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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES IN ABSOLUTE VALUE ARE CONVEX WITH APPLICATIONS

MUHAMMAD AMER LATIF

Abstract. In this paper some new Hadamard-type inequalities for functions whose derivatives in absolute values are convex are established. Some applications to special means of real numbers are given. Finally, we also give some applications of our obtained results to get new error bounds for the sum of the midpoint and trapezoidal formulae.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamar's inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard's inequality [4] has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see [1]-[13] and the references therein.

The main aim of this paper is to establish some new Hermite-Hadamard type inequalities for functions whose derivatives in absolute value are convex.

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2. Main Results

To prove our results we need the following lemma:

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° , the interior of I, where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f(x) + \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{2}{b-a} \int_a^b f(u)\, du$$
$$= \frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right) dt - \frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right) dt$$
$$- \frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right) dt + \frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right) dt, \tag{2.1}$$

for all $x \in [a,b]$.

Proof. By integration by parts and making use of the substitution $u = \frac{1+t}{2}x + \frac{1-t}{2}a$, we have

$$\frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right) dt$$
$$= \frac{(x-a)^2}{b-a} \left[\left. \frac{t f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)}{x-a} \right|_0^1 - \frac{1}{x-a} \int_0^1 f\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right) dt \right]$$
$$= \frac{x-a}{b-a} f(x) - \frac{2}{b-a} \int_{\frac{x+a}{2}}^x f(u)\, du. \tag{2.2}$$

Analogously, we also have the following equalities:

$$-\frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right) dt = \frac{x-a}{b-a} f(a) - \frac{2}{b-a} \int_a^{\frac{x+a}{2}} f(u)\, du, \tag{2.3}$$

$$-\frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right) dt = \frac{b-x}{b-a} f(x) - \frac{2}{b-a} \int_x^{\frac{x+b}{2}} f(u)\, du \tag{2.4}$$

and

$$\frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right) dt = \frac{b-x}{b-a} f(b) - \frac{2}{b-a} \int_{\frac{x+b}{2}}^b f(u)\, du. \tag{2.5}$$

Adding (2.2)-(2.5), we get the desired equality. This completes the proof of the lemma. □

Using the Lemma 1 the following results can be obtained:

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following* *inequality holds:*

$$\left| f(x) + \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{2}{b-a} \int_a^b f(u)\, du \right| \leq \frac{(x-a)^2}{b-a} \left[\frac{|f'(x)| + |f'(a)|}{4} \right] + \frac{(b-x)^2}{b-a} \left[\frac{|f'(x)| + |f'(b)|}{4} \right] \quad (2.6)$$

for all $x \in [a,b]$.

Proof. Using Lemma 1 and taking the modulus, we have

$$\begin{aligned} &\left| f(x) + \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{2}{b-a} \int_a^b f(u)\, du \right| \\ &\leq \frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} \left| f'\left(\frac{1+t}{2} x + \frac{1-t}{2} a \right) \right| dt + \frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} \left| f'\left(\frac{1-t}{2} x + \frac{1+t}{2} a \right) \right| dt \\ &+ \frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} \left| f'\left(\frac{1+t}{2} x + \frac{1-t}{2} b \right) \right| dt + \frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} \left| f'\left(\frac{1-t}{2} x + \frac{1+t}{2} b \right) \right| dt. \end{aligned} \quad (2.7)$$

Using the convexity of $|f'|$ on $[a,b]$, we get from the inequality (2.7) that

$$\begin{aligned} &\left| f(x) + \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{2}{b-a} \int_a^b f(u)\, du \right| \\ &\leq \frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} \left[\frac{1+t}{2} |f'(x)| + \frac{1-t}{2} |f'(a)| \right] dt \\ &+ \frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} \left[\frac{1-t}{2} |f'(x)| + \frac{1+t}{2} |f'(a)| \right] dt \\ &+ \frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} \left[\frac{1+t}{2} |f'(x)| + \frac{1-t}{2} |f'(b)| \right] dt \\ &+ \frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} \left[\frac{1-t}{2} |f'(x)| + \frac{1+t}{2} |f'(b)| \right] dt, \end{aligned} \quad (2.8)$$

holds for all $x \in [a,b]$.
Evaluating each integral on right side of the inequality (2.8), we get (2.6). This completes the proof of the theorem. □

