Received 28/03/12

SOME INEQUALITIES FOR POWER SERIES WITH APPLICATIONS

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Abstract. In this paper, we derive new inequalities for power series with real coefficients and apply them for some complex functions of interest such as exponential, trigonometric, hyperbolic, hypergeometric, polylogarithm and Bessel functions.

1. Introduction

Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be sequences of complex numbers. Then, the well-known Cauchy-Bunyakovsky-Schwarz (CBS) inequality [**19**] for complex case states that

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \le \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2, \tag{1.1}$$

where the equality holds in (1.1) if and only if there is a complex number $c \in \mathbb{C}$ such that $a_k = c\bar{b}_k$ for all $k \in \{1, 2, ..., n\}$.

A weighted version of the Cauchy-Bunyakovsky-Schwarz (CBS) inequality also holds, namely

$$\left|\sum_{k=1}^{n} p_k a_k b_k\right|^2 \le \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2, \tag{1.2}$$

where $p_k \ge 0$ and $a_k, b_k \in \mathbb{C}, k \in \{1, 2, ..., n\}$.

If we consider an analytic function defined by power series, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with nonnegative coefficients a_n and convergent on the open unit disc $D(0, R) \subset \mathbb{C}, R > 0$ and apply the inequality (1.2), then we can state that

$$|f(zw)|^2 \le \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} \le f\left(|z|^2\right) f\left(|w|^2\right), \tag{1.3}$$

for any $z, w \in \mathbb{C}$ with $zw, |z|^2, |w|^2 \in D(0, R)$.

This result (1.3) has numerous refinements that has been studied by many authors (see [**7**], [**12**], [**13**] and references therein). Also, some generalizations,

2000 *Mathematics Subject Classification.* 26D15; 26A51.

Key words and phrases. Power series, Cauchy-Bunyakovsky-Schwarz inequality, Polylogarithm, Hypergeometric, Bessel function.

This paper is in final form and no version of it will be submitted for publication elsewhere.

extensions and refinements of the above inequality (1.1) concerning power series can be found in the literature (see [**2**], [**7**], [**9**], [**10**], [**14**] and references therein).

Further, if we assume that the power series $f(z)=\sum_{n=0}^{\infty}a_n z^n,\ z\in D(0,R)$ have real coefficient a_n, then we can construct another power series with absolute values of the coefficients, namely

$$f_A(z)=\sum_{n=0}^{\infty}|a_n|\,z^n,$$

where $a_n=|a_n|\,sgn(a_n)$ with $sgn(x)$ is the sign function defined to be 1 if $x>0$, -1 if $x<0$ and 0 if $x=0$. It is obvious that this new power series $f_A(z)$ have the same radius of convergence as the original power series $f(z)$.

Some inequalities which connect the function $f(z)$ with its transform $f_A(z)$ have been established by several authors. For instance, in [**7**], on utilizing the de Bruijn inequality [**8**], Cerone and Dragomir showed the following inequality for power series with real coefficients,

$$|f(az)|^2\le\frac{1}{2}f_A\left(a^2\right)\left[f_A\left(|z|^2\right)+\left|f_A\left(z^2\right)\right|\right], \tag{1.4}$$

where $a\in\mathbb{R}, z\in\mathbb{C}$ with $az,a^2,z^2,|z|^2\in D(0,R), R>0$.

Refinement of (1.4) by using the Buzano inequality [**6**] in inner product spaces was given by Ibrahim and Dragomir in [**12**], namely

$$\left|f\left(\alpha\overline{x}\right)f\left(\overline{\beta}x\right)\right|\le\frac{1}{2}\left(\left[f_A\left(|\alpha|^2\right)f_A\left(|\beta|^2\right)\right]^{1/2}+\left|f_A\left(\alpha\overline{\beta}\right)f_A\left(|x|^2\right)\right|\right), \tag{1.5}$$

for $\alpha,\beta,x\in\mathbb{C}$ such that $\alpha\overline{x},\overline{\beta}x,\alpha\overline{\beta},|\alpha|^2,|\beta|^2,|x|^2\in D(0,R)$.

The same authors (see [**13**]) have obtained further refinement of (1.4) by utilizing a refinement of Schwarz inequality in inner product spaces over the complex field, namely

$$\begin{aligned}&\left[f_A\left(|x|^2\right)f_A\left(|y|^2\right)\right]^{1/2}f_A\left(|z|^2\right)-\left|f\left(x\overline{z}\right)f\left(z\overline{y}\right)\right|\\&\ge\left|f_A\left(x\overline{y}\right)f_A\left(|z|^2\right)-f\left(x\overline{z}\right)f\left(z\overline{y}\right)\right|,\end{aligned} \tag{1.6}$$

for $x,y,z\in\mathbb{C}$ such that $|x|^2,|y|^2,|z|^2,x\overline{z},z\overline{y},x\overline{y}\in D(0,R)$.

Motivated by the above results, (1.4), (1.5) and (1.6), by utilizing a different technique based on the continuity of the modulus, in this paper we obtain new inequalities for power series. In particular, we get an improvement of (1.3) as well as some similar inequalities to (1.5) and (1.6). Applications for some complex functions of interest such as the exponential, trigonometric, hyperbolic, hypergeometric, polylogarithm, Bessel and modified Bessel functions for the first kind are also presented.

2. Some inequalities for power series with real coefficients

In this section, we provide some inequalities which connect the function f with its transform f_A, where f is a function defined by power series with real coefficients and convergent on the disk $D(0,R)$.

First, we state the following result.

THEOREM 1. *Assume that the power series $f(z)=\sum_{n=0}^{\infty} p_n z^n$ with real coefficients is convergent on the disk $D(0,R)$, $R>0$. If $x,z\in\mathbb{C}$ are such that $x, xz, |x|\,|z|^2 \in D(0,R)$, then we have the inequality*

$$\begin{aligned} f_A\left(|x|\,|z|^2\right) f_A(|x|) - |f_A(|x|\,z)|^2 \\ \geq |f(x)f(x\,|z|\,z) - f(xz)f(x\,|z|)| \geq 0. \end{aligned} \tag{2.1}$$

PROOF. If $z\in D(0,R)$, then

$$\begin{aligned} \left|z^n - z^j\right|^2 &= \left|z^n - z^j\right| \left|z^n - z^j\right| \\ &\geq \left|z^n - z^j\right| \left||z|^n - |z|^j\right| \end{aligned} \tag{2.2}$$

for any $n,j\in\mathbb{N}$.

