Received 24/04/12

A NEW GENERALIZATION OF GRÜSS TYPE INEQUALITIES FOR THE STIELTJES INTEGRAL AND APPLICATIONS

MOHAMMAD W. ALOMARI

Abstract. In order to prove some Grüss' type inequalities for Riemann–Stieltjes integral, a new functional is introduced. Some related functionals are discussed and therefore, several bounds are proved. Applications to the approximation problem of the Riemann-Stieltjes integral are also pointed out.

1. Introduction

In order to approximate the Stieltjes integral $\int_a^b f(x)\,du(x)$ by the Riemann integral $\int_a^b f(t)\,dt$, Dragomir and Fedotov [7], have established the following functional:

$$\mathcal{D}(f;u) := \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt, \tag{1.1}$$

provided that the Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

In the same paper [7], the authors have proved the following inequality:

Theorem 1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $K>0$. Then we have*

$$|\mathcal{D}(f;u)| \le \frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{1.2}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In his interesting work [13], Dragomir has obtained the following inequality:

Theorem 2. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(y)-u(x)| \le L|x-y|, \forall x,y\in[a,b], \quad (L>0)$$

and f is Riemann integrable on $[a,b]$.

If $m,M\in\mathbb{R}$, are such that $m\le f(x)\le M$, for any $x\in[a,b]$, then the inequality

$$|\mathcal{D}(f;u)| \le \frac{1}{2}L(M-m)(b-a) \tag{1.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Date: April 24, 2012.
2000 *Mathematics Subject Classification.* 26D15, 26D20.
Key words and phrases. Grüss inequality, Riemann–Stieltjes integral, functional.

For other recent inequalities for the Riemann–Stieltjes integral, see [1]–[18] and the references therein.

1.1. **A weighted Dragomir functional.** In order to compare the Stieltjes integral mean with the weighted Riemann integral mean, we define the functional $\mathcal{OD}(f,g;u)$, as follows:

$$\mathcal{OD}(f,g;u) := \frac{1}{u(b)-u(a)} \cdot \int_a^b f(x)\,du(x) - \frac{1}{\int_a^b g(t)\,dt} \cdot \int_a^b f(t)\,g(t)\,dt, \tag{1.4}$$

provided that the both integrals exist and $g(t) \neq 0$, for all $t \in [a,b]$.

In particular, as special cases; we are interested in two functionals:

1: The Dragomir functional:

$$\begin{aligned} \mathcal{D}(f;u) &:= \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt \\ &= [u(b)-u(a)] \cdot \mathcal{OD}(f,1;u), \end{aligned} \tag{1.5}$$

2: The weighted integral functional:

$$\begin{aligned} \mathcal{E}(f,g;w) &:= \frac{\int_a^b f(t)\,w(t)\,dt}{\int_a^b w(t)\,dt} - \frac{\int_a^b f(t)\,g(t)\,dt}{\int_a^b g(t)\,dt} \\ &= \frac{1}{\int_a^b w(t)\,dt \int_a^b g(t)\,dt} \cdot \mathcal{OD}\left(f,g;\int_a^x w(s)\,ds\right). \end{aligned} \tag{1.6}$$

where, $u(x) = \int_a^x w(s)\,ds$, $w : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, and $g(t), w(t) \neq 0$, for all $t \in [a,b]$.

In fact, the functional $\mathcal{OD}(f,g;u)$ is a natural generalization of the Dragomir functional $\mathcal{D}(f;u)$.

In this paper, we study the functional $\mathcal{OD}(f,g;u)$, and obtain several new bounds via various type of integrators. More specifically, the obtained results deal with integrands of r-H–Hölder type, and integrators of bounded variation, Lipschitzian and monotonic. Consequently, for the functionals (1.5) and (1.6), several inequalities are deduced and investigated. Applications to approximation problem of the Stieltjes integral are pointed out.

