $F:X\to X$ which satisfy $F(\mathcal{A})=\mathcal{B}$.

Without loss of generality we suppose that $F(a_i)=b_i$ for all $i\in\{1,\ldots,n\}$.

**Proposition 1.12.** *Let $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_n\}$ be two sets of distinct points in $X$ which are isometrically equivalent and $F$ be the mapping which establish the equivalence. Then, the following statements are equivalent:*

(*i*) $x_0\in\mathcal{T}_X\{a_1,\ldots,a_n\}$;

*(ii)* $F(x_0) \in \mathcal{T}_X\{b_1, \dots, b_n\}$,

*where* $x_0 \in X$.

*Proof.* Suppose that $x_0 \in \mathcal{T}_X\{a_1, \dots, a_n\}$. Then for all $x \in X$ we have $T(x_0) \le T(x)$, which gives us

$$\sum_{i=1}^{n} \|x_0 - a_i\| \le \sum_{i=1}^{n} \|x - a_i\| \quad \text{for all} \quad x \in X.$$

Since $F$ is distance preserving, and $\mathcal{A}$ and $\mathcal{B}$ are isometrically equivalent, we have

$$\sum_{i=1}^{n} \|F(x_0) - b_i\| = \sum_{i=1}^{n} \|F(x_0) - F(a_i)\| = \sum_{i=1}^{n} \|x_0 - a_i\|$$

and

$$\sum_{i=1}^{n} \|F(x) - b_i\| = \sum_{i=1}^{n} \|F(x) - F(a_i)\| = \sum_{i=1}^{n} \|x - a_i\|$$

thus we get

$$\sum_{i=1}^{n} \|F(x_0) - b_i\| \le \sum_{i=1}^{n} \|F(x) - b_i\| \quad \text{for all} \quad x \in X.$$

Now let $y \in X$. Then there exists a unique element $x \in X$ such that $F(x) = y$ (as the map $T$ is one-to-one), thus

$$\sum_{i=1}^{n} \|F(x_0) - b_i\| \le \sum_{i=1}^{n} \|y - b_i\| \quad \text{for all} \quad y \in X,$$

which shows that $F(x_0) \in \mathcal{T}_X\{b_1, \dots, b_n\}$.

The proof for the converse goes likewise for the isometric $F^{-1}$. We will omit the details. □

**Proposition 1.13.** *Let* $\{a_1, \dots, a_n\}$ *be a set of distinct points in the normed linear space* $X$*. Then the sets*

$$\{a_1, \dots, a_n\} \quad \textit{and} \quad \{a_1 - a_i, \dots, a_{i-1} - a_i, 0, a_{i+1} - a_i, \dots, a_n - a_i\}$$

*for* $i = \overline{1,n}$ *are equivalent.*

The proof follows by observing that the map $F_i : X \to X$, $F_i(x) = x - a_i$ is isometric in $X$ for all $i = \overline{1,n}$.

## 2. The existence of Torricellian points

We will start with the following known result for which we will give a short direct proof.

**Theorem 2.1.** *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then for all $\{a_1, \ldots, a_n\}$ a set of distinct points in $X$, the Torricellian set $\mathcal{T}_X \{a_1, \ldots, a_n\}$ is nonempty in $X$.*

*Proof.* Since $T(x) \geq 0$ for all $x \in X$, then $\inf_{x \in \mathbb{X}} T(x) \in \mathbb{R}$. Denote $a = \inf_{x \in X} T(x)$. Thus, there exists a sequence of vectors $(x_n)_{n \in \mathbb{N}}$ in $X$ so that $\lim_{n \to \infty} T(x_n) = a$.