We also have

$$\begin{aligned} \left|z^n - z^j\right|^2 &= |z^n|^2 - 2\operatorname{Re}\left(z^n\overline{z}^j\right) + \left|z^j\right|^2 \\ &= |z|^{2n} - 2\operatorname{Re}\left(z^n\overline{z}^j\right) + |z|^{2j} \end{aligned}$$

and

$$\left|z^n - z^j\right| \cdot \left||z|^n - |z|^j\right| = \left|z^n\,|z|^n + z^j\,|z|^j - z^n\,|z|^j - z^j\,|z|^n\right|$$

for any $n,j\in\mathbb{N}$.

Utilizing (2.2) we get the inequality

$$|z|^{2n} - 2\operatorname{Re}\left(z^n\overline{z}^j\right) + |z|^{2j} \geq \left|z^n\,|z|^n + z^j\,|z|^j - |z|^j\,z^n - |z|^n\,z^j\right| \tag{2.3}$$

for any $n,j\in\mathbb{N}$.

If we multiply (2.3) by $|p_n|\,|x|^n\,|p_j|\,|x|^j \geq 0$ where $x\in D(0,R)$ and $n,j\in N$, then we have

$$\begin{aligned} &|p_n|\,|x|^n\,|z|^{2n}\,p_j\,|x|^j + |p_n|\,|x|^n\,p_j\,|x|^j\,|z|^{2j} \\ &- 2\operatorname{Re}\left(|p_n|\,|x|^n\,z^n\,|p_j|\,|x|^j\,\overline{z}^j\right) \\ &\geq \Big|p_n x^n\,|z|^n\,z^n p_j x^j + p_n x^n p_j x^j\,|z|^j\,z^j \\ &-p_n x^n z^n p_j x^j\,|z|^j - p_n x^n\,|z|^n\,p_j x^j z^j\Big| \end{aligned}$$

for any $n,j\in\mathbb{N}$.

Summing over n and j from 0 to k and utilizing the triangle inequality for the modulus, we have

$$\begin{aligned}(2.4)\qquad &\sum_{n=0}^{k}|p_n|\,|x|^n\,|z|^{2n}\sum_{j=0}^{k}|p_j|\,|x|^j+\sum_{n=0}^{k}|p_n|\,|x|^n\sum_{j=0}^{k}|p_j|\,|x|^j\,|z|^{2j}\\ &-2\operatorname{Re}\left(\sum_{n=0}^{k}|p_n|\,|x|^n\,z^n\sum_{j=0}^{k}|p_j|\,|x|^j\,(\overline{z})^j\right)\\ &\geq\left|\sum_{n=0}^{k}p_nx^n\,|z|^n\,z^n\sum_{j=0}^{k}p_jx^j+\sum_{n=0}^{k}p_nx^n\sum_{j=0}^{k}p_jx^j\,|z|^j\,z^j\right.\\ &\left.-\sum_{n=0}^{k}p_nx^nz^n\sum_{j=0}^{k}p_jx^j\,|z|^j-\sum_{n=0}^{k}p_nx^n\,|z|^n\sum_{j=0}^{k}p_jx^jz^j\right|.\end{aligned}$$

Since

$$\sum_{j=0}^{k}|p_j|\,|x|^j\,(\overline{z})^j=\overline{\sum_{n=0}^{k}|p_n|\,|x|^n\,z^n}$$

then

$$\operatorname{Re}\left(\sum_{n=0}^{k}|p_n|\,|x|^n\,z^n\sum_{j=0}^{k}|p_j|\,|x|^j\,(\overline{z})^j\right)=\left|\sum_{n=0}^{k}|p_n|\,|x|^n\,z^n\right|^2.$$

Hence, from the inequality (2.4) we have

$$\begin{aligned}(2.5)\qquad &\sum_{n=0}^{k}|p_n|\,|x|^n\,|z|^{2n}\sum_{n=0}^{k}|p_n|\,|x|^n-\left|\sum_{n=0}^{k}|p_n|\,|x|^n\,z^n\right|^2\\ &\geq\left|\sum_{n=0}^{k}p_nx^n\sum_{n=0}^{k}p_nx^n\,|z|^n\,z^n-\sum_{n=0}^{k}p_nx^nz^n\sum_{n=0}^{k}p_nx^n\,|z|^n\right|.\end{aligned}$$

Since all the series

$$\sum_{n=0}^{\infty}|p_n|\,|x|^n\,|z|^{2n},\quad\sum_{n=0}^{\infty}|p_n|\,|x|^n,\quad\sum_{n=0}^{\infty}p_nx^n\,|z|^n\,z^n,$$
$$\sum_{n=0}^{\infty}|p_n|\,|x|^n\,z^n,\quad\sum_{n=0}^{\infty}p_nx^n,\quad\sum_{n=0}^{\infty}p_nx^nz^n,\quad\sum_{n=0}^{\infty}p_nx^n\,|z|^n$$

are convergent, then by taking the limit over $k\to\infty$ in (2.5), we deduce the desired inequality (2.1). □

Corollary 1. *If $\sum_{n=0}^{\infty}|p_n|<\infty$, i.e., $f_A(1)<\infty$, then for any $\zeta\in\mathbb{C}$ with $|\zeta|=1$ we have*

$$(2.6)\qquad f_A\left(|z|^2\right)f_A(1)-|f_A(z)|^2\geq|f(\zeta)f(\zeta\,|z|\,z)-f(\zeta z)f(\zeta\,|z|)|\geq0.$$

In particular, for $\zeta=1$ we have

$$(2.7)\qquad f_A\left(|z|^2\right)f_A(1)-|f_A(z)|^2\geq|f(1)f(|z|\,z)-f(z)f(|z|)|\geq0$$

for any $z,|z|^2\in D(0,R)$.

Some applications of the inequalities (2.1) and (2.7) are as follows:

(1) If we apply the inequality (2.1) for the function $f(z)=\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$, $z\in D(0,1)$, then we get

$$\begin{aligned}&\frac{|1-z|}{(1-|x|)\left(1-|x|\,|z|^2\right)|1-|x|\,z|^2}\\&\geq\left|\frac{1-|z|}{(1-x)(1-xz)(1-x\,|z|)(1-x\,|z|\,z)}\right|\end{aligned}$$

for any $x,z\in\mathbb{C}$ with $x,|x|\,|z|^2\in D(0,1)$.