2. The case of bounded variation integrators

The following result holds:

Theorem 3. *Let $f,u,g : [a,b] \to \mathbb{R}$ be mappings such that f is of r-H–Hölder type on $[a,b]$, where $r \in (0,1]$ and $H > 0$ are given, and u is of bounded variation on*

$[a,b]$. *Then we have the inequality:*

$$|\mathcal{OD}(f,g;u)| \leq \frac{H}{[u(b)-u(a)]\cdot\int_a^b g(t)\,dt}\begin{cases}\frac{(b-a)^{r+1}}{r+1}\cdot\|g\|_\infty\cdot\bigvee_a^b(u), & if \quad g\in L_\infty[a,b],\\ \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\|g\|_p\cdot\bigvee_a^b(u), & if \quad g\in L_p[a,b],\\ & p>1, \frac{1}{p}+\frac{1}{q}=1,\\ (b-a)^r\|g\|_1\cdot\bigvee_a^b(u), & if \quad g\in L_1[a,b].\end{cases} \tag{2.1}$$

where, $\bigvee_a^b(u)$ *is the total variation of* u *over* $[a,b]$.

Proof. It is well-known that for a continuous function $p:[a,b]\to\mathbb{R}$ and a function $\nu:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \leq \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(\nu).$$

Therefore, as u is of bounded variation on $[a,b]$, we have

$$\begin{aligned}|\mathcal{OD}(f,g;u)| &= \left|\int_a^b\left[f(x)\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right]du(x)\right|\\ &\leq \sup_{x\in[a,b]}\left|f(x)\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right|\cdot\bigvee_a^b(u)\\ &= \sup_{x\in[a,b]}\left|\int_a^b[f(x)-f(t)]g(t)\,dt\right|\cdot\bigvee_a^b(u) \qquad (2.2)\end{aligned}$$

As f is of r-H-Hölder type on $[a,b]$ and $g\in L_\infty[a,b]$, then we have

$$\begin{aligned}\left|\int_a^b[f(x)-f(t)]g(t)\,dt\right| &\leq \int_a^b|f(x)-f(t)||g(t)|\,dt\\ &\leq H\int_a^b|x-t|^r|g(t)|\,dt\\ &\leq H\sup_{t\in[a,b]}|g(t)|\cdot\int_a^b|x-t|^r\,dt\\ &= \frac{H}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right]\cdot\|g\|_\infty \qquad (2.3)\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{x\in[a,b]}\left|\int_a^b[f(x)-f(t)]g(t)\,dt\right| &\leq \frac{H}{r+1}\cdot\|g\|_\infty\cdot\sup_{x\in[a,b]}\left[(x-a)^{r+1}+(b-x)^{r+1}\right]\\ &\leq \frac{H}{r+1}(b-a)^{r+1}\cdot\|g\|_\infty. \qquad (2.4)\end{aligned}$$

Combining (2.2) and (2.4), we get the first inequality in (2.1).

To prove the second inequality in (2.1). As f is of r-H-Hölder type on $[a,b]$, then we have

$$\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| \le \int_a^b |f(x)-f(t)|\,|g(t)|\,dt \le H\int_a^b |x-t|^r\,|g(t)|\,dt.$$

Now, as $g\in L_p[a,b]$ therefore, by applying the well-known Hölder integral inequality, we get

$$\begin{aligned}\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| &\le H\int_a^b |x-t|^r\,|g(t)|\,dt\\ &\le H\left(\int_a^b |x-t|^{rq}\,dt\right)^{1/q}\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}\\ &= \frac{H}{(qr+1)^{1/q}}\left[(x-a)^{qr+1}+(b-x)^{qr+1}\right]^{1/q}\cdot\|g\|_p.\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{x\in[a,b]}\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| &\le \frac{H}{(qr+1)^{1/q}}\cdot\|g\|_p\cdot\sup_{x\in[a,b]}\left[(x-a)^{qr+1}+(b-x)^{qr+1}\right]^{1/q}\\ &\le H\frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\|g\|_p. \qquad (2.5)\end{aligned}$$

Combining (2.2) and (2.5), we get the second inequality in (2.1).

