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SOME INEQUALITIES FOR POWER SERIES OF TWO OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities for functions defined by power series concerning two operators in both the noncommutative and commutative case are given. Natural examples for fundamental functions that can be represented by power series are presented as well.

1. Introduction

For power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ with complex coefficients we can naturally construct another power series which have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series have the same radius of convergence as the original series, and that if all coefficients $a_n\geq 0$, then $f_a=f$.

With this notation S.S. Dragomir [4] (also see [5]) showed the following:

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset C$, $R>0$. Let $T\in B(H)$, $\alpha,\beta\geq 0$ with $\alpha+\beta\geq 1$ and such that

$$\|T\|^{2\alpha},\|T\|^{2\beta}<R.$$

Then

$$\left|\left\langle Tf\left(|T|^{\alpha+\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq\left\langle f_a\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle f_a\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle$$

for any $x,y\in H$.

This is an extension of the following inequality for a bounded linear operator $T\in B(H)$ by Furuta [7] (also see [8]):

$$\left|\left\langle T|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2\leq\left\langle |T|^{2\alpha}x,x\right\rangle\left\langle |T^*|^{2\beta}y,y\right\rangle,\ x,y\in H.$$

Motivated by this result for one operator, we investigate in the current paper some inequalities for functions defined by power series concerning two operators in both the noncommutative and commutative case. In particular, for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ we show that

$$\|f(AB)\|\leq f_a^{1/p}(\|A\|^p)\,f_a^{1/q}(\|B\|^q).$$

Moreover we prove this inequality is also valid for every unitarily invariant norm.

The following is one among some examples given in this paper:

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If $\|A\|^p, \|B\|^q < 1$, then

$$\left\|(1_H \pm AB)^{-1}\right\| \le (1 - \|A\|^p)^{-1/p} (1 - \|B\|^q)^{-1/q}.$$

2. Some General Norm Inequalities

The following result concerning norm inequalities for two bounded operators may be stated:

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(\neq 0)$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If A and B are two bounded operators on the Hilbert space H and for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$*

$$\|A\|^p, \|B\|^q < R, \tag{2.1}$$

then

$$\|f(AB)\| \le \min\{K_1(p,q), K_2(p,q)\} \tag{2.2}$$

where

$$K_1(p,q) := f_a^{1/p}(\|A\|^p) f_a^{1/q}(\|B\|^q), \tag{2.3}$$

and

$$K_2(p,q) := \frac{f_a(\|A\|^p) f_a(\|B\|^q)}{f_a\left(\|A\|^{p-1} \|B\|^{q-1}\right)}. \tag{2.4}$$

Proof. By the properties of operator norm, observe that, for any $j \in \mathbb{N}$ we have

$$\left\|(AB)^j\right\| \le \|A\|^j \|B\|^j.$$

If we multiply with $|a_j|$ and use the generalized triangle inequality we have

$$\left\|\sum_{j=0}^{n} a_j (AB)^j\right\| \le \sum_{j=0}^{n} |a_j| \|A\|^j \|B\|^j \tag{2.5}$$

for any $n \in \mathbb{N}$.

Now, by Hölder's inequality we have

$$\sum_{j=0}^{n} |a_j| \|A\|^j \|B\|^j \le \left(\sum_{j=0}^{n} |a_j| \|A\|^{jp}\right)^{1/p} \left(\sum_{j=0}^{n} |a_j| \|B\|^{jq}\right)^{1/q} \tag{2.6}$$

for any $n \in \mathbb{N}$, and by (2.5) we get

$$\left\|\sum_{j=0}^{n} a_j (AB)^j\right\| \le \left(\sum_{j=0}^{n} |a_j| \|A\|^{jp}\right)^{1/p} \left(\sum_{j=0}^{n} |a_j| \|B\|^{jq}\right)^{1/q}. \tag{2.7}$$

Since the series whose partial sums are involved in (2.7) are convergent, then by taking $n \to \infty$ in (2.7) we deduce the first inequality in (2.3).

Further, by utilizing the following Hölder's type inequality obtained by Dragomir and Sándor in 1990 [6] (see also [2, Corollary 2.34]):

$$\sum_{k=0}^{n} m_k |x_k|^p \sum_{k=0}^{n} m_k |y_k|^q \ge \sum_{k=0}^{n} m_k |x_k y_k| \sum_{k=0}^{n} m_k |x_k|^{p-1} |y_k|^{q-1}, \tag{2.8}$$

that holds for nonnegative numbers m_k and complex numbers x_k, y_k where $k \in \{0, ..., n\}$, we observe that the convergence of the series $\sum_{k=0}^{\infty} m_k |x_k|^p$ and $\sum_{k=0}^{\infty} m_k |y_k|^q$ imply the convergence of the series $\sum_{k=0}^{\infty} m_k |x_k|^{p-1} |y_k|^{q-1}$.