(2) If we apply the inequality (2.1) for the function $f(z)=\frac{1}{1+z}=\sum_{n=0}^{\infty}(-1)^n z^n$, $z\in D(0,1)$, then we get the inequality

$$\begin{aligned}&\frac{|1-z|}{(1-|x|)\left(1-|x|\,|z|^2\right)|1-|x|\,z|^2}\\&\geq\left|\frac{1-|z|}{(1+x)(1+xz)(1+x\,|z|)(1+x\,|z|\,z)}\right|\end{aligned}$$

for any $x,z\in\mathbb{C}$ with $x,xz,|x|\,|z|^2\in D(0,1)$.

(3) If we apply the inequality (2.7) for the function $f(z)=\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z\in\mathbb{C}$, then we get the inequality

$$\exp\left(|z|^2+1\right)-|\exp(z)|^2\geq|\exp(z\,|z|+1)-\exp(z+|z|)|$$

for any $z\in\mathbb{C}$.

(4) If we take the function $f(z)=\cosh(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}$, $z\in\mathbb{C}$, then applying the inequality (2.7) we have

$$\begin{aligned}&\left(e^2+1\right)\cosh\left(|z|^2\right)-2e\,|\cosh(z)|^2\\&\geq\left|\left(e^2+1\right)\cosh(|z|\,z)-2e\cosh(z)\cosh(|z|)\right|.\end{aligned}\tag{2.8}$$

for any $z\in\mathbb{C}$.

In particular, if we choose $z=ib$, $b\in\mathbb{R}$ in (2.8), then we obtain the inequality

$$\begin{aligned}&\left(e^2+1\right)\cosh\left(b^2\right)-2e\cos^2(b)\\&\geq\left|\left(e^2+1\right)\cos(|b|\,b)-2e\cos(b)\cosh(|b|)\right|\end{aligned}$$

for any $b\in\mathbb{R}$.

(5) Further, if we take the function $f(z)=\sinh(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}$, $z\in\mathbb{C}$, then applying the inequality (2.7) we get

$$\begin{aligned}&\left(e^2-1\right)\sinh\left(|z|^2\right)-2e\,|\sinh(z)|^2\\&\geq\left|\left(e^2-1\right)\sinh(|z|\,z)-2e\sinh(z)\sinh(|z|)\right|.\end{aligned}\tag{2.9}$$

for any $z\in\mathbb{C}$.

In particular, if we choose $z = ib$, $b \in \mathbb{R}$ in (2.9), then we obtain the inequality

$$\begin{aligned} & \left(e^2-1\right)\sinh\left(b^2\right) - 2e\sin^2(b) \\ & \geq \left|\left(e^2-1\right)\sin\left(|b|\,b\right) - 2e\sin(b)\sinh(|b|)\right|. \end{aligned}$$

for any $b \in \mathbb{R}$.

Remark 1. *The inequality (2.1) can also be written in the form*

$$\det\begin{bmatrix} f_A\left(|x|\,|z|^2\right) & f_A\left(|x|\,z\right) \\ f_A\left(|x|\,\overline{z}\right) & f_A\left(|x|\right) \end{bmatrix} \geq \left|\det\begin{bmatrix} f(x) & f(xz) \\ f(x\,|z|) & f\left(x\,|z|\,z\right) \end{bmatrix}\right|$$

for any $x, z \in \mathbb{C}$ with $x, xz, |x|\,|z|^2 \in D(0,R)$.

Theorem 2. *Assume that the power series $f(z) = \sum_{n=0}^{\infty} p_n z^n$ with real coefficients is convergent on the disk $D(0,R)$, $R > 0$. If $x, z \in \mathbb{C}$ are such that $x, xz, |x|\,|z|^2 \in D(0,R)$, then we have the inequality*

$$\begin{aligned} & f_A\left(|x|\right) f_A\left(|x|\,|z|^2\right) - \operatorname{Re}\left[f_A^2\left(|x|\,z\right)\right] \\ & \geq \frac{1}{2}\left|f(x)f\left(x\,|z|\,z\right) + f(x)f\left(x\,|z|\,\overline{z}\right) - f(xz)f(x\,|z|) - f\left(x\,|z|\right)f\left(x\overline{z}\right)\right|. \end{aligned} \tag{2.10}$$

Proof. If $z \in D(0,R)$, then

$$\begin{aligned} \left|z^n - (\overline{z})^j\right|^2 & = \left|z^n - (\overline{z})^j\right|\left|z^n - (\overline{z})^j\right| \\ & \geq \left|z^n - (\overline{z})^j\right|\left||z|^n - |z|^j\right| \\ & = \left||z|^n z^n + (\overline{z})^j |z|^j - |z|^j z^n - |z|^n (\overline{z})^j\right| \end{aligned} \tag{2.11}$$

for any $n, j \in \mathbb{N}$.

We also have

$$\begin{aligned} \left|z^n - (\overline{z})^j\right|^2 & = |z^n|^2 - 2\operatorname{Re}\left(z^n z^j\right) + \left|z^j\right|^2 \\ & = |z|^{2n} - 2\operatorname{Re}\left(z^n z^j\right) + |z|^{2j} \end{aligned}$$

for any $n, j \in \mathbb{N}$.

Utilizing (2.11) we have the inequality

$$\begin{aligned} & |z|^{2n} - 2\operatorname{Re}\left(z^n z^j\right) + |z|^{2j} \\ & \geq \left||z|^n z^n + (\overline{z})^j z^n - |z|^j z^n - |z|^n (\overline{z})^j\right| \end{aligned}$$

for any $n, j \in \mathbb{N}$.

