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# ON HERMITE HADAMARD-TYPE INEQUALITIES FOR $\varphi_h$-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce the notion of $\varphi_h$-convex functions and present some properties and representation of such functions. Finaly, a version of Hermite Hadamard-type inequalities for $\varphi_h$-convex functions are established.

## 1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[4],[8, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([3]-[10]) and the references cited therein.

Let $I$ be an interval in $\mathbb{R}$ and $h : (0, 1) \to (0, \infty)$ be a given function. A function $f : I \to \mathbb{R}$ is said to be $h$-convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.2}$$

for all $x, y \in I$ and $t \in (0, 1)$ [15]. This notion unifies and generalizes the known classes of functions, $s$-convex functions, Gudunova-Levin functions and $P$-functions, which are obtained by putting in (1.2), $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, respectively. Many properties of them can be found, for instance, in [6],[7],[11],[13]-[15].

Let us consider a function $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the $\varphi$-convex functions in [12]:

**Definition 1.** *A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi$- convex on $[a, b]$ if for every two points $x \in [a, b], y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)).$$

In [2], Cristescu proved the followig results for the $\varphi$-convex functions

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**Lemma 1.** *For $f:[a,b]\to\mathbb{R}$, the following statements are equivalent:*

*(i) $f$ is $\varphi$-convex functions on $[a,b]$,*

*(ii) for every $x,y\in[a,b]$, the mapping $g:[0,1]\to\mathbb{R},\ g(t)=f(t\varphi(x)+(1-t)\varphi(y))$ is classically convex on $[0,1]$.*

Obviously, if function $\varphi$ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the $\varphi$-convex functions can be found, for instance, in [1], [2],[12].

In this paper, we introduce the notion of $\varphi_h$-convex functions defined and present some properties of them. In particular, we obtain a representation of $\varphi_h$-convex functions that involves the notion of $\varphi$-convex function. Finally, a version of Hermite–Hadamard-type inequalities for $\varphi_h$-convex functions is presented. Our results generalize the Hermite–Hadamard-type inequalities obtained in [2], [6], [7],[13].

## 2. Main Results

Now we will give a new definition for convex functions. Let $I$ be an interval in $\mathbb{R}$ and $h:(0,1)\to(0,\infty)$ be a given function. We say that a function $f:I\to[0,\infty)$ is $\varphi_h$-convex if

$$f(t\varphi(x)+(1-t)\varphi(y))\leq h(t)f(\varphi(x))+h(1-t)f(\varphi(y)) \tag{2.1}$$

for all $x,y\in I$ and $t\in(0,1)$. If inequality (2.1) is reversed, then $f$ is said to be $\varphi_h$-concave. In particular, if $f$ satisfies (2.1) with $h(t)=t$, $h(t)=t^s$ $(s\in(0,1))$, $h(t)=\frac{1}{t}$, and $h(t)=1$, then $f$ is said to be $\varphi$-convex, $\varphi_s$-convex, $\varphi$-Gudunova-Levin function and $\varphi$-$P$-function, respectively. We start with the following lemma which give some relationships between $\varphi_h$-convex functions and $h$-convex functions.

**Remark 1.** *Let $h:(0,1)\to(0,\infty)$ be a given function such that $h(t)\geq t$ for all $t\in(0,1)$. If $f$ is $\varphi$-convex on $I$, then for $x,y\in I$ and $t\in(0,1)$*

$$\begin{aligned} f(t\varphi(x)+(1-t)\varphi(y)) &\leq tf(\varphi(x))+(1-t)f(\varphi(y)) \\ &\leq h(t)f(\varphi(x))+h(1-t)f(\varphi(y)) \end{aligned}$$

*i.e $f:I\to[0,\infty)$ is $\varphi_h$-convex. Similarly, if the function $h$ has the property: $h(t)\leq t$ for all $t\in(0,1)$, then $f:I\to[0,\infty)$ is $\varphi_h$-concave function.*

**Lemma 2.** *Let $h_1,h_2:(0,1)\to(0,\infty)$ be a given functions such that $h_2(t)\leq h_1(t)$ for all $t\in(0,1)$. If $f$ is $\varphi_{h_2}$-convex on $I$, then for $x,y\in I$, $f$ is $\varphi_{h_1}$-convex on $I$.*

*Proof.* Since $f$ is $\varphi_{h_2}$-convex on $I$, thus for $x,y\in I$ and $t\in(0,1)$, we have

$$\begin{aligned} f(t\varphi(x)+(1-t)\varphi(y)) &\leq h_2(t)f(\varphi(x))+h_2(1-t)f(\varphi(y)) \\ &\leq h_1(t)f(\varphi(x))+h_1(1-t)f(\varphi(y)). \end{aligned}$$

□

**Lemma 3.** *Let $h:(0,1)\to(0,\infty)$ be a given functions. If $f,g:I\to[0,\infty)$ are $\varphi_h$-convex function on $I$ and $\alpha>0$, then for all $t\in(0,1)$. $f+g$ and $\alpha f$ are $\varphi_h$-convex on $I$.*