Finally, to prove the third inequality in (2.1). By assumptions we have:

$$\begin{aligned}\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| &\le \int_a^b |f(x)-f(t)|\,|g(t)|\,dt\\ &\le H\int_a^b |x-t|^r\,|g(t)|\,dt\\ &\le H\sup_{t\in[a,b]}\{|x-t|^r\}\int_a^b |g(t)|\,dt\\ &= H\|g\|_1\max_{t\in[a,b]}\{(x-a)^r,(b-x)^r\}\\ &= H\|g\|_1\left[\max_{t\in[a,b]}\{(x-a),(b-x)\}\right]^r\\ &= H\|g\|_1\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r\\ &\le H\|g\|_1(b-a)^r. \qquad (2.6)\end{aligned}$$

Combining (2.2) and (2.6), we get the third inequality in (2.1) and thus the theorem is proved. □

Corollary 1. *Let u as in Theorem 3 and $f:[a,b]\to\mathbb{R}$ be an L–Lipschitzian mapping on $[a,b]$. Then we have the inequality*

(2.7)
$$\left|\mathcal{OD}\left(f,g;u\right)\right|\leq L\begin{cases}\frac{(b-a)^2}{2}\cdot\left\|g\right\|_\infty\cdot\bigvee_a^b\left(u\right), & if \quad g\in L_\infty[a,b],\\ \\ \frac{(b-a)^{(q+1)/q}}{(q+1)^{1/q}}\cdot\left\|g\right\|_p\cdot\bigvee_a^b\left(u\right), & if \quad g\in L_p[a,b],\\ & \quad p>1,\frac{1}{p}+\frac{1}{q}=1,\\ \\ (b-a)\left\|g\right\|_1\cdot\bigvee_a^b\left(u\right), & if \quad g\in L_1[a,b].\end{cases}$$

Corollary 2. *Assume f as in Theorem 3. Let $u\in C^{(1)}[a,b]$. Then we have the inequality*

(2.8)
$$\left|\mathcal{OD}\left(f,g;u\right)\right|\leq H\begin{cases}\frac{(b-a)^{r+1}}{r+1}\cdot\left\|g\right\|_\infty\cdot\left\|u'\right\|_{1,[a,b]}, & if \quad g\in L_\infty[a,b],\\ \\ \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\left\|g\right\|_p\cdot\left\|u'\right\|_{1,[a,b]}, & if \quad g\in L_p[a,b],\\ & \quad p>1,\frac{1}{p}+\frac{1}{q}=1,\\ \\ (b-a)^r\left\|g\right\|_1\cdot\left\|u'\right\|_{1,[a,b]}, & if \quad g\in L_1[a,b].\end{cases}$$

where $\left\|\cdot\right\|_1$ is the L_1 norm, namely $\left\|u'\right\|_{1,[a,b]}:=\int_a^b\left|u'\left(t\right)\right|dt$.

Corollary 3. *Assume f as in Theorem 3. Let $u:[a,b]\to\mathbb{R}$ be a K-Lipschitzian mapping with the constant $K>0$. Then we have the inequality*

(2.9)
$$\left|\mathcal{OD}\left(f,g;u\right)\right|\leq HK\left(b-a\right)\begin{cases}\frac{(b-a)^{r+1}}{r+1}\cdot\left\|g\right\|_\infty, & if \quad g\in L_\infty[a,b],\\ \\ \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\left\|g\right\|_p, & if \quad g\in L_p[a,b],\\ & \quad p>1,\frac{1}{p}+\frac{1}{q}=1,\\ \\ (b-a)^r\left\|g\right\|_1, & if \quad g\in L_1[a,b].\end{cases}$$

Corollary 4. *Assume f as in Theorem 3. Let $u:[a,b]\to\mathbb{R}$ be a monotonic mapping. Then we have the inequality*

(2.10)
$$\left|\mathcal{OD}\left(f,g;u\right)\right|\leq H\cdot\left|u\left(b\right)-u\left(a\right)\right|\begin{cases}\frac{(b-a)^{r+1}}{r+1}\cdot\left\|g\right\|_\infty, & if \quad g\in L_\infty[a,b],\\ \\ \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\left\|g\right\|_p, & if \quad g\in L_p[a,b],\\ & \quad p>1,\frac{1}{p}+\frac{1}{q}=1,\\ \\ (b-a)^r\left\|g\right\|_1, & if \quad g\in L_1[a,b].\end{cases}$$

Remark 1. *For the last three inequalities, one may deduce several inequalities for L-Lipschitzian integrands by setting $r=1$ and replace H by L. We left the details to the interested reader.*

Remark 2. *Under the assumptions of Theorem 3, one may deduce several inequalities for the functionals (1.5) and (1.6).*