Now, on utilizing a similar argument to the one in the proof of Theorem 1 above, we deduce the desired result (2.10). The details are omitted. □

Corollary 2. *If $z = \overline{x}$ in (2.10) then we have*

$$\begin{aligned} & f_A\left(|x|\right) f_A\left(|x|^3\right) - \operatorname{Re}\left[f_A^2\left(|x|\,\overline{x}\right)\right] \\ & \geq \frac{1}{2}\left|f(x)f\left(|x|^3\right) + f(x)f\left(|x|\,x^2\right) - f(|x|^2)f(|x|\,x) - f\left(|x|\,x\right)f\left(x^2\right)\right| \end{aligned} \tag{2.12}$$

for any $x \in \mathbb{C}$ such that $x, |x|\,x, |x|\,x^2 \in D(0,R)$.

In the following, we give some applications of above inequality (2.12) for particular complex functions of interest.

(1) If we take the function $f(z) = \dfrac{1}{1+z}$, $z \in D(0,1)$, then $f_A(z) = \sum\limits_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$, $z \in D(0,1)$. Applying (2.12), we get the following inequality

$$\begin{aligned}
&\frac{1}{(1-|x|)\left(1-|x|^3\right)} - \operatorname{Re}\left(\frac{1}{1-|x|\,\overline{x}}\right)^2 \\
&\geq \frac{1}{2}\left|\frac{2+|x|\,x^2+|x|^3}{(1+x)\left(1+|x|^3\right)(1+|x|\,x^2)}\right. \\
&\left.-\frac{2+x^2+|x|^2}{(1+x^2)(1+|x|\,x)\left(1+|x|^2\right)}\right|
\end{aligned}$$

for any $x, |x|\,x, |x|\,x^2 \in D(0,R)$.

(2) If we apply the inequality (2.12) for the function $f(z) = \exp(z)$, $z \in \mathbb{C}$, then we get

$$\begin{aligned}
&\exp\left(|x| + |x|^3\right) - \operatorname{Re}\left[\exp\left(2\,|x|\,\overline{x}\right)\right] \\
&\geq \frac{1}{2}\left|\exp(x+|x|^3) + \exp(x+|x|\,x^2)\right. \\
&\left.-\exp(|x|^2+|x|\,x) - \exp(|x|\,x+x^2)\right|
\end{aligned}$$

for any $x \in \mathbb{C}$.

Theorem 3. *Assume that the power series $f(z) = \sum_{n=0}^{\infty} p_n z^n$ with real coefficients is convergent on the disk $D(0,R)$, $R > 0$. If $x, y \in \mathbb{C}$ are such that $|x|^2, |y|^2 < R$, then we have the inequality*

$$\begin{aligned}
&f_A\left(|x|^2\right) f_A\left(|y|^2\right) - \left|f_A(xy)\right|^2 \\
&\geq \left|f(|x|\,x) f(|y|\,\overline{y}) - f(|y|\,x) f(|x|\,\overline{y})\right|.
\end{aligned} \tag{2.13}$$

Proof. If $x, y \in \mathbb{C}$, then we have

$$\begin{aligned}
\left|x^n (\overline{y})^j - x^j (\overline{y})^n\right|^2 &= \left|x^n (\overline{y})^j - x^j (\overline{y})^n\right| \left|x^n (\overline{y})^j - x^j (\overline{y})^n\right| \\
&\geq \left|x^n (\overline{y})^j - x^j (\overline{y})^n\right| \left||x|^n |y|^j - |x|^j |y|^n\right|
\end{aligned} \tag{2.14}$$

for any $n, j \in \mathbb{N}$.

We have upon simple calculations that

$$\begin{aligned}
&|x|^{2n} |y|^{2j} - 2\operatorname{Re}\left(x^n y^n (\overline{x})^j (\overline{y})^j\right) + |y|^{2n} |x|^{2j} \\
&\geq \left||x|^n x^n |y|^j (\overline{y})^j + |y|^n (\overline{y})^n |x|^j x^j\right. \\
&\left.-|y|^n x^n |x|^j (\overline{y})^j - |x|^n (\overline{y})^n |y|^j x^j\right|
\end{aligned} \tag{2.15}$$

for any $n, j \in \mathbb{N}$.

If we multiply the inequality (2.15) with $|p_n|\,|p_j| \geq 0$ and summing over n and j from 0 to k we have

$$\begin{aligned}(2.16)\qquad & \sum_{n=0}^{k}|p_n|\,|x|^{2n}\sum_{j=0}^{\infty}|p_j|\,|y|^{2j}+\sum_{n=0}^{k}|p_n|\,|y|^{2n}\sum_{j=0}^{k}|p_j|\,|x|^{2j} \\ & -2\operatorname{Re}\left(\sum_{n=0}^{k}|p_n|\,x^n y^n\sum_{j=0}^{k}|p_j|\,(\overline{x})^j\,(\overline{y})^j\right) \\ & \geq \left|\sum_{n=0}^{k}p_n\,|x|^n\,x^n\sum_{j=0}^{k}p_j\,|y|^j\,(\overline{y})^j+\sum_{n=0}^{k}p_n\,|y|^n\,(\overline{y})^n\sum_{j=0}^{k}p_j\,|x|^j\,x^j\right. \\ & \left.-\sum_{n=0}^{k}p_n\,|y|^n\,x^n\sum_{j=0}^{k}p_j\,|x|^j\,(\overline{y})^j-\sum_{n=0}^{k}p_n\,|x|^n\,(\overline{y})^n\sum_{j=0}^{k}p_j\,|y|^j\,x^j\right|.\end{aligned}$$

Due to the fact that

$$\sum_{n=0}^{k}|p_n|\,x^n y^n\sum_{j=0}^{\infty}|p_j|\,(\overline{x})^j\,(\overline{y})^j=\left|\sum_{n=0}^{k}|p_n|\,x^n y^n\right|^2,$$

the inequality (2.16) is equivalent with

$$\begin{aligned}(2.17)\qquad & \sum_{n=0}^{k}|p_n|\,|x|^{2n}\sum_{n=0}^{k}|p_n|\,|y|^{2n}-\left|\sum_{n=0}^{k}|p_n|\,x^n y^n\right|^2 \\ & \geq \left|\sum_{n=0}^{k}p_n\,|x|^n\,x^n\sum_{n=0}^{k}p_n\,|y|^n\,(\overline{y})^n-\sum_{n=0}^{k}p_n\,|y|^n\,x^n\sum_{n=0}^{k}p_n\,|x|^n\,(\overline{y})^n\right|.\end{aligned}$$

Since all the series with the partial sums involved in (2.17) are convergent, then by taking the limit over $k\to\infty$ in (2.17), we deduce the desired result form (2.13). $\square$

Remark 2. *The inequality (2.13) is also equivalent to*

$$\det\begin{bmatrix} f_A\left(|x|^2\right) & f_A(xy) \\ f_A(\overline{xy}) & f_A\left(|y|^2\right)\end{bmatrix} \geq \left|\det\begin{bmatrix} f(|x|\,x) & f(|y|\,x) \\ f(|x|\,\overline{y}) & f(|y|\,\overline{y})\end{bmatrix}\right|$$

for any $x,y\in\mathbb{C}$ *with* $|x|^2,|y|^2<R$.

