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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES IN ABSOLUTE VALUE ARE CONVEX WITH APPLICATIONS

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Abstract. In this paper some new Hadamard-type inequalities for functions whose derivatives in absolute values are convex are established. Some applications to special means of real numbers are given. Finally, we also give some applications of our obtained results to get new error bounds for the sum of the midpoint and trapezoidal formulae.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamar's inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard's inequality [4] has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see [1]-[14] and the references therein.

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In a recent paper [12], K. L. Tseng et al., established the following result which gives a refinement of (1.1):

$$f\left(\frac{a+b}{2}\right)\leq\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\leq\frac{1}{b-a}\int_a^b f(x)dx \leq\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2}, \quad (1.2)$$

where $f:[a,b]\to\mathbb{R}$, is a convex function (see [12, Remark 2.11, page7.]).

The main aim of this paper is to establish some new Hermite-Hadamard type inequalities which give an estimate between $\frac{1}{b-a}\int_a^b f(x)dx$ and $\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}$ for functions whose derivatives in absolute value are convex and as a consequence we will get refinements of those results which have been established to estimate the difference between the middle and the leftmost terms in (1.1).

2. Main Results

To prove our results we need the following lemma:

Lemma 1. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° , the interior of I, where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\\&=\frac{b-a}{16}\left[\int_0^1 tf'\left(t\frac{3a+b}{4}+(1-t)\,a\right)dt\right.\\&+\int_0^1(t-1)\,f'\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)dt\\&+\int_0^1 tf'\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)dt\\&\left.+\int_0^1(t-1)\,f'\left(tb+(1-t)\,\frac{a+3b}{4}\right)dt\right].\quad(2.1)\end{aligned}$$

Proof. By integration by parts and by making use of the substitution $u=t\frac{3a+b}{4}+(1-t)\,a$, we have

$$\begin{aligned}&\frac{b-a}{16}\int_0^1 tf'\left(t\frac{3a+b}{4}+(1-t)\,a\right)dt\\&=\frac{b-a}{16}\left[\left.\frac{4tf'\left(t\frac{3a+b}{4}+(1-t)\,a\right)}{b-a}\right|_0^1-\frac{4}{b-a}\int_0^1 f\left(t\frac{3a+b}{4}+(1-t)\,a\right)dt\right]\\&=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}f(u)\,du.\quad(2.2)\end{aligned}$$

Analogously, we also have the following equalities:

$$\frac{b-a}{16}\int_0^1 (t-1)\, f'\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)dt$$
$$=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f\,(u)\,du, \quad (2.3)$$

$$\frac{b-a}{16}\int_0^1 tf'\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)dt$$
$$=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{1}{b-a}\int_{\frac{x+b}{2}}^{\frac{x+3b}{4}} f\,(u)\,du \quad (2.4)$$

and

$$\frac{b-a}{16}\int_0^1 (t-1)\, f'\left(tb+(1-t)\,\frac{a+3b}{4}\right)dt$$
$$=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{1}{b-a}\int_{\frac{x+3b}{4}}^{b} f\,(u)\,du. \quad (2.5)$$

Adding (2.2)-(2.5), we get the desired equality. This completes the proof of the lemma. □

Using the Lemma 1 the following results can be obtained:

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f\,(u)\,du\right| \le \left(\frac{b-a}{96}\right)\left[\left|f'\,(a)\right|+4\left|f'\left(\frac{3a+b}{4}\right)\right|\right.$$
$$\left.+2\left|f'\left(\frac{a+b}{2}\right)\right|+4\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\,(b)\right|\right]. \quad (2.6)$$

Proof. Using Lemma 1 and taking the modulus, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f\,(u)\,du\right|$$
$$\le \frac{b-a}{16}\left[\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|dt\right.$$
$$+\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)\right|dt$$
$$+\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)\right|dt$$
$$\left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\,\frac{a+3b}{4}\right)\right|dt\right]. \quad (2.7)$$

Using the convexity of $|f'|$ on $[a,b]$, we observe that the following inequality holds:

$$\begin{aligned}\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|dt \\ \leq \left|f'\left(\frac{3a+b}{4}\right)\right|\int_0^1 t^2dt+|f'(a)|\int_0^1 t\,(1-t)\,dt \\ =\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|+\frac{1}{6}\,|f'(a)|\,. \quad (2.8)\end{aligned}$$