Utilising (2.8) we then have

$$\sum_{j=0}^{n} |a_j| \|A\|^j \|B\|^j \leq \frac{\sum_{j=0}^{n} |a_j| \|A\|^{jp} \sum_{j=0}^{n} |a_j| \|B\|^{jq}}{\sum_{j=0}^{n} |a_j| \|A\|^{j(p-1)} \|B\|^{j(q-1)}}$$

which together with (2.5) gives

$$\left\| \sum_{j=0}^{n} a_j (AB)^j \right\| \leq \frac{\sum_{j=0}^{n} |a_j| \|A\|^{jp} \sum_{j=0}^{n} |a_j| \|B\|^{jp}}{\sum_{j=0}^{n} |a_j| \|A\|^{j(p-1)} \|B\|^{j(q-1)}}, \tag{2.9}$$

for any $n \in \mathbb{N}$.

Since all the series whose partial sums are involved in (2.9) are convergent, then by taking $n \to \infty$ in (2.9) we deduce the second inequality in (2.2). □

Remark 1. *The case $p = q = 2$ produces the Schwarz's type inequality*

$$\|f(AB)\|^2 \leq f_a\left(\|A\|^2\right) f_a\left(\|B\|^2\right), \tag{2.10}$$

provided $\|A\|^2, \|B\|^2 < R$.

The finite-dimensional case is as follows:

Theorem 2. *Theorem 1 also holds for every unitarily invariant norm $|||\cdot|||$ on a finite matrix algebra. Moreover, we have the inequalities*

$$|||f(AB)||| \leq \min\{L_1(p,q), L_2(p,q)\} \tag{2.11}$$

where

$$L_1(p,q) := f_a^{1/p}(|||\,|A|^p|||)\, f_a^{1/q}(|||\,|B|^q|||) \tag{2.12}$$

and

$$L_2(p,q) := \frac{f_a(|||\,|A|^p|||)\, f_a(|||\,|B|^q|||)}{f_a\left(|||\,|A|^p|||^{1/q}\, |||\,|B|^q|||^{1/p}\right)}, \tag{2.13}$$

provided $|||\,|A|^p\,|||, |||\,|B|^q\,||| < R$.

Proof. Since $|||AB||| \leq |||A||| \cdot |||B|||$ and $|||AB||| \leq |||\,|A|^p\,|||^{1/p} \cdot |||\,|B|^q\,|||^{1/q}$ where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ (see for instance [1, p. 95]), we have by the Hölder inequality that

$$\begin{aligned} \left|\left|\left| \sum_{j=0}^{n} a_j (AB)^j \right|\right|\right| &\leq \sum_{j=0}^{n} |a_j|\, |||AB|||^j \leq \sum_{j=0}^{n} |a_j|\, |||\,|A|^p\,|||^{j/p} \cdot |||\,|B|^q\,|||^{j/q} \\ &\leq \left(\sum_{j=0}^{n} |a_j|\, |||\,|A|^p\,|||^j\right)^{1/p} \left(\sum_{j=0}^{n} |a_j|\, |||\,|B|^q\,|||^j\right)^{1/q} \end{aligned} \tag{2.14}$$

for any for any $n \in \mathbb{N}$.

Since all the series whose partial sums are involved in (2.14) are convergent, then by taking $n \to \infty$ in (2.14) we deduce the first inequality in (2.11).

Utilising the inequality (2.8) we also have

$$\begin{aligned}(2.15)\quad &\sum_{j=0}^{n}|a_j|\,|||\,|A|^p\,|||^{j/p}\cdot|||\,|B|^q\,|||^{j/q}\\ &\leq\frac{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p\,|||^{j}\sum_{j=0}^{n}|a_j|\,|||\,|B|^q\,|||^{j}}{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p\,|||^{j\frac{p-1}{p}}\cdot|||\,|B|^q\,|||^{j\frac{q-1}{q}}}\\ &=\frac{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p\,|||^{j}\sum_{j=0}^{n}|a_j|\,|||\,|B|^q\,|||^{j}}{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p\,|||^{\frac{j}{q}}\cdot|||\,|B|^q\,|||^{\frac{j}{p}}}\end{aligned}$$

for any $n\in\mathbb{N}$.

Since all the series whose partial sums are involved in (2.15) are convergent, then by taking $n\to\infty$ in (2.15) we deduce the first inequality in (2.11). □

Remark 2. *The case $p=q=2$ produces the Schwarz's type inequality*

$$|||f(AB)|||^2\leq f_a\left(\left|\left|\left||A|^2\right|\right|\right|\right)f_a\left(\left|\left|\left||B|^2\right|\right|\right|\right),$$

provided $\left|\left|\left||A|^2\right|\right|\right|,\left|\left|\left||B|^2\right|\right|\right|<R.$

A refinement of the inequality (2.10) may be found in the following theorem:

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ $(\neq 0)$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If A and B are two bounded operators on the Hilbert space H and*

$$\|A\|^2,\|B\|^2<R, \tag{2.16}$$

then

$$\begin{aligned}(2.17)\quad \|f(AB)\|^2&\leq f_a\left(\|A\|^{1+\alpha}\|B\|^{1-\alpha}\right)f_a\left(\|A\|^{1-\alpha}\|B\|^{1+\alpha}\right)\\ &\leq f_a\left(\|A\|^2\right)f_a\left(\|B\|^2\right),\end{aligned}$$

where $\alpha\in[0,1]$.