Corollary 1. *In Theorem 1, if we choose* $x = \frac{a+b}{2}$ *and then using the convexity of* $|f'|$, *we get the following inequality:*

$$\left| f\left(\frac{a+b}{2} \right) + \frac{f(b) + f(a)}{2} - \frac{2}{b-a} \int_a^b f(u)\, du \right| \leq \left(\frac{b-a}{8} \right) \left[|f'(a)| + |f'(b)| \right]. \quad (2.9)$$

Theorem 2. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is convex on* $[a,b]$ *for some fixed*

$q>1$, then the following inequality holds:

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{2}{q}+1}$$
$$\times\left\{\frac{(x-a)^2}{b-a}\left[\left(3|f'(x)|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(|f'(x)|^q+3|f'(a)|^q\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\frac{(b-x)^2}{b-a}\left[\left(3|f'(x)|^q+|f'(b)|^q\right)^{\frac{1}{q}}+\left(|f'(x)|^q+3|f'(b)|^q\right)^{\frac{1}{q}}\right]\right\},\quad (2.10)$$

for all $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality, we have

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|$$
$$\le\frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}},\quad (2.11)$$

for all $x\in[a,b]$.
Since $|f'|^q$ is convex on $[a,b]$, we have

$$\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\le\int_0^1\left[\frac{1+t}{2}|f'(x)|^q+\frac{1-t}{2}|f'(a)|^q\right]dt=\frac{3|f'(x)|^q+|f'(a)|^q}{4}$$

Similarly,

$$\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\le\frac{|f'(x)|^q+3|f'(a)|^q}{4},$$

$$\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\le\frac{3|f'(x)|^q+|f'(b)|^q}{4}$$

and

$$\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\le\frac{|f'(x)|^q+3|f'(b)|^q}{4}.$$

Using the last four inequalities in (2.11), we get the inequality (2.10), which completes the proof of the theorem. □

Corollary 2. *In Theorem 2, if we choose $x = \frac{a+b}{2}$ and then using the convexity of $|f'|^q$, we get the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{2}{q}+1}\left(\frac{b-a}{4}\right)$$
$$\times\left\{\left[|f'(a)|^q+3\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+3|f'(a)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[3\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{\frac{1}{q}}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+3|f'(b)|^q\right]^{\frac{1}{q}}\right\}$$
$$\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}+1}\left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\left(\frac{b-a}{4}\right)\left[|f'(a)|+|f'(b)|\right]. \quad (2.12)$$

The second inequality is obtained by using the fact that

$$\sum_{k=1}^{n}(u_k+v_k)^s \le \sum_{k=1}^{n}(u_k)^s+\sum_{k=1}^{n}(v_k)^s,\, u_k,v_k\ge 0, 1\le k\le n, 0\le s<1.$$

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|$$
$$\le\left(\frac{1}{4}\right)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left\{\frac{(x-a)^2}{b-a}\left[\left(5|f'(x)|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(|f'(x)|^q+5|f'(a)|^q\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\frac{(b-x)^2}{b-a}\left[\left(5|f'(x)|^q+|f'(b)|^q\right)^{\frac{1}{q}}+\left(|f'(x)|^q+5|f'(b)|^q\right)^{\frac{1}{q}}\right]\right\}, \quad (2.13)$$

for all $x \in [a,b]$.

Proof. Suppose that $q \ge 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|$$
$$\le\frac{(x-a)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}, \quad (2.14)$$

for all $x \in [a,b]$.
Since $|f'|^q$ is convex on $[a,b]$, we have

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt \\ \leq \int_0^1 \frac{t}{2}\left[\frac{1+t}{2}\left|f'(x)\right|^q+\frac{1-t}{2}\left|f'(a)\right|^q\right]=\frac{5\left|f'(x)\right|^q+\left|f'(a)\right|^q}{24}.$$

Similarly,

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt \leq \frac{\left|f'(x)\right|^q+5\left|f'(a)\right|^q}{24},$$

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt \leq \frac{5\left|f'(x)\right|^q+\left|f'(b)\right|^q}{24}$$

and

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt \leq \frac{\left|f'(x)\right|^q+5\left|f'(b)\right|^q}{24}.$$

By making use of the last four inequalities in (2.14), we get (2.13). Hence the proof of the theorem is complete. □

Corollary 3. *In Theorem 2, if we choose $x=\frac{a+b}{2}$ and using similar arguments as in Corollary 2, we get the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \\ \leq \left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{2}{q}}\left[1+5^{\frac{1}{q}}+7^{\frac{1}{q}}+11^{\frac{1}{q}}\right]\left(\frac{b-a}{16}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \quad (2.15)$$

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$, then the following inequality holds:*