As $\lim_{\|x\| \to \infty} T(x) = \infty$ we deduce that $(x_n)_{n \in \mathbb{N}}$ is bounded in $X$. Since the Banach space is reflexive, the Banach-Alaglou and Eberlein-Smullian theorems (cf. Castillo [14, p. 77]) assert that there is a weakly convergent subsequence of $(x_n)_{n \in \mathbb{N}}$. Thus, there exists an element $x_0 \in X$ such that $x_{n_k} \overset{\text{weak}}{\longrightarrow} x_0$. By the continuity of $T$ it follows that $a = \lim_{k \to \infty} T(x_{n_k}) \geq T(x_0)$, thus $a = T(x_0)$ which means that $x_0 \in \mathcal{T}_X \{a_1, \ldots, a_n\}$ and the theorem is proved. □

We will try to point out, now, some other results of existence for normed linear spaces which are not necessarily reflexive.

**Theorem 2.2.** *Let $(X, \|\cdot\|)$ be a normed linear space and $\mathcal{A} = \{a_1, \ldots, a_n\}$ a set of distinct points from $X$. If $\mathcal{A} \subset Y$ and $Y \subset X$ is a nonexpansive linear subspace in $X$, i.e., there exists a mapping $P : X \to Y$ such that:*

( *i* ) *$P$ is the identity of $Y$, i.e., $Px = x$ for all $x \in Y$;*

( *ii* ) *$\|Px - Py\| \leq \|x - y\|$ for all $x, y \in X$,*

*then we have the inclusion:*

$$\mathcal{T}_Y \{a_1, \ldots, a_n\} \subseteq \mathcal{T}_X \{a_1, \ldots, a_n\}. \tag{5}$$

*Proof.* First of all, we observe that $T(Px) \leq T(x)$ for all $x \in X$, as

$$T(Px) = \sum_{i=1}^{n} \|Px - a_i\| = \sum_{i=1}^{n} \|Px - Pa_i\| \leq \sum_{i=1}^{n} \|x - a_i\| = T(x)$$

for all $x \in X$. By definition, if $x_0 \in \mathcal{T}_Y \{a_1, \ldots, a_n\}$, then $T(x_0) \leq T(x)$ for all $x \in Y$. Now, let $x \in X \backslash Y$. Then $Px \in Y$ and thus $T(x_0) \leq T(Px)$. On the other hand, $T(Px) \leq T(x)$ which gives us $T(x_0) \leq T(x)$ for all $x \in X$, i.e., $x_0 \in \mathcal{T}_X \{a_1, \ldots, a_n\}$; this completes the proof. □

The following corollary contains a sufficient condition for the existence of the Torricellian points.

**Corollary 2.3.** *Let $(X, \|\cdot\|)$ be a normed linear space and $\mathcal{A} = \{a_1, \ldots, a_n\}$ a system of distinct points in $X$. If the subspace $T_n := Sp\{a_1, \ldots, a_n\}$ spanned by the set $\mathcal{A}$ is nonexpansive in $X$, then $\mathcal{T}_X \{a_1, \ldots, a_n\}$ is nonempty.*

*Proof.* Let us observe that the finite-dimensional space $T_n$ can be regarded as a reflexive Banach space. Thus, by Theorem 2.1 it follows that $\mathcal{T}_{T_n} \{a_1, \ldots, a_n\} \neq \varnothing$ and from Theorem 2.2 it follows that $\mathcal{T}_{T_n} \{a_1, \ldots, a_n\} \subseteq \mathcal{T}_X \{a_1, \ldots, a_n\}$ which proves the corollary. □

The following theorem contains some examples of nonexpansive linear subspaces in inner product spaces.

**Theorem 2.4.** *Let $(X; \langle \cdot, \cdot \rangle)$ be an inner product space. If $G$ is a Čebyševian linear subspace in $X$, i.e., every element $x_0 \in X$ has a unique best approximant in $G$, then $G$ is nonexpansive in $X$.*

*Proof.* Let $x_0 \in X$. If $x_0 \in G$, then $x_0 = x_0 + 0$ with $0 \in G^\perp$. If $x_0 \notin G$, then there exists an element $x_1 \in G$ such that $d(x_0, G) = d(x_0, x_1)$, i.e., $\inf_{y \in G} \|x_0 - y\| = \|x_0 - x_1\|$.