Similarly, we also have that the following inequalities hold:

$$\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)\right|dt \leq \frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|, \quad (2.9)$$

$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)\right|dt \leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|+\frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|, \quad (2.10)$$

and

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\,\frac{a+3b}{4}\right)\right|dt \leq \frac{1}{6}\,|f'(b)|+\frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|. \quad (2.11)$$

Utilizing the inequalities (2.8)-(2.11), we get (2.6).
This completes the proof of the theorem. □

Corollary 1. *Suppose all the conditions of Theorem 1 are satisfied. Then*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{b-a}{16}\right)\left[|f'(a)|+|f'(b)|\right]. \quad (2.12)$$

Moreover, if $|f'(x)| \leq M$, for all $x \in [a,b]$, then we have also the following inequality:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{b-a}{8}\right)M. \quad (2.13)$$

Proof. It follows from Theorem 1 and using the convexity of $|f'|$. □

Theorem 2. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right) \\ \times\left\{\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ \left.+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right\}, \quad (2.14)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{16}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (2.15)$$

Since $|f'|^q$ is convex on $[a,b]$, we have

$$\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt$$
$$\leq \left|f'\left(\frac{3a+b}{4}\right)\right|^q\int_0^1 tdt+|f'(a)|^q\int_0^1(1-t)\,dt$$
$$=\frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{2}|f'(a)|^q.$$

Similarly,

$$\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\leq\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$
$$\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\leq\frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q$$

and

$$\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\leq\frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{2}|f'(b)|^q.$$

Using the last four inequalities in (2.15), we get the inequality (2.14), which completes the proof of the theorem. □

Corollary 2. *Suppose all the conditions of Theorem 2 are satisfied. Then*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}}\left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\left(\frac{b-a}{16}\right)\left[|f'(a)|+|f'(b)|\right]. \quad (2.16)$$

Proof. It follows from Theorem 2 using the convexity of $|f'|^q$ and thae fact

$$\sum_{k=1}^n (u_k+v_k)^s\leq\sum_{k=1}^n (u_k)^s+\sum_{k=1}^n (v_k)^s\,,u_k,v_k\geq 0,1\leq k\leq n,0\leq s<1.$$

□

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right)$$
$$\times\left\{\left(|f'(a)|^q+2\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{\frac{1}{q}}+\left(2\left|f'\left(\frac{a+3b}{4}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right\}. \tag{2.17}$$

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{16}\left[\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \tag{2.18}$$

Since $|f'|^q$ is convex on $[a,b]$, we have

$$\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt$$
$$\leq \left|f'\left(\frac{3a+b}{4}\right)\right|^q\int_0^1 t^2 dt+|f'(a)|^q\int_0^1 t(1-t)\,dt$$
$$=\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{6}|f'(a)|^q.$$

Analogously, we alos have that the following inequalities:

$$\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt$$
$$\leq \frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{3}\left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$

$$\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{6}\left|f'\left(\frac{a+b}{2}\right)\right|^q$$

and

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q \leq \frac{1}{3}\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \frac{1}{6}\left|f'(b)\right|^q.$$

By making use of the last four inequalities in (2.18), we get (2.17). Hence the proof of the theorem is complete. □

Corollary 3. *Suppose all the conditions of Theorem 2 are satisfied. Then using similar arguments as in Corollary 2, we get the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)\left[1+2^{\frac{1}{q}}+\left(\frac{1}{2}\right)^{\frac{1}{q}}+\left(\frac{5}{2}\right)^{\frac{1}{q}}\right]\left(\frac{b-a}{16}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \quad (2.19)$$

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$, then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{16}\right)\left\{\left|f'\left(\frac{7a+b}{8}\right)\right|+\left|f'\left(\frac{5a+3b}{8}\right)\right| + \left|f'\left(\frac{3a+5b}{8}\right)\right|+\left|f'\left(\frac{a+7b}{8}\right)\right|\right\}, \quad (2.20)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality for $q > 1$ and $p = \frac{q}{q-1}$, we have

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| &\leq \left(\frac{b-a}{16}\right)\left[\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &+\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\left.+\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (2.21)\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$ so by using the inequality (1.1), we obtain:

$$\int_0^1 \left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|^q dt \le \left|f'\left(\frac{\frac{3a+b}{4}+a}{2}\right)\right|^q = \left|f'\left(\frac{7a+b}{8}\right)\right|^q$$

