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SOME COMPANIONS OF FEJÉR'S INEQUALITY FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper some companions of Fejér's inequality for double integrals are established which generalize the inequalities of Hermite-Hadamard type from [4] and [9].

1. Introduction

It is well known in literature that a function $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$ if the inequality:

$$f(\lambda x+(1-\lambda)\,y)\leq\lambda f(x)+(1-\lambda)\,f(y),$$

holds for all $x,\,y\in[a,b]$ and $\lambda\in[0,1]$.

Many inequalities have been established for convex functions in past few years but the most famous is the Hermite-Hadamard's inequality (see [7, 8]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2},\tag{1.1}$$

due to its rich geometrical significance and applications.

The following inequality gives the weighted generalization of (1.1):

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx\leq\frac{1}{b-a}\int_a^b f(x)p(x)dx\leq\frac{f(a)+f(b)}{2}\int_a^b p(x)dx,\tag{1.2}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function and $p:[a,b]\to[0,\infty)$ is integrable and symmetric about $x=\frac{a+b}{2}$. The inequality (1.2) is known as Fejér's inequality for convex function.

The inequalities (1.1) and (1.2) have been generalized, extended and refined in a number of ways see for example [2, 3, 4, 6, 11, 12, 13, 14, 15, 16, 17, 18] and the references therein.

Let us consider a bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality:

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\leq\alpha f(x,y)+(1-\alpha)f(z,w),$$

holds, for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$.

Dragomir [4] (see also [2]) introduced a new concept of convexity, which is called the coordinated convexity, as follows:

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A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b], y \in [c, d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1. *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality:*

$$\begin{aligned} &f(tx + (1-t)y, su + (1-s)w) \\ &\leq tsf(x, u) + t(1-s)f(x, w) + s(1-t)f(y, u) + (1-t)(1-s)f(y, w) \end{aligned}$$

holds for all $t, s \in [0, 1]$ and $(x, y), (u, w) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see [4] or [2]).

In [4] an inequality of Hermite-Hadamard type for co-ordinated convex mappings on a rectangle from the plane was established and some properties of mappings associated to it were also discussed. D. Y. Hwang, K. L. Tseng and G. S. Yang considered a monotonic nondecreasing mapping connected with Hadamard type inequalities in two variables and some Hadamard type inequalities for Lipschitzian mapping in two variables were established as well in [17]. Recently M. Alomari and M. Darus [1], proved a Fejér inequality for double integrals and considered some mappings related to it to establish some inequalities for Lipschitzian mappings.

The main purpose of the present paper is to establish new companions of Fejér-type inequalities for co-ordinated convex functions on rectangle from the plane and hence generalizing the results from [4] and [9].

2. Main Results

In what follows let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a co-ordinated convex function and $p : [a, b] \times [c, d] \to [0, \infty)$ be integrable and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. We now define the following mappings on $[0, 1]^2$, associated with Fejér type inequality for double integrals proved in [1], by:

$$\begin{aligned} G(t, s) = \frac{1}{4} \Bigg[& f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) \\ & + f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) \\ & + f\left(tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) \\ & + f\left(tb + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) \Bigg], \end{aligned}$$

$$H(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dydx,$$

$$\begin{aligned}
I(t,s) &= \frac{1}{4}\int_a^b\int_c^d \left[f\left(t\frac{a+x}{2}+(1-t)\frac{a+b}{2}, s\frac{c+y}{2}+(1-s)\frac{c+d}{2}\right)\right.\\
&+ f\left(t\frac{a+x}{2}+(1-t)\frac{a+b}{2}, s\frac{y+d}{2}+(1-s)\frac{c+d}{2}\right)\\
&+ f\left(t\frac{b+x}{2}+(1-t)\frac{a+b}{2}, s\frac{c+y}{2}+(1-s)\frac{c+d}{2}\right)\\
&\left.+ f\left(t\frac{b+x}{2}+(1-t)\frac{a+b}{2}, s\frac{y+d}{2}+(1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx,
\end{aligned}$$

$$\begin{aligned}
L(t,s) &= \frac{1}{4(b-a)(d-c)}\int_a^b\int_c^d [f(ta+(1-t)x, sc+(1-s)y)\\
&+ f(ta+(1-t)x, td+(1-s)y) + f(tb+(1-t)x, sc+(1-s)y)\\
&+ f(tb+(1-t)x, sd+(1-s)y)]
\end{aligned}$$

and

$$\begin{aligned}
S_p(t,s) &= \frac{1}{4}\int_a^b\int_c^d \left[f\left(ta+(1-t)\frac{x+a}{2}, sc+(1-s)\frac{y+c}{2}\right)\right.\\
&+ f\left(ta+(1-t)\frac{x+a}{2}, sc+(1-s)\frac{y+d}{2}\right)\\
&+ f\left(ta+(1-t)\frac{x+b}{2}, sc+(1-s)\frac{y+c}{2}\right)\\
&+ f\left(ta+(1-t)\frac{x+b}{2}, sc+(1-s)\frac{y+d}{2}\right)\\
&+ f\left(tb+(1-t)\frac{x+a}{2}, sc+(1-s)\frac{y+c}{2}\right)\\
&+ f\left(tb+(1-t)\frac{x+a}{2}, sc+(1-s)\frac{y+d}{2}\right)\\
&+ f\left(tb+(1-t)\frac{x+b}{2}, sc+(1-s)\frac{y+c}{2}\right)\\
&\left.+ f\left(tb+(1-t)\frac{x+b}{2}, sc+(1-s)\frac{y+d}{2}\right)\right] p(x,y)dydx.
\end{aligned}$$

We will use the following lemma in the sequel of the paper:

Lemma 1. [1] *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and let*

$$a\le y_1\le x_1\le x_2\le y_2\le b \text{ with } x_1+x_2=y_1+y_2,$$

$$a\le w_1\le v_1\le v_2\le w_2\le b \text{ with } v_1+v_2=w_1+w_2.$$

Then, for the convex partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$, for all $x\in[a,b], y\in[c,d]$, respectively, the following hold:

$$f(x_1,v)+f(x_2,v)\le f(y_1,v)+f(y_2,v), \text{ for all } v\in[c,d], \tag{2.1}$$

and

$$f(t,v_1)+f(u,v_2)\le f(u,w_1)+f(u,w_2), \text{ for all } u\in[a,b]. \tag{2.2}$$

Theorem 1. *Let f, p, I be defined as above. Then the following inequality holds:*

$$
\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \\
&\leq 4\Bigg[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} f(x,y)\,p(4x-2a-b,4y-2c-d)\,dydx \\
&+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} f(x,y)\,p(4x-2a-b,4y-c-2d)\,dydx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+3b}{2}}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} f(x,y)\,p(4x-a-2b,4y-2c-d)\,dydx \\
&+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} f(x,y)\,p(4x-a-2b,4y-c-2d)\,dydx\Bigg] \\
&\leq \int_0^1\int_0^1 I(t,s)\,dsdt \\
&\leq \frac{1}{4}\Bigg[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \\
&+\int_a^b\int_c^d \frac{1}{4}\Bigg[f\left(\frac{a+x}{2},\frac{c+y}{2}\right) \\
&+f\left(\frac{a+x}{2},\frac{d+y}{2}\right) \\
&+f\left(\frac{b+x}{2},\frac{c+y}{2}\right) \\
&+f\left(\frac{b+x}{2},\frac{d+y}{2}\right) \\
&+f\left(\frac{a+b}{2},\frac{c+y}{2}\right) \\
&+f\left(\frac{a+b}{2},\frac{d+y}{2}\right) \\
&+f\left(\frac{x+a}{2},\frac{c+d}{2}\right) \\
&+f\left(\frac{x+b}{2},\frac{c+d}{2}\right)\Bigg]\,p(x,y)\,dydx\Bigg] \qquad (2.3)
\end{aligned}
$$

Proof. Using simple techniques of integration, under the assumptions on p, we have the following identities:

$$
\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \\
&=16\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)p(2x-a,2y-c)dsdtdydx, \qquad (2.4)
\end{aligned}
$$

$$4\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} f(x,y)\,p(4x-2a-b,4y-2c-d)\,dydx\right.$$
$$+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} f(x,y)\,p(4x-2a-b,4y-c-2d)\,dydx$$
$$+\int_{\frac{a+b}{2}}^{\frac{a+3b}{2}}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}} f(x,y)\,p(4x-a-2b,4y-c-2d)\,dydx$$
$$\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\int_{\frac{c+d}{2}}^{\frac{c+3d}{4}} f(x,y)\,p(4x-a-2b,4y-c-2d)\,dydx\right]$$
$$=4\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\int_{\frac{3c+d}{4}}^{\frac{c+d}{2}}\left[f(x,y)+f(a+b-x,y)+f(x,c+d-y)\right.$$
$$\left.+f(a+b-x,c+d-y)\right]p(4x-2a-b,4y-2c-d)\,dydx$$
$$=4\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac12}\int_0^{\frac12}\left[f\left(\frac{x}{2}+\frac{a+b}{4},\frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{x}{2}+\frac{a+b}{4},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right.$$
$$\left.+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right]$$
$$\times p(2x-a,2y-c)\,dsdtdydx \quad (2.5)$$

$$\int_0^1\int_0^1 I(t,s)dsdt=\int_a^b\int_c^d\int_0^{\frac12}\int_0^{\frac12}\frac14\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2},s\frac{c+y}{2}+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2},(1-s)\frac{c+y}{2}+s\frac{c+d}{2}\right)$$
$$+f\left((1-t)\frac{a+x}{2}+t\frac{a+b}{2},s\frac{c+y}{2}+(1-s)\frac{c+d}{2}\right)$$
$$\left.+f\left((1-t)\frac{a+x}{2}+t\frac{a+b}{2},(1-s)\frac{c+y}{2}+s\frac{c+d}{2}\right)\right]p(x,y)\,dsdtdydx$$
$$+\int_a^a\int_c^b\int_0^{\frac12}\int_0^{\frac12}\frac14\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2},s\frac{d+y}{2}+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2},(1-s)\frac{d+y}{2}+s\frac{c+d}{2}\right)$$
$$+f\left((1-t)\frac{a+x}{2}+t\frac{a+b}{2},s\frac{d+y}{2}+(1-s)\frac{c+d}{2}\right)$$
$$\left.+f\left((1-t)\frac{a+x}{2}+t\frac{a+b}{2},(1-s)\frac{d+y}{2}+s\frac{c+d}{2}\right)\right]p(x,y)\,dsdtdydx$$
$$+\int_a^b\int_c^d\int_0^{\frac12}\int_0^{\frac12}\frac14\left[f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2},s\frac{c+y}{2}+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2},(1-s)\frac{c+y}{2}+s\frac{c+d}{2}\right)$$
$$+f\left((1-t)\frac{x+b}{2}+t\frac{a+b}{2},s\frac{c+y}{2}+(1-s)\frac{c+d}{2}\right)$$

$$+f\left((1-t)\frac{b+x}{2}+t\frac{a+b}{2},(1-s)\frac{c+y}{2}+s\frac{c+d}{2}\right)\Bigg]p(x,y)\,dsdtdydx$$

$$+\int_a^b\int_c^d\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{1}{4}\Bigg[f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2},s\frac{d+y}{2}+(1-s)\frac{c+d}{2}\right)$$

$$+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2},(1-s)\frac{d+y}{2}+s\frac{c+d}{2}\right)$$

$$+f\left((1-t)\frac{x+b}{2}+t\frac{a+b}{2},s\frac{d+y}{2}+(1-s)\frac{c+d}{2}\right)$$

$$+f\left((1-t)\frac{b+x}{2}+t\frac{a+b}{2},(1-s)\frac{d+y}{2}+s\frac{c+d}{2}\right)\Bigg]p(x,y)\,dsdtdydx$$