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \\ \leq \frac{1}{2}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left\{\frac{(x-a)^2}{b-a}\left[\left|f'\left(\frac{3x+a}{4}\right)\right|+\left|f'\left(\frac{3a+x}{4}\right)\right|\right]\right. \\ \left.+\frac{(b-x)^2}{b-a}\left[\left|f'\left(\frac{3x+b}{4}\right)\right|+\left|f'\left(\frac{x+3b}{4}\right)\right|\right]\right\} \quad (2.16)$$

for all $x \in [a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$\begin{aligned}
&\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}, \quad (2.17)
\end{aligned}$$

for all $x\in[a,b]$.
Since $|f'|^q$ is concave on $[a,b]$, we can use the Jensen's integral inequality to obtain:

$$\begin{aligned}
\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt &= \int_0^1 t^0\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt \\
&\leq \left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{1}{\int_0^1 t^0 dt}\int_0^1\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)dt\right)\right|^q \\
&= \left|f'\left(\frac{3x+a}{4}\right)\right|^q.
\end{aligned}$$

Analogously, we have that the following inequalities:

$$\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt \leq \left|f'\left(\frac{x+3a}{4}\right)\right|^q,$$

$$\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt \leq \left|f'\left(\frac{3x+b}{4}\right)\right|^q$$

and

$$\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt \leq \left|f'\left(\frac{x+3b}{4}\right)\right|^q.$$

Using the last four inequalities in (2.17), we get (2.16). This completes the proof of the theorem. □

Corollary 4. *If in Theorem 4, we choose $x=\frac{a+b}{2}$ and assume that $|f'|$ is a linear map, then we get the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{4}\right)|f'(a+b)|. \quad (2.18)$$

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \geq 1$, then the following inequality holds:*

$$\left| f(x) + \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{2}{b-a}\int_a^b f(u)\,du \right| \leq \frac{(x-a)^2}{4(b-a)}\left[\left|f'\left(\frac{5x+a}{6}\right)\right| + \left|f'\left(\frac{x+5a}{6}\right)\right|\right] + \frac{(b-x)^2}{4(b-a)}\left[\left|f'\left(\frac{5x+b}{6}\right)\right| + \left|f'\left(\frac{x+5b}{6}\right)\right|\right], \quad (2.19)$$

for all $x \in [a,b]$.

Proof. First, by the concavity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we note that

$$|f(\lambda x + (1-\lambda) y)|^q \geq \lambda |f(x)|^q + (1-\lambda)|f(y)|^q \geq (\lambda |f(x)| + (1-\lambda)|f(y)|)^q$$

and hence

$$|f(\lambda x + (1-\lambda) y)| \geq \lambda |f(x)| + (1-\lambda)|f(y)|,$$

for all $\lambda \in [0,1]$ and x, $y \in [a,b]$. This shows that $|f'|$ is also concave on $[a,b]$. Accordingly, using Lemma 1 and the Jensens's integral inequality, we have

$$\begin{aligned}
&\left| f(x) + \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{2}{b-a}\int_a^b f(u)\,du \right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\right| dt + \frac{(x-a)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right| dt \\
&+ \frac{(b-x)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right| dt + \frac{(b-x)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right| dt \\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 \frac{t}{2}dt\right)\left|f'\left(\frac{\int_0^1 \frac{t}{2}\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)dt}{\int_0^1 \frac{t}{2}dt}\right)\right| \\
&+ \frac{(x-a)^2}{b-a}\left(\int_0^1 \frac{t}{2}dt\right)\left|f'\left(\frac{\int_0^1 \frac{t}{2}\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)dt}{\int_0^1 \frac{t}{2}dt}\right)\right| \\
&+ \frac{(b-x)^2}{b-a}\left(\int_0^1 \frac{t}{2}dt\right)\left|f'\left(\frac{\int_0^1 \frac{t}{2}\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)dt}{\int_0^1 \frac{t}{2}dt}\right)\right| \\
&+ \frac{(b-x)^2}{b-a}\left(\int_0^1 \frac{t}{2}dt\right)\left|f'\left(\frac{\int_0^1 \frac{t}{2}\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)dt}{\int_0^1 \frac{t}{2}dt}\right)\right|,
\end{aligned}$$

for all $x \in [a,b]$, which is equivalent to (2.19) and the proof of the theorem is complete. □

Corollary 5. *If in Theorem 5, we choose $x = \frac{a+b}{2}$ and assume that $|f'|$ is a linear map, then we have the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \leq \frac{b-a}{8}\left|f'(a+b)\right|. \tag{2.20}$$

3. Applications to Special Means

Now, we consider the applications of our Theorems to the special means. We consider the means for arbitrary real numbers $a,\ b \in \mathbb{R}$. We take