Set $x_2 := x_0 - x_1$. Then for all $\lambda \in \mathbb{R}$ and $y \in G$ we have:

$$\|x_2 + \lambda y\| = \|x_0 - x_1 + \lambda y\| \geq \|x_0 - x_1\| = \|x_2\|$$

for all $\lambda \in \mathbb{R}$ and $y \in G$, which implies that $\langle x_2, y \rangle = 0$ for all $y \in G$, i.e., $x_2 \in G^\perp$. Thus, for all $x \in X$, there exists $x_1 \in G$ and $x_2 \in G^\perp$ such that $x = x_1 + x_2$.

We note that this decomposition is unique. If $x = x_1 + x_2$, $x = y_1 + y_2$ with $x_1, y_1 \in G$ and $x_2, y_2 \in G^\perp$, then $G \ni x_1 - y_1 = y_2 - x_2 \in G^\perp$ and since $G \cap G^\perp = \{0\}$, we get that $x_1 = y_1$ and $x_2 = y_2$.

Define the projection of $X$ on $G$, i.e., the mapping $P: X \to G$ given by $P(x) = x_1$, where $x_1$ is the best approximation of $x$ in $G$. Then $P$ is the identity on $G$ and for all $x, y \in X$ we can write:

$$\|x - y\|^2 = \|x_1 + x_2 - y_1 - y_2\|^2 = \|(x_1 - y_1) + (x_2 - y_2)\|^2 .$$

Since $(x_1 - y_1) \perp (x_2 - y_2)$, then by the Pythagorean identity we get

$$\begin{aligned} \|(x_1 - y_1) + (x_2 - y_2)\|^2 &= \|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 \\ &\geq \|x_1 - y_1\|^2 = \|Px - Py\|^2 \end{aligned}$$

and thus we obtain the desired inequality $\|Px - Py\| \leq \|x - y\|$, and the theorem is proved. □

The above theorem gives us the possibility to obtain the following theorem of existence for the Torricellian points (see also Dragomir and Comănescu [7]).

**Theorem 2.5.** *Let* $(X; \langle \cdot, \cdot \rangle)$ *be an inner product space. Then for all* $\{a_1, \ldots, a_n\}$ *a set of distinct points in* $X$, $\mathcal{T}_X \{a_1, \ldots, a_n\}$ *is nonempty.*

*Proof.* Consider the space $T_n := Sp\{a_1, \ldots, a_n\}$ which is Čebyševian and nonexpansive on $X$. Then $\mathcal{T}_{T_n} \{a_1, \ldots, a_n\}$ is nonempty and

$$\mathcal{T}_{T_n} \{a_1, \ldots, a_n\} \subseteq \mathcal{T}_X \{a_1, \ldots, a_n\},$$

which proves the statement. □

## 3. The case of collinear points in normed linear spaces

In this section we shall use the following definition:

**Definition 3.1.** The set of $n$ distinct points $\{a_1, \ldots, a_n\}$ are collinear in the real normed space, i.e., there exist two distinct elements $a, b \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$a_i = \lambda_i a + (1 - \lambda_i) b, \qquad i = \overline{1, n}.$$

Without loss of generality, we will assume that $\lambda_1 < \cdots < \lambda_n$. The following proposition holds.

**Proposition 3.2.** *Let* $(X, \|\cdot\|)$ *be a real normed space and* $a_1, \dots, a_{2k+1}$ *be* $2k+1$ *collinear points. Then* $\mathcal{T}_X \{a_1, \dots, a_{2k+1}\} = \{a_{k+1}\}$.