The inequality (2.13) has some applications for particular complex functions of interest which will be pointed out as follows.

(1) If we apply the inequality (2.13) for the function $f(z)=\dfrac{1}{1-z}$, $z\in D(0,1)$, then we get the inequality

$$\begin{aligned}(2.18)\qquad & \frac{1}{\left(1-|x|^2\right)\left(1-|y|^2\right)}-\frac{1}{|1-xy|^2} \\ & \geq \left|\frac{1}{(1-x\,|x|)\,(1-|y|\,\overline{y})}-\frac{1}{(1-|y|\,x)\,(1-|x|\,\overline{y})}\right|\end{aligned}$$

for any $x,y\in\mathbb{C}$.

In particular, if we choose $y=0$ in (2.18), then we obtain the simpler inequality

$$\frac{1}{1-|x|^2}-1\geq\left|\frac{1}{1-x\,|x|}-1\right|$$

for any $x\in\mathbb{C}$.

(2) Also, if we take the function $f(z)=\dfrac{1}{1+z}$, $z\in D(0,1)$, then we have $f_A=\dfrac{1}{1-z}$, $z\in D(01)$. Applying the inequality (2.13) then we get

$$\begin{aligned}&\frac{1}{\left(1-|x|^2\right)\left(1-|y|^2\right)}-\frac{1}{|1-xy|^2}\\&\geq\left|\frac{1}{(1+x\,|x|)\,(1+|y|\,\overline{y})}-\frac{1}{(1+|y|\,x)\,(1+|x|\,\overline{y})}\right|\end{aligned}\tag{2.19}$$

for any $x,y\in\mathbb{C}$.

If in (2.19) we choose $y=0$, then we obtain the inequality

$$\frac{1}{1-|x|^2}-1\geq\left|\frac{1}{1+x\,|x|}-1\right|$$

for any $x\in\mathbb{C}$.

(3) If we apply the inequality (2.13) for the function $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we have

$$\begin{aligned}&\exp\left(|x|^2+|y|^2\right)-|\exp(xy)|^2\\&\geq|\exp(x\,|x|+|y|\,\overline{y})-\exp(|y|\,x+|x|\,\overline{y})|\end{aligned}\tag{2.20}$$

for any $x,y\in\mathbb{C}$.

In particular, if in (2.20) we choose $y=0$, then we get

$$\exp\left(|x|^2\right)-1\geq|\exp(x\,|x|)-1|$$

for any $x\in\mathbb{C}$.

(4) If we take the function $f(z)=\cos(z)$, $z\in\mathbb{C}$, then $f_A(z)=\cosh(z)$, $z\in\mathbb{C}$. Utilizing the inequality (2.13) for f as above gives

$$\begin{aligned}&\cosh\left(|x|^2\right)\cosh\left(|y|^2\right)-|\cosh(xy)|^2\\&\geq|\cos(x\,|x|)\cos(|y|\,\overline{y})-\cos(|y|\,x)\cos(|x|\,\overline{y})|\end{aligned}\tag{2.21}$$

for any $x,y\in\mathbb{C}$.

In particular, we have, with $y=0$ in (2.21),

$$\cosh\left(|x|^2\right)-1\geq|\cos(x\,|x|)-1|$$

for any $x\in\mathbb{C}$.

(5) The choice of another fundamental power series, $f(z)=\sinh(z)$, $z\in\mathbb{C}$ will produce via (2.13) the following result:

$$\begin{aligned}&\sinh\left(|x|^2\right)\sinh\left(|y|^2\right)-|\sinh(xy)|^2\\&\geq|\sinh(x\,|x|)\sinh(|y|\,\overline{y})-\sinh(|y|\,x)\sinh(|x|\,\overline{y})|\end{aligned}\tag{2.22}$$

for any $x,y\in\mathbb{C}$.

The choice $y=1$ in (2.22) generates the following inequality,

$$\begin{aligned}
&\left(e^2-1\right)\sinh\left(|x|^2\right)-2e\left|\sinh(x)\right|^2\\
&\geq\left|\left(e^2-1\right)\sinh(x|x|)-2e\sinh(x)\sinh(|x|)\right|
\end{aligned}$$

for any $x\in\mathbb{C}$.

THEOREM 4. *Assume that the power series $f(z)=\sum_{n=0}^{\infty}p_nz^n$ with real coefficients is convergent on the disk $D(0,R)$, $R>0$. If $x,y\in\mathbb{C}$ are such that $|x|^2,|y|^2<R$, then we have the inequality*

$$\begin{aligned}
&f_A\left(|x|^2\right)f_A\left(|y|^2\right)-\operatorname{Re}\left[f_A^2\left(x\overline{y}\right)\right]\\
&\geq\frac{1}{2}\left|f(|x|\,x)f\left(|y|\,\overline{y}\right)+f(|x|\,\overline{x})f\left(|y|\,y\right)\right.\\
&\left.-f(|x|\,y)f(|y|\,\overline{x})-f(|y|\,x)f(|x|\,\overline{y})\right|.
\end{aligned}\tag{2.23}$$

PROOF. If $x,y\in D(0,R)$, then we have

$$\begin{aligned}
\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^jy^n\right|^2&=\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^jy^n\right|\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^jy^n\right|\\
&\geq\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^jy^n\right|\left||x|^n|y|^j-|x|^j|y|^n\right|
\end{aligned}\tag{2.24}$$

for any $n,j\in\mathbb{N}$.