3. The case of Lipschitzian integrators

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, and $u : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, where r and $H, L > 0$ are given. Then we have the inequality*

$$|\mathcal{OD}(f,g;u)| \le LH \begin{cases} \frac{2(b-a)^{r+2}}{(r+1)(r+2)} \cdot \|g\|_\infty, & if \quad g \in L_\infty[a,b], \\ \frac{2q}{(qr+1)^{1/q}} \cdot \frac{(b-a)^{(q(r+1)+1)/q}}{(q(r+1)+1)} \cdot \|g\|_p, & if \quad g \in L_p[a,b], \\ & \quad p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \frac{(2^{r+1}-1)}{2^r(r+1)}(b-a)^{r+1}\|g\|_1, & if \quad g \in L_1[a,b]. \end{cases} \tag{3.1}$$

Proof. It is well-known that for a Riemann integrable function $p : [a,b] \to \mathbb{R}$ and L–Lipschitzian function $\nu : [a,b] \to \mathbb{R}$, one has the inequality

$$\left| \int_a^b p(t)\, d\nu(t) \right| \le L \int_a^b |p(t)|\, dt.$$

Therefore, as u is L-Lipschitzian on $[a,b]$, we have

$$\begin{aligned} |\mathcal{OD}(f,g;u)| &= \left| \int_a^b \left[f(x) \int_a^b g(t)\, dt - \int_a^b f(t)\, g(t)\, dt \right] du(x) \right| \\ &\le L \int_a^b \left| f(x) \int_a^b g(t)\, dt - \int_a^b f(t)\, g(t)\, dt \right| dx \\ &= L \int_a^b \left| \int_a^b [f(x) - f(t)]\, g(t)\, dt \right| dx \qquad (3.2) \end{aligned}$$

As f is of r-H-Hölder type on $[a,b]$ and $g \in L_\infty[a,b]$, by (2.3) we have

$$\left| \int_a^b [f(x) - f(t)]\, g(t)\, dt \right| \le \frac{H}{r+1} \left[(x-a)^{r+1} + (b-x)^{r+1} \right] \cdot \|g\|_\infty \tag{3.3}$$

It follows that

$$\begin{aligned} \int_a^b \left| \int_a^b [f(x) - f(t)]\, g(t)\, dt \right| dx &\le \frac{H}{r+1} \cdot \|g\|_\infty \cdot \int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1} \right] dx \\ &\le \frac{2H}{(r+1)(r+2)} (b-a)^{r+2} \cdot \|g\|_\infty . \qquad (3.4) \end{aligned}$$

Combining (3.2) and (3.4), we get the first inequality in (3.1).

To prove the second inequality in (3.1). As f is of r-H-Hölder type on $[a,b]$, then we have

$$\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| \le \int_a^b |f(x)-f(t)|\,|g(t)|\,dt \le H\int_a^b |x-t|^r\,|g(t)|\,dt.$$

Now, as $g\in L_p[a,b]$ therefore, by applying the well-known Hölder integral inequality, we get

$$\begin{aligned}\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| &\le H\int_a^b |x-t|^r\,|g(t)|\,dt\\ &\le H\left(\int_a^b |x-t|^{rq}\,dt\right)^{1/q}\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}\\ &= \frac{H}{(qr+1)^{1/q}}\left[(x-a)^{qr+1}+(b-x)^{qr+1}\right]^{1/q}\cdot\|g\|_p\,.\\ &= \frac{H}{(qr+1)^{1/q}}\left[\left((x-a)^{r+\frac{1}{q}}\right)^q+\left((b-x)^{r+\frac{1}{q}}\right)^q\right]^{1/q}\cdot\|g\|_p \qquad (3.5)\end{aligned}$$

Using the fact that $(A^s+B^s)^{1/s}\le(A+B)$, for all $A,B\ge 0$ and $s\ge 1$, it follows that

$$\begin{aligned}&\int_a^b\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right|dx\\ &\le \frac{H}{(qr+1)^{1/q}}\cdot\|g\|_p\int_a^b\left[\left((x-a)^{r+\frac{1}{q}}\right)^q+\left((b-x)^{r+\frac{1}{q}}\right)^q\right]^{1/q}dx\\ &\le \frac{H}{(qr+1)^{1/q}}\cdot\|g\|_p\cdot\int_a^b\left[(x-a)^{r+\frac{1}{q}}+(b-x)^{r+\frac{1}{q}}\right]dx\\ &\le H\frac{2q}{(qr+1)^{1/q}}\cdot\frac{(b-a)^{(q(r+1)+1)/q}}{(q(r+1)+1)}\cdot\|g\|_p\,. \qquad (3.6)\end{aligned}$$

Combining (3.2) and (3.6), we get the second inequality in (3.1).