*Proof.* We have successively:

$$\begin{aligned} T(a_{k+1}) &= \sum_{i=1}^{2k+1} \|a_{k+1} - a_i\| \\ &= \sum_{i=1}^{2k+1} \|\lambda_{k+1} a + (1 - \lambda_{k+1}) b - \lambda_i a - (1 - \lambda_i) b\| \\ &= \sum_{i=1}^{2k+1} |\lambda_{k+1} - \lambda_i| \, \|a - b\| \\ &= \sum_{i=1}^{k} (\lambda_{2k+2-i} - \lambda_i) \|a - b\| = \sum_{i=1}^{k} \|a_{2k+2-i} - a_i\| . \end{aligned}$$

Now, assume that $x \in X \setminus \{a_{k+1}\}$. Then for all $i \in \{1, \dots, k\}$ one has the inequality:

$$\|x - a_i\| + \|x - a_{2k+2-i}\| \geq \|a_{2k+2-i} - a_i\| .$$

Since $\|x - a_{k+1}\| > 0$, we get

$$\begin{aligned} T(x) &= \sum_{i=1}^{k} (\|x - a_i\| + \|x - a_{2k+2-i}\|) + \|x - a_{k+1}\| \\ &> \sum_{i=1}^{k} \|a_{2k+2-i} - a_i\| = T(a_{k+1}) . \end{aligned}$$

Consequently $T(x) > T(a_{k+1})$ for all $x \in X \setminus \{a_{k+1}\}$ which shows that $a_{k+1}$ is the unique Torricellian point associated with $\{a_1, \dots, a_{2k+1}\}$. □

**Remark 3.3.** Proposition 3.2 recaptures the results of Simons [2, p. 61] for the case of collinear points in Euclidean spaces.

The following proposition holds.

**Proposition 3.4.** *Let* $(X, \|\cdot\|)$ *be a normed linear space and* $\{a_1, \dots, a_{2k}\}$ *a set of* $2k$ *collinear distinct points in* $X$. *Then* $[a_k, a_{k+1}] \subset \mathcal{T}_X \{a_1, \dots, a_{2k}\}$.

*Proof.* Let $x \in [a_k, a_{k+1}]$. Then $x = \lambda a + (1 - \lambda) b$ with $\lambda \in [\lambda_k, \lambda_{k+1}]$. A simple computation shows that:

$$\begin{aligned} T(x) &= \sum_{i=1}^{2k} \|x - a_i\| \\ &= \sum_{i=1}^{2k} \|\lambda a + (1-\lambda) b - \lambda_i a - (1 - \lambda_i) b\| \\ &= \sum_{i=1}^{2k} |\lambda - \lambda_i| \, \|a - b\| \\ &= \sum_{i=1}^{k} (\lambda_{2k+1-i} - \lambda_i) \|a - b\| = \sum_{i=1}^{k} \|a_{2k+1-i} - a_i\| . \end{aligned}$$

Now, suppose that $y \in X$. Then:

$$\begin{aligned} T(y) &= \sum_{i=1}^{2k} \|y - a_i\| \\ &= \sum_{i=1}^{k} (\|y - a_i\| + \|y - a_{2k+1-i}\|) \\ &\geq \sum_{i=1}^{k} \|a_i - a_{2k+1-i}\| = T(x) , \end{aligned}$$

which shows that $[a_k, a_{k+1}] \subset \mathcal{T}_X \{a_1, \ldots, a_{2k}\}$. This completes the proof. □

**Remark 3.5.** Proposition 3.4 recaptures the results of Simons [2, p. 61] for the case of collinear points in Euclidean spaces.