(1) The arithmetic mean:

$$A(a,b) = \frac{a+b}{2};\ a,\ b \in \mathbb{R}.$$

(2) The harmonic mean:

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}};\ a, b \in \mathbb{R}\backslash\{0\}.$$

(3) The logarithmic mean:

$$L(a,b) = \frac{\ln|b| - \ln|a|}{b-a}; a, b \in \mathbb{R}, |a| \neq |b|, a, b \neq 0.$$

(4) Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}};\ a,\ b \in \mathbb{R},\ n \in \mathbb{Z}\backslash\{-1,0\},\ a \neq b.$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let $a,\ b \in \mathbb{R},\ a < b,\ 0 \notin [a,b]$ and $n \in \mathbb{Z},\ |n| \geq 2$. Then*

$$\left|A^n(a,b) + A(a^n,b^n) - 2L_n^n(a,b)\right| \leq |n|\left(\frac{b-a}{4}\right) A\left(|a|^{n-1}, |b|^{n-1}\right). \tag{3.1}$$

Proof. The assertion follows from Corollary 1 when applied to the function $f(x) = x^n,\ x \in \mathbb{R},\ n \in \mathbb{Z},\ |n| \geq 2$. □

Proposition 2. *Let $a,\ b \in \mathbb{R},\ a < b,\ 0 \notin [a,b]$ and $n \in \mathbb{Z},\ |n| \geq 2$. Then for $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, we have*

$$\left|A^n(a,b) + A(a^n,b^n) - 2L_n^n(a,b)\right|$$
$$\leq |n|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}+1}\left[1+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\left(\frac{b-a}{2}\right) A\left(|a|^{n-1}, |b|^{n-1}\right). \tag{3.2}$$

Proof. The assertion follows from Corollary 2 when applied to the function $f(x) = x^n,\ x \in \mathbb{R},\ n \in \mathbb{Z},\ |n| \geq 2$. □

Proposition 3. *Let $a,\ b \in \mathbb{R},\ a < b,\ 0 \notin [a,b]$ and $n \in \mathbb{Z},\ |n| \geq 2$. Then $q \geq 1$, we have*

$$\left|A^n(a,b) + A(a^n,b^n) - 2L_n^n(a,b)\right|$$
$$\leq |n|\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{2}{q}}\left[1+5^{\frac{1}{q}}+7^{\frac{1}{q}}+11^{\frac{1}{q}}\right]\left(\frac{b-a}{8}\right) A\left(|a|^{n-1}, |b|^{n-1}\right). \tag{3.3}$$

Proof. The assertion follows from Corollary 3 when applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 2$. □

Proposition 4. *Let a, $b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$. Then*

$$\left|A^{-1}(a,b) + A\left(a^{-1}, b^{-1}\right) - 2L(a,b)\right| \leq \left(\frac{b-a}{4}\right) A\left(|a|^{-2}, |b|^{-2}\right). \tag{3.4}$$

Proof. It is a direct consequence of Corollary 1 when applied to the function, $f(x) = \frac{1}{x}$, $x \in [a, b]$. □

Proposition 5. *Let a, $b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$, then for all $p > 1$, we have*

$$\begin{aligned} &\left|A^{-1}(a,b) + A\left(a^{-1}, b^{-1}\right) - 2L(a,b)\right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{3}{q}+1} \left[1 + 3^{\frac{1}{q}} + 5^{\frac{1}{q}} + 7^{\frac{1}{q}}\right] \left(\frac{b-a}{2}\right) A\left(|a|^{-2}, |b|^{-2}\right). \end{aligned} \tag{3.5}$$

Proof. It follows directly from Corollary 2 for the function, $f(x) = \frac{1}{x}$, $x \in [a, b]$. □

Proposition 6. *Let a, $b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$. Then for all $q \geq 1$, we have the inequality*

$$\begin{aligned} &\left|A^{-1}(a,b) + A\left(a^{-1}, b^{-1}\right) - 2L(a,b)\right| \\ &\leq \left(\frac{1}{3}\right)^{\frac{1}{q}} \left(\frac{1}{2}\right)^{\frac{2}{q}} \left[1 + 5^{\frac{1}{q}} + 7^{\frac{1}{q}} + 11^{\frac{1}{q}}\right] \left(\frac{b-a}{8}\right) A\left(|a|^{-2}, |b|^{-2}\right). \end{aligned} \tag{3.6}$$

Proof. It follows directly from Corollary 3 for the function, $f(x) = \frac{1}{x}$, $x \in [a, b]$. □