Analogously, we have that the following inequalities:

$$\int_0^1 \left|f'\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)\right|^q dt \le \left|f'\left(\frac{5a+3b}{8}\right)\right|^q,$$

$$\int_0^1 \left|f'\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)\right|^q dt \le \left|f'\left(\frac{3a+5b}{8}\right)\right|^q$$

and

$$\int_0^1 \left|f'\left(tb+(1-t)\,\frac{a+3b}{4}\right)\right|^q dt \le \left|f'\left(\frac{a+7b}{8}\right)\right|^q.$$

Using the last four inequalities in (2.21), we get (2.20). This completes the proof of the theorem. □

Corollary 4. *Suppose all the assumptions of Theorem 4 are satisfied and assume that $|f'|$ is a linear map, then we get the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{8}\right)\left|f'(a+b)\right|. \tag{2.22}$$

Proof. It is a direct consequence of Theorem 4 and using the linearity of $|f'|$. □

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| &\le \left(\frac{b-a}{32}\right)\left[\left|f'\left(\frac{13a+3b}{12}\right)\right|+\left|f'\left(\frac{11a+5b}{12}\right)\right|\right.\\ &\left.+\left|f'\left(\frac{5a+13b}{12}\right)\right|+\left|f'\left(\frac{3a+13b}{12}\right)\right|\right].\end{aligned} \tag{2.23}$$

Proof. First, by the concavity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we note that

$$\begin{aligned}|f(\lambda x+(1-\lambda)\,y)|^q &\ge \lambda\,|f(x)|^q+(1-\lambda)\,|f(y)|^q\\ &\ge (\lambda\,|f(x)|+(1-\lambda)\,|f(y)|)^q\end{aligned}$$

and hence

$$|f(\lambda x+(1-\lambda)\,y)| \ge \lambda\,|f(x)|+(1-\lambda)\,|f(y)|,$$

for all $\lambda \in [0,1]$ and $x,\, y \in [a,b]$. This shows that $|f'|$ is also concave on $[a,b]$. Accordingly, using Lemma 1 and the Jensens's integral inequality, we have

$$\begin{aligned}
&\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq \left(\frac{b-a}{16}\right)\left[\int_0^1 t\left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|dt\right. \\
&+\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt \\
&\int_0^1 t\left|f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \\
&\left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|dt\right] \\
&\leq \frac{b-a}{16}\left[\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{3a+b}{4}+(1-t)\,a\right)dt}{\int_0^1 tdt}\right)\right|\right] \\
&+\left(\int_0^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_0^1 (1-t)\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)dt}{\int_0^1 (1-t)\,dt}\right)\right| \\
&+\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)dt}{\int_0^1 tdt}\right)\right| \\
&+\left(\int_0^1 (1-t)\,dt\right)\left|f'\left(\frac{\int_0^1 (1-t)\left(tb+(1-t)\frac{a+3b}{4}\right)dt}{\int_0^1 (1-t)\,dt}\right)\right|,
\end{aligned}$$

which is equivalent to (2.23) and the proof of the theorem is complete. □

Corollary 5. *Suppose all the assumptions of Theorem 5 are satisfied and assume that $|f'|$ is a linear map, then we have the following inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{b-a}{12}\right)\left|f'(a+b)\right|. \tag{2.24}$$

Proof. It follows from Theorem 5 and using the linearity of $|f'|$. □

3. Applications to Special Means

Now, we consider the applications of our Theorems to the special means. We consider the means for arbitrary real numbers $a,\, b \in \mathbb{R}$. We take

(1) The arithmetic mean:

$$A(a,b)=\frac{a+b}{2};\ a,\, b\in\mathbb{R}.$$

(2) The harmonic mean:

$$H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}};\ a,b\in\mathbb{R}\backslash\{0\}.$$

(3) The logarithmic mean:

$$L(a,b)=\frac{\ln|b|-\ln|a|}{b-a};a,b\in\mathbb{R},|a|\neq|b|,a,b\neq 0.$$

(4) Generalized log-mean:

$$L_n(a,b)=\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}};\ a,\ b\in\mathbb{R},\ n\in\mathbb{Z}\backslash\{-1,0\},\ a\neq b.$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let $a,\ b\in\mathbb{R},\ a<b,\ 0\notin[a,b]$ and $n\in\mathbb{Z},\ |n|\geq 2$. Then*

$$\left|A^n(a,b)+A(a^n,b^n)-2L_n^n(a,b)\right|\leq|n|\left(\frac{b-a}{4}\right)A\left(|a|^{n-1},|b|^{n-1}\right). \tag{3.1}$$