We will use the following notation in the following characterisation theorem. Let $a$ and $b$ be two distinct vectors in $X$, we define

$$dr(a,b) := \{\lambda a + (1-\lambda)b \mid \lambda \in \mathbb{R}\}.$$

**Theorem 3.6.** *Let $(X, \|\cdot\|)$ be a real normed space. The following statements are equivalent:*

*(i)* $(X, \|\cdot\|)$ *is strictly convex;*

*(ii)* *For every $k \in \mathbb{N}^*$, and for every $\{a_i, \ldots, a_{2k}\}$, $2k$ collinearly distinct point in $X$, we have*

$$\mathcal{T}_X \{a_1, \ldots, a_{2k}\} = [a_k, a_{k+1}] .$$

*Proof.* "(i) $\Longrightarrow$ (ii)". Let $(X, \|\cdot\|)$ be a strictly convex normed space and $\{a_1, \ldots, a_{2k}\}$ be a set of $2k$ collinearly distinct points in $X$. By Proposition 3.4 we have that $[a_k, a_{k+1}] \subset \mathcal{T}_X \{a_1, \ldots, a_{2k}\}$.

Let us suppose that there exists a point $x \in \mathcal{T}_X \{a_1, \ldots, a_{2k}\} \setminus [a_k, a_{k+1}]$. We want to show that

$$\|x - a_i\| + \|x - a_{2k+1-i}\| = \|a_{2k+1-i} - a_i\| \tag{6}$$

for all $i \in \{1, \ldots, k\}$. Suppose that there exists an element $i_0 \in \{1, \ldots, k\}$ such that

$$\|x - a_{i_0}\| + \|x - a_{2k+1-i_0}\| > \|a_{2k+1-i_0} - a_{i_0}\|$$

we have

$$\begin{aligned}
T(x) &= \sum_{i=1}^{2k} \|x - a_i\| \\
&= \sum_{i=1}^{k} (\|x - a_i\| + \|x - a_{2k+1-i}\|) \\
&> \sum_{i=1}^{k} \|a_{2k+1-i} - a_i\| = T(x_0),
\end{aligned}$$

where $x_0 \in [a_k, a_{k+1}]$ (see Proposition 3.2), which contradicts the fact that $x$ minimises the Torricellian map $T$. Then (6) holds for all $i \in \{1, \ldots, k\}$.

As the space $(X, \|\cdot\|)$ is assumed to be strictly convex, then by (6) we have $\theta_i \in \mathbb{R}$ such that

$$x - a_{2k+1-i} = \theta_i (x - a_i)$$

If $x \neq a_i$ for all $i \in \{1, \ldots, k\}$. Then by the above equality we have:

$$x(1 - \theta_i) = a_{2k+1-i} - \theta_i a_i, \qquad (\theta_i \neq 1),$$

which gives us:

$$\begin{aligned}
x &= \frac{a_{2k+1-i}}{1 - \theta_i} - \frac{\theta_i a_i}{1 - \theta_i} \\
&= \frac{(\lambda_{2k+1-i} a + (1 - \lambda_{2k+1-i}) b)}{1 - \theta_i} - \frac{\theta_i (\lambda_i a + (1 - \lambda_i) b)}{1 - \theta_i} \\
&= \frac{\lambda_{2k+1-i} - \theta_i \lambda_i}{1 - \theta_i} a + \left( \frac{1 - \lambda_{2k+1-i} - \theta_i (1 - \lambda_i)}{1 - \theta_i} \right) b
\end{aligned}$$

which shows that $x \in dr(a,b)$. If there exists an $i_0 \in \{1,\ldots,k\}$ so that $x = a_{i_0}$, then the conclusion $x \in dr(a,b)$ remains true. We shall omit the details.