$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg[f\left(t\frac{a+b}{2}+(1-t)x,s\frac{c+d}{2}+(1-s)y\right)$$

$$+f\left(t\frac{a+b}{2}+(1-t)x,sy+(1-s)\frac{c+d}{2}\right)+f\left(tx+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)y\right)$$

$$+f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\Bigg]p(2x-a,2y-c)\,dsdtdydx$$

$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg[f\left(t\frac{a+b}{2}+(1-t)x,s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$

$$+f\left(t\frac{a+b}{2}+(1-t)x,s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$

$$+f\left(tx+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$

$$+f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\Bigg]p(2x-a,2y-c)\,dsdtdydx$$

$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg[f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)y\right)$$

$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),sy+(1-s)\frac{c+d}{2}\right)$$

$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)y\right)$$

$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\Bigg]p(2x-a,2y-c)\,dsdtdydx$$

$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\Bigg[f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$

$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$

$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$

$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\Bigg]p(2x-a,2y-c)\,dsdtdydx \tag{2.6}$$

$$\frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx+\int_a^b\int_c^d\frac{1}{4}\left[f\left(\frac{a+x}{2},\frac{c+y}{2}\right)\right.\right.$$
$$+f\left(\frac{a+x}{2},\frac{d+y}{2}\right)+f\left(\frac{b+x}{2},\frac{c+y}{2}\right)+f\left(\frac{b+x}{2},\frac{d+y}{2}\right)+f\left(\frac{a+b}{2},\frac{c+y}{2}\right)$$
$$\left.\left.+f\left(\frac{a+x}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{d+y}{2}\right)+f\left(\frac{b+x}{2},\frac{c+d}{2}\right)\right]p(x,y)\,dydx\right.$$
$$=\frac{1}{4}\int_a^b\int_c^d\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+x}{2},\frac{c+y}{2}\right)\right.$$
$$+f\left(\frac{a+x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+y}{2}\right)+f\left(\frac{a+b}{2},\frac{c+y}{2}\right)$$
$$+f\left(\frac{a+x}{2},\frac{c+d}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+d}{2}\right)$$
$$\left.f\left(\frac{a+b}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)\right]p(x,y)\,dsdtdydx$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f(x,y)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.+f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},y\right)\right]p(2x-a,2y-c)\,dsdtdydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f(a+b-x,y)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.+f\left(a+b-x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},y\right)\right]p(2x-a,2y-c)\,dsdtdydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f(x,c+d-y)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.+f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c+d-y\right)\right]p(2x-a,2y-c)\,dsdtdydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f(a+b-x,c+d-y)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.+f\left(a+b-x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c+d-y\right)\right]p(2x-a,2y-c)\,dsdtdydx \quad (2.7)$$

By Lemma 1.1, the following inequalities hold for all $(t,s)\in\left[0,\frac{1}{2}\right]^2$ and $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$:

By setting $y_1=\frac{x}{2}+\frac{a+b}{4}$, $x_1=x_2=\frac{a+b}{2}$, $y_2=\frac{3(a+b)}{4}-\frac{x}{2}$ in (2.1) for $(x,v)\in[a,\frac{a+b}{2}]\times[c,\frac{c+d}{2}]$, we observe that

$$4f\left(\frac{a+b}{2},v\right)\le 2\left[f\left(\frac{x}{2}+\frac{a+b}{4},v\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2},v\right)\right], \quad (2.8)$$

holds.

Multiplying both sides of the inequality (2.8) by 4, replacing $v=\frac{c+d}{2}$ and then applying (2.2) for $w_1=\frac{y}{2}+\frac{c+d}{4}$, $v_1=v_2=\frac{c+d}{2}$, $w_2=\frac{3(c+d)}{4}-\frac{y}{2}$ for both of the expressions on right-side of (2.8), we have that

$$16f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\le 4\left[f\left(\frac{x}{2}+\frac{a+b}{4},\frac{y}{2}+\frac{c+d}{4}\right)\right.$$
$$+f\left(\frac{x}{2}+\frac{a+b}{4},\frac{3(c+d)}{4}-\frac{y}{2}\right)$$
$$+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)$$
$$\left.+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right], \quad (2.9)$$

holds.

Now by choosing $y_1=t\frac{a+b}{2}+(1-t)\,x$, $x_1=x_2=\frac{x}{2}+\frac{a+b}{4}$, $y_2=tx+(1-t)\,\frac{a+b}{2}$ in (2.1) for all $(x,v)\in[a,\frac{a+b}{2}]\times[c,\frac{c+d}{2}]$, the following inequality holds:

$$2f\left(\frac{x}{2}+\frac{a+b}{4},v\right)$$
$$\le 2f\left(t\frac{a+b}{2}+(1-t)\,x,v\right)+f\left(tx+(1-t)\,\frac{a+b}{2},v\right). \quad (2.10)$$

By replacing $y_1=t\,(a+b-x)+(1-t)\,\frac{a+b}{2}$, $x_1=x_2=\frac{3(a+b)}{4}-\frac{x}{2}$, $y_2=t\frac{a+b}{2}+(1-t)\,(a+b-x)$ in (2.1), for $v\in[c,\frac{c+d}{2}]$, we notice that

$$2f\left(\frac{3(a+b)}{4}-\frac{x}{2},v\right)$$
$$\le f\left(t\,(a+b-x)+(1-t)\,\frac{a+b}{2},v\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)\,(a+b-x),v\right), \quad (2.11)$$

holds.