Finally, to prove the third inequality in (3.1). By assumptions we have:

$$\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| \le H\,\|g\|_1\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r \qquad (3.7)$$

It follows that

$$\begin{aligned}\int_a^b\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right|dx &\le H\,\|g\|_1\cdot\int_a^b\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r dx\\ &\le H\,\|g\|_1\,\frac{(2^{r+1}-1)}{2^r(r+1)}(b-a)^{r+1}\,. \qquad (3.8)\end{aligned}$$

Combining (3.2) and (3.9), we get the third inequality in (3.1) and thus the theorem is proved.

□

Corollary 5. *Let u as in Theorem 4 and $f : [a,b] \to \mathbb{R}$ be an K–Lipschitzian mapping on $[a,b]$. Then we have the inequality*

(3.9)
$$|\mathcal{OD}(f,g;u)| \le LK \begin{cases} \frac{(b-a)^3}{3} \cdot \|g\|_\infty, & if \quad g \in L_\infty[a,b], \\ \\ \frac{2q}{(q+1)^{1/q}} \cdot \frac{(b-a)^{(2q+1)/q}}{(2q+1)} \cdot \|g\|_p, & if \quad g \in L_p[a,b],\ p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{3}{4}(b-a)^2 \|g\|_1, & if \quad g \in L_1[a,b]. \end{cases}$$

Remark 3. *Under the assumptions of Theorem 4, one may deduce several inequalities for the functionals (1.5) and (1.6).*

4. The case of monotonic integrators

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, and $u : [a,b] \to \mathbb{R}$ be a monotonic mapping on $[a,b]$, where r and $H > 0$ are given. Then we have the inequality*

(4.1)
$$|\mathcal{OD}(f,g;u)| \le H \begin{cases} 2\frac{(b-a)^{r+1}}{r+1} \cdot \|g\|_\infty \cdot [u(b)-u(a)], & if \quad g \in L_\infty[a,b], \\ \\ \frac{2(b-a)^{r+\frac{1}{q}}}{(qr+1)^{1/q}} \cdot \|g\|_p \cdot [u(b)-u(a)], & if \quad g \in L_p[a,b]; \\ & \qquad p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ (b-a)^r \|g\|_1 \cdot [u(b)-u(a)], & if \quad g \in L_1[a,b]. \end{cases}$$

Proof. It is well-known that for a monotonic non-decreasing function $\nu : [a,b] \to \mathbb{R}$ and continuous function $p : [a,b] \to \mathbb{R}$, one has the inequality

$$\left| \int_a^b p(t)\, d\nu(t) \right| \le \int_a^b |p(t)|\, d\nu(t).$$

Therefore, as u is monotonic non-decreasing on $[a,b]$, we have

$$\begin{aligned} |\mathcal{OD}(f,g;u)| &= \left| \int_a^b \left[f(x) \int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt \right] du(x) \right| \\ &\le \int_a^b \left| f(x) \int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt \right| du(x) \\ &= \int_a^b \left| \int_a^b [f(x)-f(t)]\,g(t)\,dt \right| du(x) \end{aligned} \tag{4.2}$$

As f is of r-H-Hölder type on $[a,b]$ and $g \in L_\infty[a,b]$, by (2.3) we have

$$\left| \int_a^b [f(x)-f(t)]\,g(t)\,dt \right| \le \frac{H}{r+1} \left[(x-a)^{r+1} + (b-x)^{r+1} \right] \cdot \|g\|_\infty \tag{4.3}$$

It follows that

$$\int_a^b \left| \int_a^b [f(x) - f(t)] g(t)\,dt \right| du(x) \leq \frac{H}{r+1} \cdot \|g\|_\infty \cdot \int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1} \right] du(x) \tag{4.4}$$