If $\sum_{n=0}^{\infty}|a_n|<\infty$ and in addition to the condition (2.16) we have $\|A\|,\|B\|<R$, then

$$\|f(AB)\|\leq f_a(1)\cdot\frac{f_a\left(\|A\|^2\right)f_a\left(\|B\|^2\right)}{f_a(\|A\|)f_a(\|B\|)}. \tag{2.18}$$

Proof. We utilize the *Callebaut inequality* (see for instance [2, Remark 3.31])

$$\left(\sum_{j=1}^{n}p_ja_jb_j\right)^2\leq\sum_{j=1}^{n}p_ja_j^{1+\alpha}b_j^{1-\alpha}\sum_{j=1}^{n}p_ja_j^{1-\alpha}b_j^{1+\alpha}\leq\sum_{j=1}^{n}p_ja_j^2\sum_{j=1}^{n}p_jb_j^2$$

that holds for $\alpha\in[0,1]$ and the nonnegative numbers a_j,b_j,p_j with $j\in\{1,...,n\}$.

Therefore

$$\left(\sum_{j=0}^{n}|a_j|\,\|A\|^{j}\,\|B\|^{j}\right)^2 \leq \sum_{j=0}^{n}|a_j|\,\|A\|^{(1+\alpha)j}\,\|B\|^{(1-\alpha)j}\sum_{j=0}^{n}|a_j|\,\|A\|^{(1-\alpha)j}\,\|B\|^{(1+\alpha)j}$$
$$\leq \sum_{j=0}^{n}|a_j|\,\|A\|^{2j}\sum_{j=0}^{n}|a_j|\,\|B\|^{2j}$$

and by (2.5) we get

$$\left\|\sum_{j=0}^{n}a_j\,(AB)^j\right\|^2 \leq \sum_{j=0}^{n}|a_j|\,\|A\|^{(1+\alpha)j}\,\|B\|^{(1-\alpha)j}\sum_{j=0}^{n}|a_j|\,\|A\|^{(1-\alpha)j}\,\|B\|^{(1+\alpha)j} \leq \sum_{j=0}^{n}|a_j|\,\|A\|^{2j}\sum_{j=0}^{n}|a_j|\,\|B\|^{2j}, \tag{2.19}$$

for any $n\in\mathbb{N}$.

Since all the series whose partial sums are involved in (2.19) are convergent, then by taking $n\to\infty$ in (2.19) we deduce the inequality (2.17).

For the second part, we use the following inequality obtained by S.S. Dragomir in 1984 [3] (see also [2, Theorem 2.20]):

$$\frac{\sum_{j=1}^{n}p_j a_j b_j\sum_{j=1}^{n}p_j a_j\sum_{j=1}^{n}p_j b_j}{\sum_{j=1}^{n}p_j}\leq\sum_{j=1}^{n}p_j a_j^2\sum_{j=1}^{n}p_j b_j^2$$

that holds for the nonnegative numbers a_j, b_j, p_j with $j\in\{1,...,n\}$ and $\sum_{j=1}^{n}p_j>0$.

Utilising this inequality, we have

$$\sum_{j=0}^{n}|a_j|\,\|A\|^{j}\,\|B\|^{j}\leq\sum_{j=0}^{n}|a_j|\cdot\frac{\sum_{j=0}^{n}|a_j|\,\|A\|^{2j}\sum_{j=0}^{n}|a_j|\,\|B\|^{2j}}{\sum_{j=0}^{n}|a_j|\,\|A\|^{2}\sum_{j=0}^{n}|a_j|\,\|B\|^{2}}$$

which together with (2.5) produces

$$\left\|\sum_{j=0}^{n}a_j\,(AB)^j\right\|\leq\sum_{j=0}^{n}|a_j|\cdot\frac{\sum_{j=0}^{n}|a_j|\,\|A\|^{2j}\sum_{j=0}^{n}|a_j|\,\|B\|^{2j}}{\sum_{j=0}^{n}|a_j|\,\|A\|^{2}\sum_{j=0}^{n}|a_j|\,\|B\|^{2}}. \tag{2.20}$$

Since all the series whose partial sums are involved in (2.20) are convergent, then by taking $n\to\infty$ in (2.20) we deduce the inequality (2.18). □

Remark 3. *The condition $f_a(1)<\infty$ can be avoided if a complex parameter $|z|<R$ is introduced. Namely, we can obtain the following generalization of (2.18)*

$$\|f(zAB)\|\leq f_a(|z|)\cdot\frac{f_a\left(|z|\,\|A\|^2\right)f_a\left(|z|\,\|B\|^2\right)}{f_a(|z|\,\|A\|)\,f_a(|z|\,\|B\|)}, \tag{2.21}$$

provided $|z|\,\|A\|^2, |z|\,\|B\|^2, |z|\,\|A\|, |z|\,\|B\|<R$.