Doing simple calculations we get that

$$\begin{aligned}
&|x|^{2n}|y|^{2j}-2\operatorname{Re}\left[x^n\left(\overline{y}\right)^jx^j\left(\overline{y}\right)^n\right]+|x|^{2j}|y|^{2n}\\
&\geq\left||x|^nx^n|y|^j\left(\overline{y}\right)^j+|x|^j\left(\overline{x}\right)^j|y|^ny^n-|x|^ny^n|y|^j\left(\overline{x}\right)^j-|y|^nx^n|x|^j\left(\overline{y}\right)^j\right|
\end{aligned}\tag{2.25}$$

for any $n,j\in\mathbb{N}$.

If we multiply (2.25) with $|p_n|\,|p_j|\geq0$ and summing over n and j from 0 to k we get

$$\begin{aligned}
&\sum_{n=0}^{k}|p_n|\,|x|^{2n}\sum_{j=0}^{k}|p_j|\,|y|^{2j}-2\operatorname{Re}\left[\sum_{n=0}^{k}|p_n|\,x^n\left(\overline{y}\right)^n\sum_{j=0}^{k}|p_j|\,x^j\left(\overline{y}\right)^j\right]\\
&+\sum_{n=0}^{k}|p_n|\,|y|^{2n}\sum_{j=0}^{k}|p_j|\,|x|^{2j}\\
&\geq\left|\sum_{n=0}^{k}p_n|x|^nx^n\sum_{j=0}^{k}p_j|y|^j\left(\overline{y}\right)^j+\sum_{n=0}^{k}p_ny^n|y|^n\sum_{j=0}^{k}p_j|x|^j\left(\overline{x}\right)^j\right.\\
&\left.-\sum_{n=0}^{k}p_n|x|^ny^n\sum_{j=0}^{k}p_j|y|^j\left(\overline{x}\right)^j-\sum_{n=0}^{k}p_n|y|^nx^n\sum_{j=0}^{k}p_j|x|^j\left(\overline{y}\right)^j\right|
\end{aligned}\tag{2.26}$$

for any $n,j\in\mathbb{N}$.

Since all the series whose partial sums are involed in (2.26) are convergent, then by taking the limit over $k\to\infty$ in (2.26) we deduce the desired result (2.23). $\square$

The inequality (2.23) is also a valuable source of particular inequalities for complex functions of interest, that will be outlined in the following.

(1) In (2.23) we take the function $f(z) = \exp(z)$, $z \in \mathbb{C}$, then we can state that

$$\begin{aligned} &\exp\left(|x|^2 + |y|^2\right) - \operatorname{Re}\left[\exp\left(2x\overline{y}\right)\right] \\ &\geq \frac{1}{2}\left|\exp\left(|x|\,x + |y|\,\overline{y}\right) + \exp\left(|x|\,\overline{x}\right) + |y|\,y\right) \\ &- \exp\left(|x|\,y\right) + |y|\,\overline{x}\right) - \exp\left(|y|\,x + |x|\,\overline{y}\right)\right| \end{aligned} \tag{2.27}$$

for any $x, y \in \mathbb{C}$.

If in (2.27) we choose $y = 0$, then we obtain the simpler result:

$$\exp\left(|x|^2\right) - 1 \geq \frac{1}{2}\left|\exp\left(|x|\,x\right) + \exp\left(|x|\,\overline{x}\right) - 2\right|$$

for any $x \in \mathbb{C}$.

(2) If we apply the inequality (2.23) for the function $f(z) = \cos(z)$, $z \in \mathbb{C}$, with its transform $f_A(z) = \cosh(z)$, $z \in \mathbb{C}$, then we get

$$\begin{aligned} &\cosh\left(|x|^2\right)\cosh\left(|y|^2\right) - \operatorname{Re}\left[\cosh^2\left(x\overline{y}\right)\right] \\ &\geq \frac{1}{2}\left|\cos(|x|\,x)\cos\left(|y|\,\overline{y}\right) + \cos(|x|\,\overline{x})\cos\left(|y|\,y\right)\right. \\ &\left.- \cos(|x|\,y)\cos(|y|\,\overline{x}) - \cos(|y|\,x)\cos(|x|\,\overline{y})\right|. \end{aligned} \tag{2.28}$$

for any $x, y \in \mathbb{C}$.

In particular, if in (2.28) we choose $y = 0$, then we obtain that

$$\cosh\left(|x|^2\right) - 1 \geq \frac{1}{2}\left|\cos(|x|\,x) + \cos(|x|\,\overline{x}) - 2\right|$$

for any $x \in \mathbb{C}$.

3. Some inequalities for the polylogarithm

Before we state our results for the polylogarithm, we recall here some concepts that will be used in sequel.

The *polylogarithm* $Li_n(z)$ also known as *Jonquiére's function* is a function defined by the power series

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \tag{3.1}$$

This series (3.1) converges absolutely for all complex values of the order n and the argument z where $|z| < 1$. The special case $n = 1$ involves the ordinary logarithm, i.e., $Li_1(z) = \ln\left(\frac{1}{1-z}\right)$ while the special case $s = 2$ is called the *dilogarithm* or *Spence's function*, namely

$$Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{3.2}$$

for $|z| < 1$.

The polylogarithm of nonnegative order n arises in the sums of the form

$$Li_{-n}(r) = \sum_{k=1}^{\infty} k^n r^k = \frac{1}{(1-r)^{n+1}} \sum_{i=0}^{s} E_{n,i} r^{n-i},$$

where $E_{n,i}$ is an *Eulerian number*, namely, we recall that

$$E_{n,k} := \sum_{j=0}^{k+1} (-1)^j \left(\begin{array}{c} n+1 \\ j \end{array} \right) (k-j+1)^n .$$

Polylogarithms also arise in sum of *generalized harmonic numbers* $H_{n,r}$ as

$$\sum_{n=1}^{\infty} H_{n,r} z^n = \frac{Li_r(z)}{1-z},$$

for $z \in D(0,1)$, where we recall that

$$H_{n,r} := \sum_{k=1}^{\infty} \frac{1}{k^r} \text{ and } H_{n,1} := H_n = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Special forms of low-order polylogarithms include

$$\begin{aligned} Li_{-2}(z) &= \frac{z(z+1)}{(1-z)^3}, \quad Li_{-1}(z) = \frac{z}{(1-z)^2}, \\ Li_0(z) &= \frac{z}{1-z} \text{ and } Li_1(z) = \ln \frac{1}{1-z} \end{aligned}$$

for all $z \in D(0,1)$.