Now, as $x \in dr(a,b)$, then $x = \mu a + (1-\mu) b$ with $\mu \in \mathbb{R}$, and we have

$$T(x) = \sum_{i=1}^{n} |\mu - \lambda_i| \, \|a-b\| .$$

We want to show that this would lead to a contradiction that $x$ minimises the Torricellian point. We shall separate this in four cases:

(**Case 1:** $\mu < \lambda_1$.) We have,

$$\sum_{i=1}^{2k} |\mu - \lambda_i| = \sum_{i=1}^{2k} \lambda_i - 2k\mu$$

and

$$\sum_{i=1}^{2k} \lambda_i - 2k\mu > -\sum_{i=1}^{k} (\lambda_{2k+1-i} - \lambda_i) \tag{7}$$

since

$$\sum_{i=1}^{k} \lambda_i - 2k\mu > -\sum_{i=1}^{k} \lambda_i$$

and this last inequality follows by the fact that:

$$\sum_{i=1}^{k} \lambda_i > k\lambda_1 > k\mu.$$

Now, let us observe that by the inequality (7) we get

$$T(x) > \sum_{i=1}^{k} (\lambda_{2k+1-i} - \lambda_i) \, \|a-b\| = T(x_0)$$

with $x_0 \in [a_k, a_{k+1}]$ (see Proposition 3.4), which contradicts the fact that $x$ is a Torricellian point.

(**Case 2:** $\mu > \lambda_{2k}$) The proof goes likewise and we shall omit the details.

(**Case 3:** $\mu \in (\lambda_j, \lambda_{j+1})$ **with** $j \neq k$.) We have:

$$\begin{aligned}\sum_{i=1}^{2k} |\mu - \lambda_i| &= \sum_{i=1}^{j} (\mu - \lambda_i) + \sum_{i=j+1}^{2k} (\lambda_i - \mu) \\ &= \mu [j - (2k - j)] + \sum_{i=j+1}^{2k} \lambda_i - \sum_{i=1}^{j} \lambda_i \\ &= 2 (j - k) \mu + \sum_{i=j+1}^{2k} \lambda_i - \sum_{i=1}^{j} \lambda_i .\end{aligned}$$

We will prove that:

$$2 (j - k) \mu + \sum_{i=j+1}^{2k} \lambda_i - \sum_{i=1}^{j} \lambda_i > \sum_{i=1}^{k} (\lambda_{2k+1-i} - \lambda_i) . \tag{8}$$

a) Suppose that $j < k$. Then the inequality (8) becomes:

$$\sum_{i=j+1}^{k} \lambda_i > (k - j) \mu$$

since $\mu > \lambda_j$ and the inequality (8) is proved.

b) The argument is similar for $j > k$ and we shall omit the details.

Now, by the inequality (8), we can state

$$T(x) > \sum_{i=1}^{k} (\lambda_{2k+1-i} - \lambda_i) \|a - b\| = T(x_0)$$

with $x_0 \in [a_k, a_{k+1}]$ which also contradicts the fact that $x$ is a Torricellian point.

(**Case 4:**) Suppose that there exists $j \neq k$, $k+1$ so that $\mu = \lambda_j$. We have:

$$\begin{aligned}\sum_{i=1}^{2k} |\mu - \lambda_i| &= \sum_{i=1}^{j} (\lambda_j - \lambda_i) + \sum_{i=j+1}^{2k} (\lambda_i - \lambda_j) \\ &= [j - (2k - j)] \lambda_j - \sum_{i=1}^{j} \lambda_i + \sum_{i=j+1}^{2k} \lambda_i .\end{aligned}$$

We want to prove the inequality

$$[j - (2k - j)] \lambda_j - \sum_{i=1}^{j} \lambda_i + \sum_{i=j+1}^{2k} \lambda_i > \sum_{i=1}^{k} (\lambda_{2k+1-i} - \lambda_i) . \tag{9}$$

The above inequality is equivalent with

$$2 (j - k) \lambda_j > - \sum_{i=j+1}^{2k} \lambda_i - \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} \lambda_{2k+1-i} + \sum_{i=1}^{j} \lambda_i .$$

a) Suppose that $j<k$. Then by the above inequality, we deduce

$$2(j-k)\lambda_j > -\sum_{i=j+1}^{k}\lambda_i - \sum_{i=j+1}^{k}\lambda_i = -2\sum_{i=j+1}^{k}\lambda_i,$$

which is equivalent with

$$(k-j)\lambda_j < \sum_{i=j+1}^{k}\lambda_i$$

which follows from the fact that $\lambda_i > \lambda_j$ for all $i=j+1,\dots k$.

b) The argument goes likewise, for $j>k+1$, and we shall omit the details.