The finite-dimensional version of Theorem 3 is as follows:

Theorem 4. *Theorem 3 also holds for every unitarily invariant norm $|||\cdot|||$ on a finite matrix algebra. Moreover, we have the inequalities*

$$\begin{aligned}(2.22)\qquad & |||f(AB)|||^2 \\ & \leq f_a\left(\left|\left|\left||A|^2\right|\right|\right|^{\frac{1+\alpha}{2}}\left|\left|\left||B|^2\right|\right|\right|^{\frac{1-\alpha}{2}}\right) f_a\left(\left|\left|\left||A|^2\right|\right|\right|^{\frac{1-\alpha}{2}}\left|\left|\left||B|^2\right|\right|\right|^{\frac{1+\alpha}{2}}\right) \\ & \leq f_a\left(|\left|\left||A|^2\right|\right|\right) f_a\left(\left|\left|\left||B|^2\right|\right|\right|\right),\end{aligned}$$

provided

$$\left|\left|\left|\,|A|^2\,\right|\right|\right|, \left|\left|\left||B|^2\right|\right|\right| < R, \tag{2.23}$$

where $\alpha \in [0,1]$.

If $\sum_{n=0}^{\infty}|a_n| < \infty$ and in addition to the condition (2.23) we have

$$\left|\left|\left||A|^2\right|\right|\right|^{1/2}, \left|\left|\left||B|^2\right|\right|\right|^{1/2} < R,$$

then

$$|||f(AB)||| \leq f_a(1)\cdot\frac{f_a\left(\left|\left|\left||A|^2\right|\right|\right|\right) f_a\left(\left|\left|\left||B|^2\right|\right|\right|\right)}{f_a\left(\left|\left|\left||A|^2\right|\right|\right|^{1/2}\right) f_a\left(\left|\left|\left||B|^2\right|\right|\right|^{1/2}\right)}. \tag{2.24}$$

The details of the proof are left to the reader.

3. Some Vector Inequalities for Normal Operators

The case of normal operators is as follows:

Theorem 5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If A and B are two commuting normal operators on the Hilbert space H, $z \in \mathbb{C}$ and*

$$|z|\,\|A\|^2, |z|\,\|B\|^2 < R, \tag{3.1}$$

then we have

$$|\langle f(zAB)x, y\rangle|^2 \leq \left\langle f_a\left(|z|\,|A|^2\right)x, x\right\rangle \left\langle f_a\left(|z|\,|B|^2\right)y, y\right\rangle \tag{3.2}$$

for any $x, y \in H$.

Proof. By utilizing Schwarz inequality we have for any $x, y \in H$ that

$$\left|\left\langle A^j x, (B^*)^j y\right\rangle\right|^2 \leq \left\langle A^j x, A^j x\right\rangle \left\langle (B^*)^j y, (B^*)^j y\right\rangle$$

for any $j \in \mathbb{N}$, which in operator modulus notations is equivalent with

$$\left|\left\langle B^j A^j x, y\right\rangle\right|^2 \leq \left\langle \left|A^j\right|^2 x, x\right\rangle \left\langle \left|(B^*)^j\right|^2 y, y\right\rangle. \tag{3.3}$$

Since A and B are normal operators, then

$$\left|A^j\right|^2 = |A|^{2j} \text{ and } \left|(B^*)^j\right|^2 = |B|^{2j}$$

for any $j \in \mathbb{N}$.

By the commutativity of A with B we also have

$$B^j A^j = (AB)^j$$

for any $j \in \mathbb{N}$ and then by (3.3) we have

$$\left|\left\langle (AB)^j x, y\right\rangle\right|^2 \le \left\langle |A|^{2j} x, x\right\rangle \left\langle |B|^{2j} y, y\right\rangle \tag{3.4}$$

for any $x, y \in H$ and for any $j \in \mathbb{N}$.

If we multiply the inequality (3.3) with $|a_j| |z|^j$, sum over j from 0 to m and use the generalized triangle inequality and the Cauchy-Bunyakovsky-Schwarz weighted inequality, we have successively

$$\begin{aligned}
&\left|\left\langle \sum_{j=0}^{m} a_j z^j (AB)^j x, y\right\rangle\right| \\
&\le \sum_{j=0}^{m} |a_j| |z|^j \left|\left\langle (AB)^j x, y\right\rangle\right| \\
&\le \sum_{j=0}^{m} |a_j| |z|^j \left\langle |A|^{2j} x, x\right\rangle^{1/2} \left\langle |B|^{2j} y, y\right\rangle^{1/2} \\
&\le \left(\sum_{j=0}^{m} |a_j| |z|^j \left\langle |A|^{2j} x, x\right\rangle\right)^{1/2} \left(\sum_{j=0}^{m} |a_j| |z|^j \left\langle |B|^{2j} y, y\right\rangle\right)^{1/2} \\
&= \left\langle \sum_{j=0}^{m} |a_j| |z|^j |A|^{2j} x, x\right\rangle^{1/2} \left\langle \sum_{j=0}^{m} |a_j| |z|^j |B|^{2j} y, y\right\rangle^{1/2}
\end{aligned} \tag{3.5}$$

for any $x, y \in H$ and for any $m \in \mathbb{N}$.