The polylogarithm also has relationship to other functions for the special cases of argument z. For instance:

$$\begin{aligned} Li_n(1) &= \zeta(n), \\ Li_n(-1) &= -\eta(n), \\ Li_n(\pm i) &= \frac{1}{2^s}\eta(n) \pm i\beta(n), \end{aligned}$$

where $\zeta(n), \eta(n)$ and $\beta(n)$ are the *Riemann zeta*, *Dirichlet eta* and *Dirichlet beta function* respectively.

It is clearly seen that from (3.1), $Li_n(z)$ is a power series with nonnegative coefficients. Therefore, all the results in the above section hold true. For instance, from (2.1) we have the inequality

$$\begin{aligned} &Li_n\left(|x|\,|z|^2\right) Li_n(|x|) - |Li_n(|x|\,z)|^2 \\ &\geq |Li_n(x)Li_n(x\,|z|\,z) - Li_n(xz)\,Li_n(x\,|z|)| \end{aligned} \tag{3.3}$$

for any $x, z \in \mathbb{C}$ with $x, xz, |x|\,|z|^2 \in D(0,1)$, where n is a negative or positive integer.

Also, from (2.13) we have

$$\begin{aligned} &Li_n\left(|x|^2\right) Li_n\left(|y|^2\right) - |Li_n(xy)|^2 \\ &\geq |Li_n(x\,|x|)Li_n(|y|\,\overline{y}) - Li_n(|y|\,x)Li_n(|x|\,\overline{y})| \end{aligned} \tag{3.4}$$

for any $x, y \in \mathbb{C}$ with $|x|^2, |y|^2 < R$.

In the following we present some results which incorporates the zeta $\zeta(s)$, Dirichlet eta $\eta(s)$ and Dirichlet zeta function $\beta(s)$, where $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$, $\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}$ and $\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}$ respectively which converge for any complex number s with a positive real part.

Corollary 3. *If we choose $x=1$ in (3.3), then we have*

$$\begin{aligned}\zeta(n)\,Li_n\left(|z|^2\right)-|Li_n(z)|^2 \\ \geq|\zeta(n)\,Li_n(|z|\,z)-Li_n(z)\,Li_n(|z|)|\end{aligned}\tag{3.5}$$

for any $z\in D(0,1)$ and n is a positive or negative integer.

In particular, if $n=2$ in (3.5), we get that

$$\frac{\pi^2}{6}Li_2\left(|z|^2\right)-|Li_2(z)|^2\geq\left|\frac{\pi^2}{6}Li_2(|z|\,z)-Li_2(z)\,Li_2(|z|)\right|$$

for any $z\in D(0,1)$ and the $Li_2(z)$ is the dilogarithm or Spence's function which is defined in (3.2).

Corollary 4. *If we choose $x=i$ in (3.3), then we have*

$$\begin{aligned}Li_n\left(|z|^2\right)\zeta(n)-|Li_n(z)|^2 \\ \geq\left|\left(\frac{1}{2^n}\eta(n)+i\beta(n)\right)Li_n(i\,|z|\,z)-Li_n(iz)\,Li_n(i\,|z|)\right|.\end{aligned}\tag{3.6}$$

for any $z\in D(0,1)$ and n is a positive or negative integer.

In particular, for $n=2$, we have from (3.6) that

$$\begin{aligned}\frac{\pi^2}{6}Li_2\left(|z|^2\right)-|Li_2(z)|^2 \\ \geq\left|\left(\frac{\pi^2}{48}+iG\right)Li_2(i\,|z|\,z)-Li_2(iz)\,Li_2(i\,|z|)\right|\end{aligned}$$

for any $z\in D(0,1)$ and $G=\beta(2)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$ is called the Catalan's constant.

Corollary 5. *If we choose $x=1$ in (3.4), then we have*

$$\zeta(n)\,Li_n\left(|y|^2\right)-|Li_n(y)|^2\geq|\zeta(n)\,Li_n(|y|\,\overline{y})-Li_n(|y|)Li_n(\overline{y})|\tag{3.7}$$

for any $z\in D(0,1)$ and n is a positive or negative integer.

In particular, for $n=2$ in (3.7), we get

$$\frac{\pi^2}{6}Li_2\left(|y|^2\right)-|Li_2(y)|^2\geq\left|\frac{\pi^2}{6}Li_2(|y|\,\overline{y})-Li_2(|y|)Li_2(\overline{y})\right|$$

for any $z\in D(0,1)$ and n is a positive or negative integer.

Corollary 6. *If we choose $x=i$ in (3.4), then we have*

$$\begin{aligned}\zeta(n)\,Li_n\left(|y|^2\right)-|Li_n(iy)|^2 \\ \geq\left|\left(\frac{1}{2^n}\eta(n)+i\beta(n)\right)Li_n(|y|\,\overline{y})-Li_n(i\,|y|)Li_n(\overline{y})\right|\end{aligned}\tag{3.8}$$

for any $y\in D(0,1)$ and n is a positive or negative integer.

In particular, we get the following inequality by choosing $n=2$ in (3.8),

$$\begin{aligned}\frac{\pi^2}{6}Li_2\left(|y|^2\right)-|Li_2(iy)|^2 \\ \geq\left|\left(\frac{\pi^2}{48}+iG\right)Li_2(|y|\,\overline{y})-Li_2(i\,|y|)Li_2(\overline{y})\right|\end{aligned}$$

for any $z \in D(0,1)$.