Proof. The assertion follows from Corollary 1 when applied to the function $f(x)=x^n,\ x\in\mathbb{R},\ n\in\mathbb{Z},\ |n|\geq 2$. □

Proposition 2. *Let $a,\ b\in\mathbb{R},\ a<b,\ 0\notin[a,b]$ and $n\in\mathbb{Z},\ |n|\geq 2$. Then for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\begin{aligned}&\left|A^n(a,b)+A(a^n,b^n)-2L_n^n(a,b)\right|\\ &\leq|n|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}+1}\left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\left(\frac{b-a}{2}\right)A\left(|a|^{n-1},|b|^{n-1}\right).\end{aligned} \tag{3.2}$$

Proof. The assertion follows from Corollary 2 when applied to the function $f(x)=x^n,\ x\in\mathbb{R},\ n\in\mathbb{Z},\ |n|\geq 2$. □

Proposition 3. *Let $a,\ b\in\mathbb{R},\ a<b,\ 0\notin[a,b]$ and $n\in\mathbb{Z},\ |n|\geq 2$. Then $q\geq 1$, we have*

$$\begin{aligned}&\left|A^n(a,b)+A(a^n,b^n)-2L_n^n(a,b)\right|\\ &\qquad\leq|n|\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{2}{q}}\left[1+5^{\frac{1}{q}}+7^{\frac{1}{q}}+11^{\frac{1}{q}}\right]\left(\frac{b-a}{8}\right)A\left(|a|^{n-1},|b|^{n-1}\right).\end{aligned} \tag{3.3}$$

Proof. The assertion follows from Corollary 3 when applied to the function $f(x)=x^n,\ x\in\mathbb{R},\ n\in\mathbb{Z},\ |n|\geq 2$. □

Proposition 4. *Let $a,\ b\in\mathbb{R},\ a<b,\ 0\notin[a,b]$. Then*

$$\left|A^{-1}(a,b)+A\left(a^{-1},b^{-1}\right)-2L^{-1}(a,b)\right|\leq\left(\frac{b-a}{4}\right)A\left(|a|^{-2},|b|^{-2}\right). \tag{3.4}$$

Proof. It is a direct consequence of Corollary 1 when applied to the function, $f(x)=\frac{1}{x},\ x\in[a,b]$. □

Proposition 5. *Let $a,\ b\in\mathbb{R},\ a<b,\ 0\notin[a,b]$, then for all $p>1$, we have*

$$\begin{aligned}&\left|A^{-1}(a,b)+A\left(a^{-1},b^{-1}\right)-2L^{-1}(a,b)\right|\\ &\qquad\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}+1}\left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\left(\frac{b-a}{2}\right)A\left(|a|^{-2},|b|^{-2}\right).\end{aligned} \tag{3.5}$$

Proof. It follows directly from Corollary 2 for the function, $f(x)=\frac{1}{x},\ x\in[a,b]$. □

Proposition 6. *Let $a,\ b \in \mathbb{R},\ a < b,\ 0 \notin [a,b]$. Then for all $q \geq 1$, we have the inequality*

$$\begin{aligned}&\left|A^{-1}(a,b) + A\left(a^{-1},b^{-1}\right) - 2L^{-1}(a,b)\right| \\ &\quad \leq \left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{2}{q}}\left[1+5^{\frac{1}{q}}+7^{\frac{1}{q}}+11^{\frac{1}{q}}\right]\left(\frac{b-a}{8}\right)A\left(|a|^{-2},|b|^{-2}\right). \quad (3.6)\end{aligned}$$

Proof. It follows directly from Corollary 3 for the function, $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

Remark 1. *We can get several inequalities for means from Corollary 4 and Corollary 5 for a particular choice of the concave function f but we omit the details for the interested reader.*

Remark 2. *Let $a,\ b \in \mathbb{R}\backslash\{0\}$, $a < b$ then $a^{-1} > b^{-1}$ and $A^{-1}\left(a^{-1},b^{-1}\right) = \frac{2}{\frac{1}{a}+\frac{1}{b}} = H(a,b)$. Hence one can get several inequalities containing harmonic mean and logarithmic means and we omit the details for the interested readers.*