Since the series $\sum_{j=0}^{\infty} |a_j| |z|^j |A|^{2j}$, $\sum_{j=0}^{\infty} |a_j| |z|^j |B|^{2j}$ and $\sum_{j=0}^{\infty} a_j z^j (AB)^j$ are convergent, then by taking the limit over $m \to \infty$ in (3.5) we deduce the desired result (3.2). □

Corollary 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If A and B are two commuting normal operators on the Hilbert space H satisfying the condition (3.1) then we have the norm inequality*

$$\|f(zAB)\|^2 \le \left\|f_a\left(|z| |A|^2\right)\right\| \left\|f_a\left(|z| |B|^2\right)\right\| \tag{3.6}$$

and the numerical radius inequality

$$w[f(zAB)] \le \frac{1}{2}\left\|f_a\left(|z| |A|^2\right) + f_a\left(|z| |B|^2\right)\right\|. \tag{3.7}$$

Proof. From (3.2) we also have the inequalities

$$\begin{aligned}
|\langle f(zAB) x, x\rangle| &\le \left\langle f_a\left(|z| |A|^2\right) x, x\right\rangle^{1/2} \left\langle f_a\left(|z| |B|^2\right) x, x\right\rangle^{1/2} \\
&\le \frac{1}{2}\left\langle \left[f_a\left(|z| |A|^2\right) + f_a\left(|z| |B|^2\right)\right] x, x\right\rangle
\end{aligned}$$

for any $x \in H$, which, by taking the supremum over $\|x\| = 1$, produces the desired result (3.7). □

Remark 4. *If A is a normal operator and $z \in \mathbb{C}$ with $|z| \|A\|^2, |z| < R$, then by taking $B = 1_H$ in (3.2) we get*

$$|\langle f(zA)x, y\rangle|^2 \le f_a(|z|)\left\langle f_a\left(|z||A|^2\right)x, x\right\rangle \|y\|^2 \tag{3.8}$$

for any $x, y \in H$.

If A is a normal operator and $z \in \mathbb{C}$ with $|z| \|A\|^2, |z| < R$, then by taking $B = A$ in (3.2) we get

$$\left|\left\langle f\left(zA^2\right)x, y\right\rangle\right|^2 \le \left\langle f_a\left(|z||A|^2\right)x, x\right\rangle \left\langle f_a\left(|z||A|^2\right)y, y\right\rangle \tag{3.9}$$

and by taking $B = A^*$ in (3.2) we also get

$$\left|\left\langle f\left(z|A|^2\right)x, y\right\rangle\right|^2 \le \left\langle f_a\left(|z||A|^2\right)x, x\right\rangle \left\langle f_a\left(|z||A|^2\right)y, y\right\rangle \tag{3.10}$$

for any $x, y \in H$.

Moreover, if U and V are two commuting unitary operators, then by taking $A = U$ and $B = V$ in (3.2) we get

$$|\langle f(zUV)x, y\rangle| \le f_a(|z|)\|x\|\|y\| \tag{3.11}$$

for any $x, y \in H$ and $z \in \mathbb{C}$ with $|z| < R$.

The following result for two power series can be stated as well:

Theorem 6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and be $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two functions defined by power series with complex coefficients and both of them convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If A and B are two normal operators on the Hilbert space H, $z, u \in \mathbb{C}$ and*

$$|z|\|A\|, |u|\|B\| \le R \tag{3.12}$$

then we have

$$\begin{aligned} &|\langle f(zA)x, g(uB)y\rangle|^2 \\ &\le f_a\left(|z|^2\right) g_a\left(|u|^2\right) \left\langle f_a\left(|A|^2\right)x, x\right\rangle \left\langle g_a\left(|B|^2\right)y, y\right\rangle \end{aligned} \tag{3.13}$$

for any $x, y \in H$.

Proof. By Schwarz's inequality we also have the following inequality for normal operators

$$\left|\left\langle A^j x, B^k y\right\rangle\right| \le \left\langle |A|^{2j} x, x\right\rangle^{1/2} \left\langle |B|^{2k} y, y\right\rangle^{1/2} \tag{3.14}$$

for any $x, y \in H$ and $j, k \in \mathbb{N}$.

If we multiply (3.14) with $|a_j|\,|z|^j\,|b_k|\,|u|^k$, sum over j and k from 0 to m and use the generalized triangle inequality, then we have successively

$$\begin{aligned}(3.15)\qquad &\left|\left\langle \sum_{j=0}^{m} a_j z^j A^j x, \sum_{k=0}^{m} b_k u^k B^k y\right\rangle\right| \\ &\le \sum_{j=0}^{m}\sum_{k=0}^{m} |a_j|\,|z|^j\,|b_k|\,|u|^k\left|\left\langle A^j x, B^k y\right\rangle\right| \\ &\le \sum_{j=0}^{m}\sum_{k=0}^{m} |a_j|\,|z|^j\,|b_k|\,|u|^k\left\langle |A|^{2j}x,x\right\rangle^{1/2}\left\langle |B|^{2k}y,y\right\rangle^{1/2} \\ &= \sum_{j=0}^{m} |a_j|\,|z|^j\left\langle |A|^{2j}x,x\right\rangle^{1/2}\sum_{k=0}^{m}|b_k|\,|u|^k\left\langle |B|^{2k}y,y\right\rangle^{1/2}\end{aligned}$$

for any $x, y \in H$ and $m \in \mathbb{N}$.