Multiplying both sides of (2.10) and (2.11) by 2, setting $v=\frac{y}{2}+\frac{c+d}{2}$ and $v=\frac{3(c+d)}{4}-\frac{y}{2}$, then using (2.2), the following hold:

$$4f\left(\frac{x}{2}+\frac{a+b}{4},\frac{y}{2}+\frac{c+d}{4}\right)$$
$$\le f\left(t\frac{a+b}{2}+(1-t)\,x,s\frac{c+d}{2}+(1-s)\,y\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)\,x,sy+(1-s)\,\frac{c+d}{2}\right)$$
$$+f\left(tx+(1-t)\,\frac{a+b}{2},s\frac{c+d}{2}+(1-s)\,y\right)$$
$$+f\left(tx+(1-t)\,\frac{a+b}{2},sy+(1-s)\,\frac{c+d}{2}\right), \quad (2.12)$$

$$4f\left(\frac{x}{2}+\frac{a+b}{4},\frac{3(c+d)}{4}-\frac{y}{2}\right)$$
$$\leq f\left(t\frac{a+b}{2}+(1-t)x,s\frac{c+d}{2}+(1-s)(c+d-y)y\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)x,s(c+d-y)+(1-s)\frac{c+d}{2}\right)+$$
$$f\left(tx+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$
$$+f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right),\quad(2.13)$$

$$4f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)$$
$$\leq f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)(c+d-y)\right)\quad(2.14)$$

and

$$4f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)$$
$$\leq f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)y\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)y\right).\quad(2.15)$$

By setting $y_1=x$, $x_1=t\frac{a+b}{2}+(1-t)x$, $x_2=tx+(1-t)\frac{a+b}{2}$, $y_2=\frac{a+b}{2}$ in (2.1), the following holds:

$$f\left(t\frac{a+b}{2}+(1-t)x,v\right)+f\left(tx+(1-t)\frac{a+b}{2},v\right)$$
$$\leq f(x,v)+f\left(\frac{a+b}{2},v\right),\quad(2.16)$$

for all $v\in[c,\frac{c+d}{2}]$.
By the choice of $y_1=\frac{a+b}{2}$, $x_1=t(a+b-x)+(1-t)\frac{a+b}{2}$, $x_2=t\frac{a+b}{2}+(1-t)(a+b-x)$,

$y_2 = a + b - x$ in (2.1) for all $v \in [c, \frac{c+d}{2}]$, the following holds:

$$f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, v\right) + f\left(t\frac{a+b}{2} + (1-t)(a+b-x), v\right) \leq f(a+b-x, v) + f\left(\frac{a+b}{2}, v\right). \quad (2.17)$$

By respective settings $v = s\frac{c+d}{2} + (1-s)y$, $v = sy + (1-s)\frac{c+d}{2}$, $v = s\frac{c+d}{2} + (1-s)(c+y-y)$, $v = s(c+y-y) + (1-s)\frac{c+d}{2}$ in (2.16), and (2.17), using (2.2) for particular choices of w_1, w_2, v_1, v_2 and then summing up the resulting inequalities, we obtain

$$\begin{aligned} & f\left(t\frac{a+b}{2} + (1-t)x, s\frac{c+d}{2} + (1-s)y\right) \\ & + f\left(t\frac{a+b}{2} + (1-t)x, sy + (1-s)\frac{c+d}{2}\right) + f\left(tx + (1-t)\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)y\right) \\ & + f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \\ & \leq f(x,y) + f\left(x, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, y\right) + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right), \end{aligned} \quad (2.18)$$

$$\begin{aligned} & f\left(t\frac{a+b}{2} + (1-t)x, s\frac{c+d}{2} + (1-s)(c+d-y)\right) \\ & + f\left(t\frac{a+b}{2} + (1-t)x, s(c+d-y) + (1-s)\frac{c+d}{2}\right) \\ & + f\left(tx + (1-t)\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)(c+d-y)\right) \\ & + f\left(tx + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2}\right) \\ & \leq f(x, c+d-y) + f\left(x, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c+d-y\right) + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right), \end{aligned} \quad (2.19)$$

$$\begin{aligned} & f\left(t\frac{a+b}{2} + (1-t)(a+b-x), s\frac{c+d}{2} + (1-s)y\right) \\ & + f\left(t\frac{a+b}{2} + (1-t)(a+b-x), sy + (1-s)\frac{c+d}{2}\right) \\ & + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)y\right) \\ & + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \\ & \leq f(a+b-x, y) + f\left(a+b-x, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, y\right) + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \end{aligned} \quad (2.20)$$

and

$$
\begin{aligned}
& f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)(c+d-y)\right) \\
& \quad + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
& \quad + f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
& \quad + f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s\frac{c+d}{2}+(1-s)(c+d-y)\right) \\
& \qquad \le f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \qquad + f(a+b-x, c+d-y) \\
& \qquad + f\left(\frac{a+b}{2}, c+d-y\right) \\
& \qquad + f\left(a+b-x, \frac{c+d}{2}\right). \qquad (2.21)
\end{aligned}
$$

Multiplying the inequalities (2.9), (2.12), (2.13), (2.14), (2.15), (2.18), (2.19), (2.20) and (2.21) by $p(2x-a, 2y-c)$ and integrating respectively over t on $[0, \frac{1}{2}]$, over s on $[0, \frac{1}{2}]$, over x on $[a, \frac{a+b}{2}]$ and over y on $[c, \frac{c+d}{2}]$ and using the identities (2.4)-(2.7), we derive (2.3).
This completes the proof of the Theorem. □

Before presenting our next theorem, we first quote the following result from [10, Theorem 1, page 3] to be used in the sequel:

Theorem 2. [10, Theorem 1, page 3] *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a co-ordinated convex function and the mappings $I : [0, 1]^2 \to \mathbb{R}$ and $p : [a, b] \times [c, d] \to [0, \infty)$ be defined as above. Then we have*