Now, using Riemann–Stieltjes integral we have

$$\int_a^b (x-a)^{r+1} du(x) = (b-a)^{r+1} u(b) - (r+1) \int_a^b (x-a)^r u(x)\,dx \tag{4.5}$$

and

$$\int_a^b (b-x)^{r+1} du(x) = -(b-a)^{r+1} u(a) + (r+1) \int_a^b (b-x)^r u(x)\,dx. \tag{4.6}$$

Adding (4.5) and (4.6), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1} \right] du(x). = (b-a)^{r+1} [u(b) - u(a)] + (r+1) \int_a^b [(b-x)^r - (x-a)^r] u(x)\,dx. \tag{4.7}$$

Now, by the monotonicity property of u we have

$$\int_a^b (x-a)^r u(x)\,dx \geq u(a) \int_a^b (x-a)^r dx = \frac{(b-a)^{r+1}}{r+1} \cdot u(a) \tag{4.8}$$

and

$$\int_a^b (b-x)^r u(x)\,dx \leq u(b) \int_a^b (b-x)^r dx = \frac{(b-a)^{r+1}}{r+1} \cdot u(b). \tag{4.9}$$

Substituting (4.8) and (4.9) in (4.7), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1} \right] du(x) \leq 2 (b-a)^{r+1} [u(b) - u(a)] \tag{4.10}$$

Substituting (4.10) in (4.4), we get

$$\int_a^b \left| \int_a^b [f(x) - f(t)] g(t)\,dt \right| du(x) \leq 2H \frac{(b-a)^{r+1}}{r+1} \cdot \|g\|_\infty \cdot [u(b) - u(a)],$$

and therefore, by (4.2) we get the first inequality in (4.1).

To prove the second inequality in (4.1). As f is of r-H-Hölder type on $[a,b]$ and $g \in L_p[a,b]$ therefore, by (3.5), we have

$$\left| \int_a^b [f(x) - f(t)] g(t)\,dt \right| \leq \frac{H}{(qr+1)^{1/q}} \left[\left((x-a)^{r+\frac{1}{q}} \right)^q + \left((b-x)^{r+\frac{1}{q}} \right)^q \right]^{1/q} \cdot \|g\|_p. \tag{4.11}$$

It follows by (3.6), that

$$\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| du(x)$$
$$\le \frac{H}{(qr+1)^{1/q}}\cdot\|g\|_p \int_a^b \left[\left((x-a)^{r+\frac{1}{q}}\right)^q + \left((b-x)^{r+\frac{1}{q}}\right)^q\right]^{1/q} du(x)$$
$$\le \frac{H}{(qr+1)^{1/q}}\cdot\|g\|_p\cdot\int_a^b \left[(x-a)^{r+\frac{1}{q}} + (b-x)^{r+\frac{1}{q}}\right] du(x) \tag{4.12}$$

Now, using Riemann–Stieltjes integral we have

$$\int_a^b (x-a)^{r+\frac{1}{q}}\,du(x) = (b-a)^{r+\frac{1}{q}}\,u(b) - \left(r+\frac{1}{q}\right)\int_a^b (x-a)^{r+\frac{1}{q}-1}\,u(x)\,dx \tag{4.13}$$

and

$$\int_a^b (b-x)^{r+\frac{1}{q}}\,du(x) = -(b-a)^{r+\frac{1}{q}}\,u(a) + \left(r+\frac{1}{q}\right)\int_a^b (b-x)^{r+\frac{1}{q}-1}\,u(x)\,dx \tag{4.14}$$

Adding (4.13) and (4.14), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x). \tag{4.15}$$
$$= (b-a)^{r+\frac{1}{q}}[u(b)-u(a)] + \left(r+\frac{1}{q}\right)\int_a^b \left[(b-x)^{r+\frac{1}{q}-1} - (x-a)^{r+\frac{1}{q}-1}\right] u(x)\,dx.$$