Further, by the Cauchy-Bunyakovsky-Schwarz inequality we also have

$$\sum_{j=0}^{m} |a_j|\,|z|^j\left\langle |A|^{2j}x,x\right\rangle^{1/2} \le \left(\sum_{j=0}^{m}|a_j|\,|z|^{2j}\right)^{1/2}\left\langle \sum_{j=0}^{m}|a_j|\,|A|^{2j}x,x\right\rangle^{1/2}$$

and

$$\sum_{k=0}^{m} |b_k|\,|u|^k\left\langle |B|^{2k}y,y\right\rangle^{1/2} \le \left(\sum_{k=0}^{m}|b_k|\,|u|^{2k}\right)^{1/2}\left\langle \sum_{k=0}^{m}|b_k|\,|B|^{2k}y,y\right\rangle^{1/2}$$

for any $x, y \in H$ and $m \in \mathbb{N}$, which together with (3.15) provide

$$\begin{aligned}(3.16)\qquad &\left|\left\langle \sum_{j=0}^{m} a_j z^j A^j x, \sum_{k=0}^{m} b_k u^k B^k y\right\rangle\right| \\ &\le \left(\sum_{j=0}^{m}|a_j|\,|z|^{2j}\right)^{1/2}\left\langle \sum_{j=0}^{m}|a_j|\,|z|^j\,|A|^{2j}x,x\right\rangle^{1/2} \\ &\times \left(\sum_{k=0}^{m}|b_k|\,|u|^{2k}\right)^{1/2}\left\langle \sum_{k=0}^{m}|b_k|\,|u|^k\,|B|^{2k}y,y\right\rangle^{1/2}\end{aligned}$$

for any $x, y \in H$ and $m \in \mathbb{N}$.

Since the series whose partial sums are involved in the inequality (3.16) are convergent, then taking the limit over $m \to \infty$ in (3.16) we deduce the desired result (3.13). □

Corollary 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and be $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two functions defined by power series with real coefficients and both of them convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If A and B are two normal operators on the Hilbert space H that satisfy condition (3.12) then we have*

$$(3.17)\qquad \left\|g\left(\overline{u}B^*\right)f(zA)\right\|^2 \le f_a\left(|z|^2\right)g_a\left(|u|^2\right)\left\|f_a\left(|A|^2\right)\right\|\left\|g_a\left(|B|^2\right)\right\|$$

and

$$w\left(g\left(\overline{u}B^{*}\right)f\left(zA\right)\right)\leq\frac{1}{2}f_{a}\left(\left|z\right|^{2}\right)g_{a}\left(\left|u\right|^{2}\right)\left\|f_{a}\left(\left|A\right|^{2}\right)+g_{a}\left(\left|B\right|^{2}\right)\right\|. \tag{3.18}$$

4. Some Examples

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f\left(z\right)&=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}}{n}z^{n}=\ln\frac{1}{1+z},\ \ z\in D\left(0,1\right); \\ g\left(z\right)&=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(2n\right)!}z^{2n}=\cos z,\ \ z\in\mathbb{C}; \\ h\left(z\right)&=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(2n+1\right)!}z^{2n+1}=\sin z,\ \ z\in\mathbb{C}; \\ l\left(z\right)&=\sum_{n=0}^{\infty}\left(-1\right)^{n}z^{n}=\frac{1}{1+z},\ \ z\in D\left(0,1\right); \end{aligned} \tag{4.1}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_{a}\left(z\right)&=\sum_{n=1}^{\infty}\frac{1}{n}z^{n}=\ln\frac{1}{1-z},\ \ z\in D\left(0,1\right); \\ g_{a}\left(z\right)&=\sum_{n=0}^{\infty}\frac{1}{\left(2n\right)!}z^{2n}=\cosh z,\ \ z\in\mathbb{C}; \\ h_{A}\left(z\right)&=\sum_{n=0}^{\infty}\frac{1}{\left(2n+1\right)!}z^{2n+1}=\sinh z,\ \ z\in\mathbb{C}; \\ l_{A}\left(z\right)&=\sum_{n=0}^{\infty}z^{n}=\frac{1}{1-z},\ \ z\in D\left(0,1\right). \end{aligned} \tag{4.2}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\begin{aligned} \exp\left(z\right)&=\sum_{n=0}^{\infty}\frac{1}{n!}z^{n}\qquad z\in\mathbb{C}, \\ \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)&=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\qquad z\in D\left(0,1\right); \\ \sin^{-1}\left(z\right)&=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\left(2n+1\right)n!}z^{2n+1},\qquad z\in D\left(0,1\right); \\ \tanh^{-1}\left(z\right)&=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\qquad z\in D\left(0,1\right) \\ {}_{2}F_{1}\left(\alpha,\beta,\gamma,z\right)&=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\alpha\right)\Gamma\left(n+\beta\right)\Gamma\left(\gamma\right)}{n!\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(n+\gamma\right)}z^{n},\alpha,\beta,\gamma>0, \\ &z\in D\left(0,1\right); \end{aligned} \tag{4.3}$$

where Γ is Gamma function.