1. *The mapping I is co-ordinated convex on $[0, 1]^2$,*
2. *The mapping I is co-ordinated monotonic nondecreasing on $[0, 1]^2$,*
3. *We have the bounds*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \\
& \quad = I(0,0) \le I(t,s) \le I(1,1) \\
& \quad = \frac{1}{4}\int_a^b\int_c^d \left[f\left(\frac{x+a}{2}, \frac{y+c}{2}\right) + f\left(\frac{x+a}{2}, \frac{y+d}{2}\right)\right. \\
& \quad \left. + f\left(\frac{x+b}{2}, \frac{y+c}{2}\right) + f\left(\frac{x+b}{2}, \frac{y+d}{2}\right)\right] p(x,y)dydx. \qquad (2.22)
\end{aligned}
$$

Theorem 3. *Let f, p, I be defined as above. Let f be twice differentiable on $[a, b] \times [c, d]$ and let the first order partial derivatives of f be co-ordinated convex.*

If p is bounded on $[a,b]\times[c,d]$, then

$$0\geq I(t,s)-\frac{1}{4}\int_a^b\int_c^d\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right.$$
$$\left.+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)dydx$$
$$\leq(1-t)(1-s)\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right.$$
$$\left.-\int_a^b\int_c^d f(x,y)\,dydx\right]\|p\|_\infty, \quad (2.23)$$

for all $(s,t)\in[0,1]^2$, where $\|p\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}|p(x,y)|$.

Proof. Utilizing the integration by parts, we have

$$\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\left[\frac{\partial^2 f(a+b-x,c+d-y)}{\partial x\partial y}\right.$$
$$\left.-\frac{\partial^2 f(a+b-x,y)}{\partial x\partial y}-\frac{\partial^2 f(x,c+d-y)}{\partial x\partial y}+\frac{\partial^2 f(x,y)}{\partial x\partial y}\right]dydx$$
$$=\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx$$
$$=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)+\int_a^b\int_c^d f(x,y)\,dydx$$
$$-\frac{d-c}{2}\int_a^b[f(x,c)+f(x,d)]\,dx-\frac{b-a}{2}\int_c^d[f(a,y)+f(b,y)]\,dy. \quad (2.24)$$

Using substitution rules for integration, under the assumptions on p, we also have the following identities:

$$\int_a^b\int_c^d\frac{1}{4}\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right.$$
$$\left.+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)dydx$$
$$=\int_a^b\int_c^d\frac{1}{4}\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{c+2d-y}{2}\right)\right.$$
$$\left.+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{y+c}{2}\right)\right]p(x,y)dydx$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}[f(x,y)+f(x,c+d-y)+f(a+b-x,y)$$
$$+f(a+b-x,c+d-y)]\,p(2x-a,2-cy)dydx \quad (2.25)$$

and

$$
\begin{aligned}
I(t,s) = \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} & \left[f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2} \right) \right. \\
& + f\left(tx + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2} \right) \\
& + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2} \right) \\
& \left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2} \right) \right] \\
& \times p(2x-a, 2y-c)dydx, \quad (2.26)
\end{aligned}
$$

for all $(t,s) \in [0,1]^2$.

Now using the co-ordinated convexity of the first order partial derivatives and that of f, under the assumptions on p, the inequality

$$
\begin{aligned}
& \left[f(a+b-x, c+d-y) - f\left(a+b-x, s(c+d-y) + (1-s)\frac{c+d}{2} \right) \right. \\
& - f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, c+d-y \right) \\
& \left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2} \right) \right] p(2x-a, 2y-c) \\
& + \left[f(a+b-x, y) - f\left(a+b-x, sy + (1-s)\frac{c+d}{2} \right) \right. \\
& - f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, y \right) \\
& \left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2} \right) \right] p(2x-a, 2y-c) \\
& + \left[f(x, c+d-y) - f\left(x, s(c+d-y) + (1-s)\frac{c+d}{2} \right) \right. \\
& - f\left(tx + (1-t)\frac{a+b}{2}, c+d-y \right) \\
& \left. + f\left(tx + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2} \right) \right] p(2x-a, 2y-c) \\
& + \left[f(x,y) - f\left(tx + (1-t)\frac{a+b}{2}, y \right) - f\left(x, sy + (1-s)\frac{c+d}{2} \right) \right. \\
& \left. - f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2} \right) \right] p(2x-a, 2y-c) \\
& \le (1-t)(1-s)\left(\frac{a+b}{2} - x \right)\left(\frac{c+d}{2} - y \right) \left[\frac{\partial^2 f(a+b-x, c+d-y)}{\partial x \partial y} \right. \\
& \left. - \frac{\partial^2 f(x, c+d-y)}{\partial x \partial y} - \frac{\partial^2 f(a+b-x, y)}{\partial x \partial y} + \frac{\partial^2 f(x,y)}{\partial x \partial y} \right] \|p\|_\infty, \quad (2.27)
\end{aligned}
$$

holds for all $(s,t) \in [0,1]^2$ and for all $(x,y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$.

Integrating (2.27) over $(x,y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$, using (2.24)-(2.26), the inequality (2.22), the inequality from [4, Theorem 1, page 778.] and the facts $I(t,1) \leq I(1,1)$, $I(1,s) \leq I(1,1)$, we have that the inequalities (2.23) hold.
Hence the theorem is established. □

We again quote a result from [10] to be used to prove our next theorem:

Corollary 1. [10] *Let f, p be defined as above. Then we have*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx \\ \leq & \frac{f\left(\frac{3a+b}{4}, \frac{3c+d}{2}\right) + f\left(\frac{3a+b}{2}, \frac{c+3d}{2}\right) + f\left(\frac{a+3b}{2}, \frac{3c+d}{2}\right) + f\left(\frac{a+3b}{2}, \frac{c+3d}{2}\right)}{4} \int_a^b \int_c^d p(x,y)dydx \\ & \leq \frac{1}{4} \int_a^b \int_c^d \left[f\left(\frac{x+a}{2}, \frac{y+c}{2}\right) + f\left(\frac{x+a}{2}, \frac{y+d}{2}\right) + f\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \right. \\ & \qquad \left. + f\left(\frac{x+b}{2}, \frac{y+d}{2}\right)\right] p(x,y)dydx \\ \leq & \frac{1}{4}\left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right] \int_a^b \int_c^d p(x,y)dydx \\ & \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \int_a^b \int_c^d p(x,y)dydx. \quad (2.28) \end{aligned}$$