4. Application to the Midpoint and Trapezoidal Formulae

Let d be a division of the interval $[a,b]$, *i.e.* $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, and consider the quadrature formulae

$$\int_a^b f(x)dx = M(f,d) + E(f,d),$$

and

$$\int_a^b f(x)dx = T'(f,d) + E'(f,d),$$

where

$$T(f,d) = \sum_{i=0}^{n-1}(x_{i+1}-x_i)\, f\left(\frac{x_i+x_{i+1}}{2}\right)$$

and

$$T'(f,d) = \sum_{i=0}^{n-1}(x_{i+1}-x_i)\,\frac{f(x_i)+f(x_{i+1})}{2}$$

are the midpoint and trapezoidal versions and $E(f,d)$ and $E'(f,d)$ are the associated errors respectively. Here, we derive some error estimates for the sum of midpoint and trapezoidal formulae.

Proposition 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then for every division d of $[a,b]$, we have:*

$$|E(f,d) + E'(f,d)| \leq \frac{1}{8}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)| + |f'(x_{i+1})|\right]. \quad (4.1)$$

Proof. By applying Corollary 1 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division d, we have

$$\begin{aligned}&\left|f\left(\frac{x_i+x_{i+1}}{2}\right) + \frac{f(x_{i+1})+f(x_i)}{2} - \frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)\,dx\right| \\ &\qquad \leq \left(\frac{x_{i+1}-x_i}{8}\right)\left[|f'(x_i)| + |f'(x_{i+1})|\right]. \quad (4.2)\end{aligned}$$

Now

$$
\begin{aligned}
&|E(f,d)+E'(f,d)| \\
&\quad = \left| \sum_{i=0}^{n-1} (x_{i+1}-x_i) f\left(\frac{x_i+x_{i+1}}{2}\right) + \sum_{i=0}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2} (x_{i+1}-x_i) \right. \\
&\qquad\qquad \left. -2\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx \right| \\
&\le \sum_{i=0}^{n-1} (x_{i+1}-x_i) \left| f\left(\frac{x_i+x_{i+1}}{2}\right) + \frac{f(x_i)+f(x_{i+1})}{2} - \frac{2}{x_{i+1}-x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right|.
\end{aligned} \tag{4.3}
$$

Using (4.2) in (4.3), we get (4.1). This completes the proof of the proposition. □

Proposition 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$, then for every division d of $[a,b]$, we have*

$$
\begin{aligned}
&|E(f,d)+E'(f,d)| \\
&\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{3}{q}+3} \left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right] \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left[|f'(x_i)|+|f'(x_{i+1})|\right],
\end{aligned} \tag{4.4}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The proof is similar to that of Proposition 1 and using Corollary 2. □

Proposition 9. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then for every division d of $[a,b]$, we have*

$$
\begin{aligned}
|E(f,d)+E'(f,d)| \le \left(\frac{1}{3}\right)^{\frac{1}{q}} \left(\frac{1}{2}\right)^{\frac{2}{q}+4} \left[1+5^{\frac{1}{q}}+7^{\frac{1}{q}}+11^{\frac{1}{q}}\right] \\
\times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left[|f'(x_i)|+|f'(x_{i+1})|\right].
\end{aligned} \tag{4.5}
$$

Proof. The proof is similar to that of Proposition 7 and using Corollary 3. □

Proposition 10. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$ and $|f'|^q$ is a linear map, then for every division d of $[a,b]$, we have*

$$
|E(f,d)+E'(f,d)| \le \frac{1}{4}\left(\frac{q-1}{2q+1}\right)^{\frac{q-1}{q}} \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left|f'(x_{i+1}+x_i)\right| \tag{4.6}
$$

Proof. The proof is similar to that of Proposition 7 and it follows from Corollary 4. □

Proposition 11. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \geq 1$ and $|f'|^q$ is a linear map, then for every division d of $[a,b]$, then the following inequality holds:*

$$|E(f,d) + E'(f,d)| \leq \frac{1}{8} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left| f'\left(x_{i+1} + x_i\right) \right|. \tag{4.7}$$

Proof. The proof is similar to that of Proposition 7 and it follows from Corollary 5. □

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