On making use of Theorem 1, we can state some particular examples as follows:

Example 1. *a) If A and B are two bounded operators on the Hilbert space H and for $p>1,\frac{1}{p}+\frac{1}{q}=1,\ \|A\|^p,\|B\|^q<1$, then*

$$\left\|\left(1_H\pm AB\right)^{-1}\right\|\leq\min\left\{S_1(p,q),S_2(p,q)\right\} \tag{4.4}$$

where

$$S_1(p,q):=\left(1-\|A\|^p\right)^{-1/p}\left(1-\|B\|^q\right)^{-1/q},$$

and

$$S_2(p,q):=\frac{1-\|A\|^{p-1}\|B\|^{q-1}}{\left(1-\|A\|^p\right)\left(1-\|B\|^q\right)}.$$

We also have the following inequality for the logarithm

$$\left\|\ln\left(1_H\pm AB\right)^{-1}\right\|\leq\min\left\{T_1(p,q),T_2(p,q)\right\} \tag{4.5}$$

where

$$T_1(p,q):=\left[\ln\left(1-\|A\|^p\right)^{-1}\right]^{1/p}\left[\ln\left(1-\|B\|^q\right)^{-1}\right]^{1/q}$$

and

$$T_2(p,q):=\frac{\left(1-\|A\|^{p-1}\|B\|^{q-1}\right)}{\left(1-\|A\|^p\right)\left(1-\|B\|^q\right)},$$

$$T_2(p,q):=\frac{\left[\ln\left(1-\|A\|^p\right)^{-1}\right]\left[\ln\left(1-\|B\|^q\right)^{-1}\right]}{\ln\left(1-\|A\|^{p-1}\|B\|^{q-1}\right)^{-1}}.$$

b) If A and B are two bounded operators on the Hilbert space H and $p>1,\frac{1}{p}+\frac{1}{q}=1$, then

$$\left\|\exp\left(AB\right)\right\|\leq\min\left\{U_1(p,q),U_2(p,q)\right\} \tag{4.6}$$

where

$$U_1(p,q):=\exp\left(\frac{1}{p}\|A\|^p+\frac{1}{q}\|B\|^q\right),$$

and

$$U_2(p,q):=\exp\left(\|A\|^p+\|B\|^q-\|A\|^{p-1}\|B\|^{q-1}\right).$$

Theorem 2 provides the following results for unitarily invariant norm $|||\cdot|||$ on a finite matrix algebra.

Example 2. *a) Let $|||\cdot|||$ be a unitarily invariant norm on a finite matrix algebra. If $|||\,|A|^p\,|||,|||\,|B|^q\,|||<1$, where $p>1,\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\left|\left|\left(I\pm AB\right)^{-1}\right|\right|\right|\leq\min\left\{V_1(p,q),V_2(p,q)\right\} \tag{4.7}$$

where

$$V_1(p,q):=\left(1-|\,|||A|^p\|\,|\right)^{-1/p}\left(1-|||B|^q\|\,|\right)^{-1/q},$$

and

$$V_2(p,q):=\frac{1-|\,|||A|^p\|\,|^{1/q}\,|||B|^q\|\,|^{1/p}}{\left(1-|\,|||A|^p\|\,|\right)\left(1-|||B|^q\|\,|\right)},$$

and

$$\left|\left|\left|\ln\left(I\pm AB\right)^{-1}\right|\right|\right|\leq\min\left\{W_1\left(p,q\right),W_2\left(p,q\right)\right\} \tag{4.8}$$

where

$$W_1\left(p,q\right):=\left[\ln\left(1-|\,\||A|^p\|\,|\right)^{-1}\right]^{1/p}\left[\ln\left(1-\||B|^q\|\,|\right)^{-1}\right]^{1/q},$$

and

$$W_2\left(p,q\right):=\frac{\ln\left(1-|\,\||A|^p\|\,|\right)^{-1}\ln\left(1-\||B|^q\|\,|\right)^{-1}}{\ln\left(1-|\,\||A|^p\|\,|^{1/q}\,\||B|^q\|\,|^{1/p}\right)^{-1}}.$$

b) For any two matrices we have

$$|||\exp\left(AB\right)|||\leq\min\left\{Z_1\left(p,q\right),Z_2\left(p,q\right)\right\} \tag{4.9}$$

where

$$Z_1\left(p,q\right):=\exp\left(\frac{1}{p}\,\||A|^p\|\,|+\frac{1}{q}\,\||B|^q\|\,|\right),$$

and

$$Z_2\left(p,q\right):=\exp\left(\||A|^p\|\,|+\||B|^q\|\,|-\||A|^p\|\,|^{1/q}\,\||B|^q\|\,|^{1/p}\right).$$

Employing the inequalities from Theorem 3 and Remark 3 we can state:

Example 3. *a) If A and B are two bounded operators on the Hilbert space H and*

$$\|A\|,\|B\|<1,$$

then

$$\begin{aligned}\left\|\left(1_H\pm AB\right)^{-1}\right\|^2&\leq\left(1-\|A\|^{1+\alpha}\|B\|^{1-\alpha}\right)^{-1}\left(1-\|A\|^{1-\alpha}\|B\|^{1+\alpha}\right)^{-1}\\&\leq\left(1-\|A\|^2\right)^{-1}\left(1-\|B\|^2\right)^{-1},\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\left\|\ln\left(1_H\pm AB\right)^{-1}\right\|^2\\&\leq\ln\left(1-\|A\|^{1+\alpha}\|B\|^{1-\alpha}\right)^{-1}\ln\left(1-\|A\|^{1-\alpha}\|B\|^{1+\alpha}\right)^{-1}\\&\leq\ln\left(1-\|A\|^2\right)^{-1}\ln\left(1-\|B\|^2\right)^{-1},\end{aligned} \tag{4.11}$$

where $\alpha\in[0,1]$.

If $|z|<1$, we also have

$$\left\|\left(1_H\pm zAB\right)^{-1}\right\|\leq\left(1-|z|\right)^{-1}\cdot\frac{\left(1-|z|\,\|A\|^2\right)^{-1}f_a\left(1-|z|\,\|B\|^2\right)^{-1}}{\left(1-|z|\,\|A\|\right)^{-1}f_a\left(1-|z|\,\|B\|\right)^{-1}} \tag{4.12}$$

and

$$\begin{aligned}&\left\|\ln\left(1_H\pm zAB\right)^{-1}\right\|\\&\leq\ln\left(1-|z|\right)^{-1}\cdot\frac{\ln\left(1-|z|\,\|A\|^2\right)^{-1}\ln\left(1-|z|\,\|B\|^2\right)^{-1}}{\ln\left(1-|z|\,\|A\|\right)^{-1}\ln\left(1-|z|\,\|B\|\right)^{-1}}.\end{aligned} \tag{4.13}$$

b) For any bounded linear operators A and B we have the inequalities

$$\|\exp(AB)\|^2 \leq \exp\left(\|A\|^{1+\alpha}\|B\|^{1-\alpha} + \|A\|^{1-\alpha}\|B\|^{1+\alpha}\right) \leq \exp\left(\|A\|^2 + \|B\|^2\right), \tag{4.14}$$

where $\alpha \in [0,1]$, and

$$\|\exp(zAB)\| \leq \exp\left(|z|\left(1 + \|A\|^2 + \|B\|^2 - \|A\| - \|B\|\right)\right), \tag{4.15}$$

where $z \in \mathbb{C}$.

Finally, by the use of the result in Theorem 5 we also have:

Example 4. *a) If A and B are two commuting normal operators on the Hilbert space H with $\|A\|, \|B\| < 1$ and $z \in D(0,1)$ then we have the inequalities*

$$\left|\left\langle (1_H \pm zAB)^{-1} x, y\right\rangle\right|^2 \leq \left\langle \left(1_H - |z|\,|A|^2\right)^{-1} x, x\right\rangle \left\langle \left(1_H - |z|\,|B|^2\right)^{-1} y, y\right\rangle, \tag{4.16}$$

$$\left|\left\langle \ln(1_H \pm zAB)^{-1} x, y\right\rangle\right|^2 \leq \left\langle \ln\left(1_H - |z|\,|A|^2\right)^{-1} x, x\right\rangle \left\langle \ln\left(1_H - |z|\,|B|^2\right)^{-1} y, y\right\rangle, \tag{4.17}$$

and

$$|\langle {}_2F_1(\alpha,\beta,\gamma,zAB)\, x, y\rangle|^2 \leq \left\langle {}_2F_1\left(\alpha,\beta,\gamma,|z|\,|A|^2\right) x, x\right\rangle \left\langle {}_2F_1\left(\alpha,\beta,\gamma,|z|\,|B|^2\right) y, y\right\rangle \tag{4.18}$$

where $\alpha, \beta, \gamma > 0$, for any $x, y \in H$.

b) If A and B are two commuting normal operators on the Hilbert space H and $z \in \mathbb{C}$ then we have the inequalities

$$|\langle \sin(zAB)\, x, y\rangle|^2, |\langle \sinh(zAB)\, x, y\rangle|^2 \leq \left\langle \sinh\left(|z|\,|A|^2\right) x, x\right\rangle \left\langle \sinh\left(|z|\,|B|^2\right) y, y\right\rangle, \tag{4.19}$$

$$|\langle \cos(zAB)\, x, y\rangle|^2, |\langle \cosh(zAB)\, x, y\rangle|^2 \leq \left\langle \cosh\left(|z|\,|A|^2\right) x, x\right\rangle \left\langle \cosh\left(|z|\,|B|^2\right) y, y\right\rangle \tag{4.20}$$

and

$$|\langle \exp(zAB)\, x, y\rangle|^2 \leq \left\langle \exp\left(|z|\,|A|^2\right) x, x\right\rangle \left\langle \exp\left(|z|\,|B|^2\right) y, y\right\rangle \tag{4.21}$$

for any $x, y \in H$

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