Theorem 4. *Let f, p, I be defined as above. Let f be twice differentiable on $[a,b] \times [c,d]$ and let the first order partial derivatives of f be co-ordinated convex. Then*

$$\begin{aligned} 0 \geq I(t,s) - & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \int_a^b \int_c^d p(x,y)dydx \\ & \leq \frac{(b-a)(d-c)}{16} \times \\ & \left[\frac{\partial^2 f(a,d)}{\partial y \partial x} + \frac{\partial^2 f(b,c)}{\partial y \partial x} - \frac{\partial^2 f(a,c)}{\partial y \partial x} - \frac{\partial^2 f(b,d)}{\partial y \partial x}\right] \int_a^b \int_c^d p(x,y)dydx. \quad (2.29) \end{aligned}$$

Proof. By the co-ordinated convexity of first order partial derivatives and that of f, the following inequalities hold:

$$\frac{-f(a,c) + f\left(\frac{a+b}{2}, c\right) + f\left(a, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{4} \leq -\frac{(b-a)(d-c)}{16} \frac{\partial^2 f(a,c)}{\partial y \partial x},$$

$$\frac{-f(a,d) + f\left(\frac{a+b}{2}, d\right) + f\left(a, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{4} \leq \frac{(b-a)(d-c)}{16} \frac{\partial^2 f(a,d)}{\partial y \partial x},$$

$$\frac{-f(b,c) + f\left(\frac{a+b}{2}, c\right) + f\left(b, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{4} \leq \frac{(b-a)(d-c)}{16} \frac{\partial^2 f(b,c)}{\partial y \partial x},$$

and

$$\frac{-f(b,d) + f\left(\frac{a+b}{2}, d\right) + f\left(b, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{4} \leq -\frac{(b-a)(d-c)}{16} \frac{\partial^2 f(b,d)}{\partial y \partial x}.$$

Adding these results, multiplying the resulting by $p(x,y)$ then integrating over $(x,y)\in[a,b]\times[c,d]$, the following holds:

$$\begin{aligned}
&-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx\\
&\quad-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx+\frac{1}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)\right.\\
&\quad\left.+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\right]\int_a^b\int_c^d p(x,y)dydx\le\frac{(b-a)(d-c)}{16}\times\\
&\left[\frac{\partial^2 f(a,d)}{\partial y\partial x}+\frac{\partial^2 f(b,c)}{\partial y\partial x}-\frac{\partial^2 f(b,d)}{\partial y\partial x}-\frac{\partial^2 f(a,c)}{\partial y\partial x}\right]\int_a^b\int_c^d p(x,y)dydx. \qquad (2.30)
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\right]\int_a^b\int_c^d p(x,y)dydx\\
&\ge\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\int_a^b\int_c^d p(x,y)dydx\\
&\qquad\ge 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx. \qquad (2.31)
\end{aligned}$$

From (2.22), (2.28), (2.30) and (2.31), we get (2.29) and this completes the proof. □

Theorem 5. *Let f, p, I and G be defined as above. Then*

(1) *The following inequality holds for all $(t,s)\in[0,1]^2$:*

$$I(t,s)\le G(t,s)\int_a^b\int_c^d p(x,y)dydx. \qquad (2.32)$$

(2) *If f is twice differentiable on $[a,b]\times[c,d]$ and let the first order partial derivatives of f be co-ordinated convex. If p is bounded on $[a,b]\times[c,d]$, then*

$$\begin{aligned}
0&\ge f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx-I(t,s)\\
&\le(b-a)(d-c)\left[H(t,s)+G(t,s)-2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\|p\|_\infty, \qquad (2.33)
\end{aligned}$$

for all $(s,t)\in[0,1]^2$, where $\|p\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}|p(x,y)|$.

Proof. (1) Using simple techniques of integration, under the assumptions on p, we have that the following identity holds on $[0,1]^2$:

$$\begin{aligned} &G(t,s)\int_a^b\int_c^d p(x,y)dydx \\ &= \int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right. \\ &\quad + f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) \\ &\quad + f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) \\ &\quad \left. +f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right] \end{aligned} \tag{2.34}$$

By the choice of $y_1 = ta+(1-t)\frac{a+b}{2}$, $x_1 = tx+(1-t)\frac{a+b}{2}$, $x_2 = t(a+b-x)+(1-t)\frac{a+b}{2}$, $y_2 = tb+(1-t)\frac{a+b}{2}$ in (2.1) for all $v\in[c,\frac{c+d}{2}]$, $x\in[a,\frac{a+b}{2}]$, $(t,s)\in[0,1]^2$, the following holds:

$$\begin{aligned} &f\left(tx+(1-t)\frac{a+b}{2}, v\right) + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, v\right) \\ &\leq f\left(ta+(1-t)\frac{a+b}{2}, v\right) + f\left(tb+(1-t)\frac{a+b}{2}, v\right). \end{aligned} \tag{2.35}$$

By respective settings $v = sy+(1-s)\frac{c+d}{2}$, $v = s(c+d-y)+(1-s)\frac{c+d}{2}$ in (2.36), summing up the resulting inequalities side by side and using (2.2) for particular choices of w_1, w_2, v_1, v_2, we have the following inequality holds:

$$\begin{aligned} &f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) \\ &\quad + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) \\ &\quad f\left(tx+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\ &\quad + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\ &\leq f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) \\ &\quad + f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) \\ &\quad + f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) \\ &\quad + f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) \end{aligned} \tag{2.36}$$

Multiplying the inequality (2.36) by $p(2x-a, 2y-c)$, integrating both sides over $(x,y)\in[c,\frac{c+d}{2}]\times[a,\frac{a+b}{2}]$ and using (2.34), we derive (2.32).