Consequently, in all cases we can state

$$T(x) > T(x_0)$$

with $x_0 \in [a_k, a_{k+1}]$ which contradicts the fact that $x$ minimises the Torricellian map $T$ and thus $\mathcal{T}_X\{a_1,\dots,a_n\} \subset [a_k, a_{k+1}]$, and the implication is proved.

"(ii) $\Longrightarrow$ (i)". Suppose that $(X, \|\cdot\|)$ is not strictly convex, i.e., there exists two elements $x, y \in X$, $x \neq y$ such that $\|x+y\| = \|x\| + \|y\|$ and $x, y$ are linearly independent. Because of the linear independence, $x \neq -y$ and if we apply statement (ii) for $a_1 = x$ and $a_2 = -y$, we can write:

$$\mathcal{T}_X\{x,-y\} = [x,-y].$$

Now, $0 \notin [x,-y]$ as if we would assume that $0 \in [x,-y]$, then there exists $\lambda_0 \in [0,1]$ such that

$$\lambda_0 x + (1-\lambda_0)(-y) = 0$$

i.e., the system of vectors $\{x,y\}$ are linearly dependent, which contradicts the above assumption.

Since $\mathcal{T}_X\{x,-y\} = [x,-y]$ and $0 \notin [x,-y]$, then we can state:

$$T(0) > T(x_0), \tag{10}$$

where $x_0 \in [x,-y]$. However,

$$T(0) = \|x\| + \|y\|$$

and

$$T(x_0) = \|x - (\lambda_0 x + (1-\lambda_0)(-y))\| + \|-y - (\lambda_0 x + (1-\lambda_0)(-y))\|$$

$$= (1-\lambda_0)\|x+y\| + \lambda_0 \|x+y\| = \|x+y\|,$$

where $\lambda_0 \in [0,1]$.

Let us observe that by the inequality (10) we deduce $\|x\| + \|y\| > \|x+y\|$ which contradicts the hypothesis $\|x\| + \|y\| = \|x+y\|$, hence $(X, \|\cdot\|)$ is strictly convex. □

**Corollary 3.7.** *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and $\{a_1, \ldots, a_{2k}\}$ are $2k$ collinearly distinct points in $H$. Then*

$$\mathcal{T}_H \{a_1, \ldots, a_{2k}\} = [a_k, a_{k+1}].$$

We note that the above corollary also recaptures the similar result of Simons [2, p.61], as stated in Remark 3.5 for the case of collinear points in Euclidean spaces.

## 4. The unicity of Torricellian points for non-collinear Sets

In Section 2 of this paper we proved amongst others the fact that in every strictly convex space $(X, \|\cdot\|)$ the Torricellian map associated with a set $\mathcal{A} = \{a_1, \ldots, a_n\}$ of $n-$non-collinear distinct points $(n \geq 3)$ in $X$ is strictly convex in $X$. We can now prove the following theorem of unicity.

**Theorem 4.1.** *Let $(X, \|\cdot\|)$ be a normed linear space and $\mathcal{A} = \{a_1, \ldots, a_n\} \subset X$ $(n \geq 3)$ a set of non-collinear points in $X$. If the space is strictly convex, then $\mathcal{T}_X \{a_1, \ldots, a_n\}$ contains at most one element.*