Remark 1. *We can get several inequalities for means from Corollary 4 and Corollary 5 for a particular choice of the concave function f but we omit the details for the interested reader.*

Remark 2. *Let a, $b \in \mathbb{R} \setminus \{0\}$, $a < b$ then $a^{-1} > b^{-1}$ and $A\left(a^{-1}, b^{-1}\right) = \frac{2}{\frac{1}{a}+\frac{1}{b}} = H^{-1}(a, b)$. Hence one can get several inequalities containing harmonic mean and logarithmic means but we omit the details for the interested readers.*

4. Application to the Midpoint and Trapezoidal Formulae

Let d be a division of the interval $[a, b]$, *i.e.* $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, and consider the quadrature formulae

$$\int_a^b f(x)dx = M(f, d) + E(f, d),$$

and

$$\int_a^b f(x)dx = T'(f, d) + E'(f, d),$$

where

$$T(f, d) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right)$$

and

$$T'(f, d) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{f(x_i) + f(x_{i+1})}{2}$$

are the midpoint and trapezoidal versions and $E(f,d)$ and $E'(f,d)$ are the associated errors respectively. Here, we derive some error estimates for the sum of midpoint and trapezoidal formulae.

Proposition 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then for every division d of $[a,b]$, we have:*

$$|E(f,d) + E'(f,d)| \leq \frac{1}{8} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[|f'(x_i)| + |f'(x_{i+1})|\right]. \tag{4.1}$$

Proof. By applying Corollary 1 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division d, we have

$$\left| f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{f(x_{i+1}) + f(x_i)}{2} - \frac{2}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)\,dx \right| \leq \left(\frac{x_{i+1} - x_i}{8}\right) \left[|f'(x_i)| + |f'(x_{i+1})|\right]. \tag{4.2}$$

Now

$$\begin{aligned} &|E(f,d) + E'(f,d)| \\ &= \left| \sum_{i=0}^{n-1} (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right) + \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) - 2 \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx \right| \\ &\leq \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left| f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{f(x_i) + f(x_{i+1})}{2} - \frac{2}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right|. \end{aligned} \tag{4.3}$$

Using (4.2) in (4.3), we get (4.1). This completes the proof of the proposition. □

Proposition 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q > 1$, then for every division d of $[a,b]$, we have*

$$\begin{aligned} &|E(f,d) + E'(f,d)| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{3}{q}+3} \left[1 + 3^{\frac{1}{q}} + 5^{\frac{1}{q}} + 7^{\frac{1}{q}}\right] \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[|f'(x_i)| + |f'(x_{i+1})|\right], \end{aligned} \tag{4.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof is similar to that of Proposition 7 and using Corollary 2. □

Proposition 9. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed*

$q\geq 1$, then for every division d of $[a,b]$, we have

$$\left|E(f,d)+E'(f,d)\right|\leq \left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{2}{q}+4}\left[1+5^{\frac{1}{q}}+7^{\frac{1}{q}}+11^{\frac{1}{q}}\right]\times\sum_{i=0}^{n-1}\left(x_{i+1}-x_{i}\right)^{2}\left[\left|f'\left(x_{i}\right)\right|+\left|f'\left(x_{i+1}\right)\right|\right]. \quad (4.5)$$

Proof. The proof is similar to that of Proposition 7 and using Corollary 3. □

Proposition 10. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{q}$ is concave on $[a,b]$ for some fixed $q>1$ and $|f'|^{q}$ is a linear map, then for every division d of $[a,b]$, we have*

$$\left|E(f,d)+E'(f,d)\right|\leq \frac{1}{4}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\sum_{i=0}^{n-1}\left(x_{i+1}-x_{i}\right)^{2}\left|f'\left(x_{i+1}+x_{i}\right)\right| \quad (4.6)$$

Proof. The proof is similar to that of Proposition 7 and it follows from Corollary 4. □

Proposition 11. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{q}$ is concave on $[a,b]$ for some fixed $q\geq 1$ and $|f'|^{q}$ is a linear map, then for every division d of $[a,b]$, then the following inequality holds:*

$$\left|E(f,d)+E'(f,d)\right|\leq \frac{1}{8}\sum_{i=0}^{n-1}\left(x_{i+1}-x_{i}\right)^{2}\left|f'\left(x_{i+1}+x_{i}\right)\right|. \quad (4.7)$$

Proof. The proof is similar to that of Proposition 7 and it follows from Corollary 5. □

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College of Science, Department of Mathematics, University of Hail, Hail 2440, Saudi Arabia

E-mail address: m_amer_latif@hotmail.com