4. Applications to hypergeometric function

The *hypergeometric function* ${}_2F_1(a,b;c;z)$ is defined for all $|z|<1$ by the series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{4.1} $$

for arbitrary $a,b,c \in \mathbb{R}$ with $c \neq 0,-1,-2,\ldots$, and the $(t)_n$, $n \in \{0,1,2,\ldots\}$ is a *Pochhammer* symbol which is defined by

$$ (t)_n = \begin{cases} 1, & \text{if } n=0, \\ t(t+1)\cdots(t+n-1), & \text{if } n>0. \end{cases} $$

Hypergeometric function (4.1) with particular arguments of a,b and c reduces to elementary functions, for example,

$$ \begin{aligned} {}_2F_1(1,1;1;z) &= {}_2F_1(1,2;2;z) = \frac{1}{1-z}, \\ {}_2F_1(1,2;1;z) &= \frac{1}{(1-z)^2}, \\ {}_2F_1(a,b;b;z) &= \frac{1}{(1-z)^a}, \\ {}_2F_1(1,1;2;z) &= \frac{1}{z}\ln\left(\frac{1}{1-z}\right), \\ {}_2F_1(1,1;2;-z) &= \frac{1}{z}\ln(1+z). \end{aligned} $$

Since, the hypergeometric function (4.1) is a power series with nonnegative coefficients, then all the above results in Section 2 hold true. For instance, from (2.13) we have the inequality

$$ \begin{aligned} &{}_2F_1\left(a,b;c;|x|^2\right){}_2F_1\left(a,b;c;|y|^2\right) - \left|{}_2F_1(a,b;c;xy)\right|^2 \\ &\geq \left|{}_2F_1(a,b;c;|x|x){}_2F_1(a,b;c;|y|\overline{y}) \right. \\ &\left. - {}_2F_1(a,b;c;|y|x){}_2F_1(a,b;c;|x|\overline{y})\right| \end{aligned} \tag{4.2} $$

for any $a,b,c \in \mathbb{R}$, with $c \neq 0,-1,-2,\ldots$ and $x,y \in \mathbb{C}$ such that $|x|,|y|<1$.

Corollary 7. *If in (4.2), we choose* $c=b$, *then we have*

$$ \begin{aligned} &\frac{1}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^a} - \frac{1}{|1-xy|^{2a}} \\ &\geq \left| \frac{1}{[(1-|x|x)(1-|y|\overline{y})]^a} - \frac{1}{[(1-|y|x)(1-|x|\overline{y})]^a} \right| \end{aligned} \tag{4.3} $$

for any $a \in \mathbb{R}, x,y \in \mathbb{C}$ *such that* $|x|,|y|<1$.

Remark 3. *For* $a=1$, *the inequality (4.3) reduces to (2.19).*

COROLLARY 8. *If in (4.2), we choose* $a = b = 1$, $c = 2$, *then we have*

$$\ln\left(\frac{1}{1-|x|^2}\right)\ln\left(\frac{1}{1-|y|^2}\right)-\left|\ln\left(\frac{1}{1-xy}\right)\right|^2 \geq \left|\ln\left(\frac{1}{1-|x|\,x}\right)\ln\left(\frac{1}{1-|y|\,\overline{y}}\right)-\ln\left(\frac{1}{1-|y|\,x}\right)\ln\left(\frac{1}{1-|x|\,\overline{y}}\right)\right| \tag{4.4}$$

for $x, y \in \mathbb{C}$ *with* $|x|, |y| < 1$.

5. Applications to Bessel function

The *Bessel functions* of the first kind $J_\alpha(z)$ are defined as the solutions to the *Bessel differential equation*, i.e.,

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+\left(z^2-\alpha^2\right)y=0 \tag{5.1}$$

for an arbitrary real or complex order α. This solution of (5.1) is an analytic function of z in $\mathbb{C}$, except for a point $z = 0$ when α is not an integer. These solutions, denoted by $J_\alpha(z)$ are defined by Taylor series expansion around the origin [**1**, p. 360], i.e.,

$$J_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\alpha+1)\,n!}\left(\frac{z}{2}\right)^{2n+\alpha} \tag{5.2}$$

where $\Gamma(x)$ is the *gamma function*.

For non-integer order α, $J_\alpha(z)$ and $J_{-\alpha}(z)$ are linearly independent, and therefore the two solutions of the differential equation (5.1). The $J_\alpha(z)$ and $J_{-\alpha}(z)$ are linearly dependent for α integer, hence the following relationship is valid [**1**, p. 358],

$$J_{-\alpha}(z)=(-1)^\alpha J_\alpha(z). \tag{5.3}$$

If z in (5.2) is replaced by the arguments $\pm iz$, then the solutions of the second order differential equation, $I_\alpha(z)$ are called the *modified Bessel functions* of the first kind. It is easily to verify from (5.2) that the modified Bessel function is defined by the following power series [**1**, p. 375],

$$I_\alpha(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+\alpha+1)\,n!}\left(\frac{z}{2}\right)^{2n+\alpha} \tag{5.4}$$

for $\alpha, z \in \mathbb{C}$.

We observe that the function $I_\alpha(z)$ has all the nonnegative coefficients. Similar to Bessel functions, the modified Bessel function (5.4) also, satisfies the following relations,

$$I_\alpha(-z)=(-1)^\alpha I_\alpha(z) \text{ and } I_{-\alpha}(z)=I_\alpha(z)$$

for $\alpha \in \mathbb{Z}, z \in \mathbb{C}$.

The modified Bessel functions of the first kind of order α, $I_\alpha(z)$, can be expressed by the Bessel function of the first kind, that is

$$J_\alpha(iz)=i^\alpha I_\alpha(z). \tag{5.5}$$

If we apply the inequality (2.13) for Bessel function (5.2) with its transform is the modified Bessel function (5.4), then we have the following corollary.

Corollary 9. *If $J_\alpha(x)$ and $I_\alpha(x)$ are the Bessel function and modified Bessel function for the first kind respectively, then we have*

$$I_\alpha\left(|x|^2\right) I_\alpha\left(|y|^2\right) - |I_\alpha(xy)|^2 \geq |J_\alpha(|x|\,x)J_\alpha(|y|\,\overline{y}) - J_\alpha(|y|\,x)J_\alpha(|x|\,\overline{y})| \tag{5.6}$$

for $\alpha \in \mathbb{R}, x, y \in \mathbb{C}$ with $|x|, |y| < 1$.

In particular, if $y = \alpha = 0$ in (5.6), then for any $|x| < 1$ we obtain that

$$I_0\left(|x|^2\right) - 1 \geq |J_0(|x|\,x) - 1)|$$

where $J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(k!)^2}\left(\frac{z}{2}\right)^{2k}$ and $I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}$, $|z| < 1$.

Other inequalities related to polylogarithm, hypergeometric, Bessel and modified Bessel functions for further reading can be found in the literature (see [**2**], [**3**], [**4**], [**5**], [**11**], [**12**], [**13**], [**14**], [**15**], [**16**], [**17**], [**18**], [**20**], [**21**], [**22**], [**23**], [**24**] and the references therein).

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