*Proof.* Suppose that there exists two distinct elements $x_1, x_2 \in \mathcal{T}_X \{a_1, \ldots, a_n\}$. Then $T(x_1) = T(x_2) = \inf_{x \in X} T(x)$. Let $\lambda \in (0,1)$ and put $x_\lambda := \lambda x_1 + (1-\lambda) x_2$. As $T$ is a strictly convex mapping on $X$ (see Proposition 1.5) we have that:

$$T(x_\lambda) < \lambda T(x_1) + (1-\lambda) T(x_2) = \inf_{x \in X} T(x),$$

which produces a contradiction and the theorem is thus proved. □

**Remark 4.2.** The converse of this statement is not always true. Consider the set $\{(1,0),(0,1),(1,1)\} \subset (\mathbb{R}^2, \|\cdot\|_{\ell^1})$ which is non-collinear. Let $z = (a,b)$ be an arbitrary vector in $\mathbb{R}^2$, then the Torricellian functional is given by

$$\begin{aligned}
\inf_{(a,b)\in\mathbb{R}^2} T(a,b) &= \inf_{(a,b)\in\mathbb{R}^2} \|(a-1,b)\|_{\ell^1} + \|(a,b-1)\|_{\ell^1} + \|(a-1,b-1)\|_{\ell^1} \\
&= \inf_{(a,b)\in\mathbb{R}^2} \left[2\left(|a-1| + |b-1|\right) + |a| + |b|\right] \\
&= \inf_{(a,b)\in[0,1]^2} \left[2\left(|a-1| + |b-1|\right) + |a| + |b|\right] \\
&= \inf_{(a,b)\in[0,1]^2} \left[4 - a - b\right]
\end{aligned}$$

which minimum is attained at $(1,1)$. However, the space $(\mathbb{R}^2, \ell^1)$ is not strictly convex.

**Remark 4.3.** In Martini, Swanepoel and Weiss [5], the definition of collinear is metric dependent. Thus, Theorem 4.1 only generalises their result when the metric agrees with our definition of collinearity (cf. Definition 3.1).

The following corollary recaptures the result in Dragomir and Comănescu [7, Theorem 1].

**Corollary 4.4.** *Let $(X;(\cdot,\cdot))$ be an inner product space and $\mathcal{A} = \{a_1,\ldots,a_n\} \subset X$ $(n \geq 3)$ a set of non-collinear points in $X$. Then $\mathcal{T}_X\{a_1,\ldots,a_n\}$ contains a unique point.*

*Proof.* The existence follows by Theorem 2.5; while the unicity follows by Theorem 4.1, taking into account that every inner product space is a strictly convex space. □

## Conclusions

We consider the Torricelli problem in the settings of normed linear spaces. Our results show that the Torricelli point(s) of a set $\mathcal{A}$ always exists in reflexive spaces, also when the set $\mathcal{A}$ spans a non-expansive linear subspace of a normed space. This implies that the Torricelli point of a set $\mathcal{A}$ always exists in inner product spaces.

Furthermore, in a normed space, if all points in the set $\mathcal{A} = \{a_1,\ldots,a_{2k+1}\}$ lie in a single straight line (collinear), then the Torricelli set is the singleton $\{a_{k+1}\}$. When all the points in the set $\mathcal{A} = \{a_1,\ldots,a_{2k}\}$ are collinear, the Torricelli set contains the segment $[a_k, a_{k+1}]$. This generalises the results of Simons [2].

This result can be used to characterise strictly convex spaces. When the Torricelli set of the collinear set $\{a_1, \dots, a_{2k}\}$ is exactly the segment $[a_k, a_{k+1}]$, then the space must be an strictly convex space and vice versa. This implies that when the space is an inner product space, the Torricelli set of the collinear set $\{a_1, \dots, a_{2k}\}$ is the segment $[a_k, a_{k+1}]$.

Finally we investigate the uniqueness of the Torricelli points for non-collinear sets. We show that the Torricelli point is unique when the space is strictly convex, which generalises the results of Martini, Swanepoel and Weiss [5]. The converse is not always true (cf. Remarks 4.2 and 4.3). This implies that the Torricellian point is unique in inner product spaces.

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