(2) By integration by parts, we have

$$
\begin{aligned}
st & \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right) \\
& \times \left[\frac{\partial^2}{\partial x \partial y} f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right. \\
& -\frac{\partial^2}{\partial x \partial y} f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) \\
& -\frac{\partial^2}{\partial x \partial y} f\left(tx+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
& \left.+\frac{\partial^2}{\partial x \partial y} f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right] dydx \\
&= st \int_a^b \int_c^d \left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right) \\
& \times \frac{\partial^2}{\partial x \partial y} f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) dydx \\
&= (b-a)(d-c)\left[G(t,s)+H(t,s)\right] \\
& -\frac{b-a}{2}\int_c^d \left\{f\left(ta+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right. \\
& \left.+f\left(tb+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right\} dy \\
& -\frac{d-c}{2}\int_a^b \left\{f\left(tx+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right. \\
& \left.+f\left(tx+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right\} dx. \qquad (2.37)
\end{aligned}
$$

By using the co-ordinated convexity of the first order partial derivatives and that of f, under the assumptions on p, the following inequality holds:

$$
\begin{aligned}
& \left[f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right. \\
& \left.-f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right] p(x,y) \\
& -\left[f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right] p(x,y) \\
& +\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right. \\
& \left.-f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right] p(x,y) \\
& -\left[f\left(\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right] p(x,y) \\
& +\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&-f\left(t(a+b-x)+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\quad-\left[f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]p(x,y)\\
&\quad+\Bigg[f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&\qquad-f\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\quad-\left[f\left(\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]p(x,y)\\
&\le st\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\Bigg[\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&\quad+\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\Bigg]\|p\|_\infty,
\end{aligned}
\tag{2.38}
$$

for all $(t,s)\in[0,1]^2$ and $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$.

Integrating the inequality (2.38) over $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$, under the assumptions on p, using the facts $I(0,s)\le I(t,s)$, $I(t,0)\le I(t,s)$ and (2.37), we get

$$
\begin{aligned}
0\ge f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\int_a^b\int_c^d p(x,y)dydx-I(t,s)\le(b-a)(d-c)\left[H(t,s)+G(t,s)\right]\|p\|_\infty\\
&-\Bigg[\frac{b-a}{2}\int_c^d\Bigg\{f\left(ta+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\
&\quad+f\left(tb+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\Bigg\}\\
&+\frac{d-c}{2}\int_a^b\Bigg\{f\left(tx+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\
&\quad+f\left(tx+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\Bigg\}dx\Bigg]\|p\|_\infty.
\end{aligned}
\tag{2.39}
$$

Since f is convex on the co-ordinates, by Jensen's inequality for integrals the followings hold:

$$
\begin{aligned}
&\frac{b-a}{2}\int_c^d f\left(ta+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\\
&\quad+\frac{b-a}{2}\int_c^d f\left(tb+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\\
&\qquad\ge(b-a)(d-c)G(t,0)\ge(b-a)(d-c)G(0,0),
\end{aligned}
\tag{2.40}
$$

and

$$\begin{aligned}
&\frac{d-c}{2}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)dx \\
&\qquad + \frac{d-c}{2}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)dx \\
&\qquad\qquad \geq (b-a)(d-c)\,G(0,s) \\
&\qquad\qquad\qquad \geq (b-a)(d-c)\,G(0,0). \qquad (2.41)
\end{aligned}$$

By using (2.40) and (2.41) in (2.39), we get (2.33).
This completes the proof of the theorem. □

Theorem 6. *Let f, p, G, I, S_p be defined as above, then S_p is co-ordinated convex on $[0,1]^2$ and we have the following inequalities:*

$$\begin{aligned}
G(t,s)\int_a^b\int_c^d p(x,y)dydx &\leq S_p(t,s) \\
&\leq (1-t)(1-s)\int_a^b\int_c^d f(x,y)p(x,y)dydx \\
&+ ts\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx \\
&+ \frac{1}{2}t(1-s)\int_a^b\int_c^d [f(a,y)+f(b,y)]\,p(x,y)dydx \\
&+ \frac{s(1-t)}{2}\int_a^b\int_c^d [f(x,c)+f(x,d)]\,p(x,y)dydx \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx, \qquad (2.42)
\end{aligned}$$

$$I(1-t,1-s) \leq S_p(t,s) \qquad (2.43)$$

and

$$\frac{I(1-t,1-s)+I(t,s)}{2} \leq S_p(t,s). \qquad (2.44)$$

Moreover, the following bound is true:

$$\sup_{(t,s)\in[0,1]^2} S_p(t,s) = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx. \qquad (2.45)$$

Proof. Co-ordinated convexity of S_p directly follows from co-ordinated convexity of f.
By simple techniques of integration, under the assumption on p, the following does

hold:

$$\begin{aligned}
S_p(t,s) = \frac{1}{4}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} & [f\left(ta+(1-t)x, sc+(1-s)y\right) \\
& + f\left(ta+(1-t)x, sc+(1-s)(c+d-y)\right) \\
& + f\left(ta+(1-t)(a+b-x), sc+(1-s)y\right) \\
& f\left(ta+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right) \\
& + f\left(ta+(1-t)(a+b-x), sd+(1-s)y\right) \\
& + f\left(ta+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right) \\
& + f\left(ta+(1-t)x, sd+(1-s)y\right) \\
& + f\left(ta+(1-t)x, sd+(1-s)(c+d-y)\right) \\
& + f\left(tb+(1-t)x, sc+(1-s)y\right) \\
& + f\left(tb+(1-t)x, sc+(1-s)(c+d-y)\right) \\
& + f\left(tb+(1-t)(a+b-x), sc+(1-s)y\right) \\
& + f\left(tb+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right) \\
& + f\left(tb+(1-t)(a+b-x), sd+(1-s)y\right) \\
& + f\left(tb+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right) \\
& + f\left(tb+(1-t)x, sd+(1-s)(c+d-y)\right) \\
& + f\left(tb+(1-t)x, sd+(1-s)y\right)]\, p(2x-a, 2y-c)dydx, \quad (2.46)
\end{aligned}$$