Now, by the monotonicity property of u we have

$$\int_a^b (x-a)^{r+\frac{1}{q}-1}\,u(x)\,dx \ge u(a)\int_a^b (x-a)^{r+\frac{1}{q}-1}\,dx$$
$$= \frac{q}{qr+1}(b-a)^{r+\frac{1}{q}}\cdot u(a) \tag{4.16}$$

and

$$\int_a^b (b-x)^{r+\frac{1}{q}-1}\,u(x)\,dx \le u(b)\int_a^b (b-x)^{r+\frac{1}{q}-1}\,dx$$
$$= \frac{q}{qr+1}(b-a)^{r+\frac{1}{q}}\cdot u(b). \tag{4.17}$$

Substituting (4.16) and (4.17) in (4.15), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x) \le 2(b-a)^{r+\frac{1}{q}}[u(b)-u(a)] \tag{4.18}$$

Substituting (4.18) in (4.12), we get

$$\int_a^b \left| \int_a^b [f(x) - f(t)] g(t) \, dt \right| du(x) \le \frac{2H}{(qr+1)^{1/q}} (b-a)^{r+\frac{1}{q}} \cdot \|g\|_p \cdot [u(b) - u(a)],$$

and therefore, by (4.2) we get the second inequality in (4.1).

Finally, to prove the third inequality in (4.1). As f is of r-H-Hölder type on $[a,b]$ and $g \in L_1[a,b]$ therefore, by (3.7), we have

$$\left| \int_a^b [f(x) - f(t)] g(t) \, dt \right| \le H \|g\|_1 \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r, \tag{4.19}$$

which gives by (4.2), that

$$\int_a^b \left| \int_a^b [f(x) - f(t)] g(t) \, dt \right| du(x) \le H \|g\|_1 \cdot \int_a^b \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r du(x). \tag{4.20}$$

Now, using Riemann–Stieltjes integral we have

$$\begin{aligned}
&\int_a^b \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r du(x) \\
&= \int_a^{\frac{a+b}{2}} (b-x)^r \, du(x) + \int_{\frac{a+b}{2}}^b (x-a)^r \, du(x) \\
&= \left(\frac{b-a}{2} \right)^r u\left(\frac{a+b}{2} \right) - (b-a)^r u(a) + r \int_a^{\frac{a+b}{2}} (b-x)^{r-1} u(x) \, dx \\
&\quad + (b-a)^r u(b) - \left(\frac{b-a}{2} \right)^r u\left(\frac{a+b}{2} \right) - r \int_{\frac{a+b}{2}}^b (x-a)^{r-1} u(x) \, dx \\
&= (b-a)^r [u(b) - u(a)] + r \left[\int_a^{\frac{a+b}{2}} (b-x)^{r-1} u(x) \, dx - \int_{\frac{a+b}{2}}^b (x-a)^{r-1} u(x) \, dx \right]
\end{aligned} \tag{4.21}$$

Now, by the monotonicity property of u we have

$$\begin{aligned}
\int_a^{\frac{a+b}{2}} (b-x)^{r-1} u(x) \, dx &\le u\left(\frac{a+b}{2} \right) \int_a^{\frac{a+b}{2}} (b-x)^{r-1} \, dx \\
&= \frac{(2^r - 1)}{r 2^r} (b-a)^r \cdot u\left(\frac{a+b}{2} \right)
\end{aligned} \tag{4.22}$$

and

$$\int_{\frac{a+b}{2}}^{b}(x-a)^{r-1}u(x)\,dx \geq u\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}(x-a)^{r-1}\,dx$$

$$= \frac{(2^r-1)}{r2^r}(b-a)^r \cdot u\left(\frac{a+b}{2}\right) \tag{4.23}$$

Substituting (4.22) and (4.23) in (4.21), we get

$$\int_a^b \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r du(x) \leq (b-a)^r\left[u(b)-u(a)\right]. \tag{4.24}$$

Substituting (4.24) in (4.20), we get

$$\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| du(x) \leq H\left\|g\right\|_1 \cdot (b-a)^r\left[u(b)-u(a)\right]$$

and therefore, by (4.2) we get the third inequality in (4.1), and thus the theorem is proved. □

Remark 4. *Under the assumptions of Theorem 4, one may deduce several inequalities for the functionals (1.5) and (1.6).*

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Department of Mathematics, Faculty of Science, Jerash University, 26150 Jerash, Jordan

E-mail address: `mwomath@gmail.com`