for all $(s,t) \in [0,1]^2$.
By setting, $y_1 = ta+(1-t)x$, $x_1 = x_2 = ta+(1-t)\frac{a+b}{2}$, $y_2 = ta+(1-t)(a+b-x)$ in (2.1) for respective settings $v = sc+(1-s)\frac{c+d}{2}$ and $v = sd+(1-s)\frac{c+d}{2}$, the following hold true:

$$\begin{aligned}
2f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) & \\
\leq f\left(ta+(1-t)x, sc+(1-s)\frac{c+d}{2}\right) & \\
+ f\left(ta+(1-t)(a+b-x), sc+(1-s)\frac{c+d}{2}\right) & \quad (2.47)
\end{aligned}$$

and

$$\begin{aligned}
2f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) & \\
\leq f\left(ta+(1-t)x, sd+(1-s)\frac{c+d}{2}\right) & \\
+ f\left(ta+(1-t)(a+b-x), sd+(1-s)\frac{c+d}{2}\right), & \quad (2.48)
\end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$ and for all $(s,t) \in [0,1]^2$. □

Multiplying (2.47) and (2.48) by 2, then adding and using (2.2) for the particular choices of w_1, w_2, v_1 and v_2, we get

$$\begin{aligned}
4&\left[f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.+f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right]\\
&\leq f\left(ta+(1-t)x, sc+(1-s)y\right)+f\left(ta+(1-t)x, sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sc+(1-s)y\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(ta+(1-t)x, sd+(1-s)y\right)+f\left(ta+(1-t)x, sd+(1-s)(c+d-y)\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sd+(1-s)y\right)\\
&\quad+f\left(ta+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right). \qquad (2.49)
\end{aligned}$$

Analogously

$$\begin{aligned}
4&\left[f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.+f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right]\\
&\leq f\left(tb+(1-t)x, sc+(1-s)y\right)+f\left(tb+(1-t)x, sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sc+(1-s)y\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right)\\
&\quad+f\left(tb+(1-t)x, sd+(1-s)y\right)+f\left(tb+(1-t)x, sd+(1-s)(c+d-y)\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sd+(1-s)y\right)\\
&\quad+f\left(tb+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right). \qquad (2.50)
\end{aligned}$$

Multiplying the inequalities (2.49) and (2.50) by $p(2x-a, 2y-c)$, integrating the resulting over $(x,y)\in\left[a, \frac{a+b}{2}\right]\times\left[c, \frac{c+d}{2}\right]$ and making use of identities (2.34) and (2.46), the first inequality of (2.42) holds true.
By using the co-ordinated convexity of f and the inequalities:

$$\int_a^b\int_c^d f(a,y)p(x,y)dydx \leq \frac{f(a,c)+f(a,d)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(b,y)p(x,y)dydx \leq \frac{f(b,c)+f(b,d)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(x,c)p(x,y)dydx \leq \frac{f(a,c)+f(b,c)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(x,d)p(x,y)dydx \leq \frac{f(a,d)+f(b,d)}{2}\int_a^b\int_c^d p(x,y)dydx$$

and

$$\int_a^b\int_c^d f(x,y)p(x,y)dydx \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx,$$

we get second inequalities of (2.42).
Again using the co-ordinated convexity of f

$$
\begin{aligned}
&I(1-t,1-s)\\
&=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\Big[f\left((1-t)(a+b-x)+t\frac{a+b}{2},(1-s)(c+d-y)+s\frac{c+d}{2}\right)\\
&\quad+f\left((1-t)x+t\frac{a+b}{2},(1-s)(c+d-y)+s\frac{c+d}{2}\right)\\
&\quad+f\left((1-t)(a+b-x)+t\frac{a+b}{2},(1-s)y+s\frac{c+d}{2}\right)\\
&\quad+f\left((1-t)x+t\frac{a+b}{2},(1-s)y+s\frac{c+d}{2}\right)\Big]p(2x-a,2y-c)dydx\\
&=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\Big[f\left(\frac{ta+(1-t)(a+b-x)}{2}+\frac{tb+(1-t)(a+b-x)}{2},\right.\\
&\quad\left.\frac{sc+(1-s)(c+d-y)}{2}+\frac{sd+(1-s)(c+d-y)}{2}\right)\\
&+f\left(\frac{ta+(1-t)x}{2}+\frac{tb+(1-t)x}{2},\frac{sc+(1-s)(c+d-y)}{2}+\frac{sd+(1-s)(c+d-y)}{2}\right)\\
&+f\left(\frac{ta+(1-t)(a+b-x)}{2}+\frac{tb+(1-t)(a+b-x)}{2},\frac{sc+(1-s)y}{2}+\frac{sd+(1-s)y}{2}\right)\\
&\quad+f\left(\frac{ta+(1-t)x}{2}+\frac{tb+(1-t)x}{2},\frac{sc+(1-s)y}{2}+\frac{sd+(1-s)y}{2}\right)\\
&\qquad p(2x-a,2y-c)dydx\le S_p(t,s).
\end{aligned}
$$

This proves 2.43.
From (2.32), (2.42) and (2.43), we get (2.44). Using (2.42), we get (2.45). This completes the proof.

Remark 1. *If $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$, then $I(t,s)=H(t,s)$ and $S_p(t,s)=L(t,s)$ for all $(t,s)\in[0,1]^2$ and hence from all the above Theorems we get the inequalities related to the mappings H, G and L*

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