*Proof.* By definition of $\varphi_h$-convexity, proof is obvious. □

**Lemma 4.** *Let $f:[a,b]\rightarrow[0,\infty)$ and $h:(0,1)\rightarrow(0,\infty)$. Then the following statements are equivalent:*

*i) A function $f$ is $\varphi_h$-convex on $[a,b]$*

*ii) For every $x,y\in[a,b]$, the mapping $g:[0,1]\rightarrow\mathbb{R}$, $g(t)=f(t\varphi(x)+(1-t)\varphi(y))$is $h$-convex on $[a,b]$.*

*Proof.* Assume that $f$ is $\varphi_h$-convex function and let us consider two points $x,y\in[a,b]$, $\lambda\in(0,1)$ and $t_1,t_2\in[0,1]$, then we obtain

$$\begin{aligned}
&g(\lambda t_1+(1-\lambda)t_2)\\
&= f([\lambda t_1+(1-\lambda)t_2]\,\varphi(x)+[1-\lambda t_1-(1-\lambda)t_2]\,\varphi(y))\\
&= f(\lambda\,[t_1\varphi(x)+(1-t_1)\varphi(x)]+(1-\lambda)\,[t_2\varphi(y)+(1-t_2)\varphi(y)])\\
&\leq h(\lambda)f(t_1\varphi(x)+(1-t_1)\varphi(x))+h(1-\lambda)f(t_2\varphi(y)+(1-t_2)\varphi(y))\\
&= h(\lambda)g(t_1)+h(1-\lambda)g(t_1)
\end{aligned}$$

which gives that $g$ is $h$-convex function.

Conversely, if $g$ is $h$-convex function, then for $x,y\in[a,b]$, $\lambda\in(0,1)$ and $t_1=1$ and $t_2=0$, we get

$$\begin{aligned}
f(\lambda\varphi(x)+(1-\lambda)\varphi(y)) &= g(\lambda 1+(1-\lambda)0)\\
&\leq h(\lambda)g(1)+h(1-\lambda)g(0)\\
&= h(\lambda)f(\varphi(x))+h(1-\lambda)f(\varphi(y))
\end{aligned}$$

which shows that $f$ is $\varphi_h$-convex function. This completes to proof. □

Now, we give a new Hermite–Hadamard-type inequalities for $\varphi_h$-convex functions as follows:

**Theorem 1.** *Let $h:(0,1)\rightarrow(0,\infty)$ be a given function. If $f:I\rightarrow[0,\infty)$ is Lebesgue integrable and $\varphi_h$-convex for continuous function $\varphi:[a,b]\rightarrow[a,b]$, then the following inequalities hold:*

(2.2)
$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)}f(x)dx\leq[f(\varphi(a))+f(\varphi(b))]\int\limits_{0}^{1}h(t)dt.$$

*Proof.* By using the $\varphi_h$-convextiy of $f$, we have

(2.3)
$$\begin{aligned}
&f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\\
&= f\left(\frac{t\varphi(a)+(1-t)\,\varphi(b)}{2}+\frac{(1-t)\,\varphi(a)+t\varphi(b)}{2}\right)\\
&\leq h(\frac{1}{2})\,[f\,(t\varphi(a)+(1-t)\,\varphi(b))+f\,((1-t)\,\varphi(a)+t\varphi(b))]\,.
\end{aligned}$$

Integrating both sides of (2.3) over the interval $(0,1)$, it follows that the inequality

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq h(\frac{1}{2})\left[\int_0^1 f\left(t\varphi(a)+(1-t)\,\varphi(b)\right)dt+\int_0^1 f\left((1-t)\,\varphi(a)+t\varphi(b)\right)dt\right].$$

In the first integral, we substitute $x=t\varphi(a)+(1-t)\,\varphi(b)$. Meanwhile, in the second integral, we also use the substitution $x=(1-t)\,\varphi(a)+t\varphi(b)$, we have

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)dx.$$

In order the prove the second inequality of (2.2), we start from the $\varphi_h$-convextiy of $f$, meaning that for every $t\in(0,1)$, one has

$$f(t\varphi(a)+(1-t)\varphi(b))\leq h(t)f(\varphi(a))+h(1-t)f(\varphi(b)).$$

Integrating both sides of the above inequality over $(0,1)$, we obtain

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))dt\leq f(\varphi(a))\int_0^1 h(t)dt+f(\varphi(b))\int_0^1 h(1-t)dt.$$

The previous substitution in the first side of this inequality leads to

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)dx\leq\left[f(\varphi(a))+f(\varphi(b))\right]\int_0^1 h(t)dt$$

which gives the second inequalities of (2.2). This completes to proof. □

**Remark 2.** *If $h(t)=t,\ t\in(0,1)$, then the inequalities (2.2) coincide with the Hermite-Hadamard type inequalities for $\varphi$- convex functions proved by Cristescu in* [2].

**Corollary 1.** *Under the assumptions of Theorem 1 with $h(t)=t^s$ $(s\in(0,1))$, $t\in(0,1)$, we have*

$$2^{s-1}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)dx\leq\frac{f(\varphi(a))+f(\varphi(b))}{s+1}. \tag{2.4}$$

These inequalities are associated Hermite-Hadamard type inequalities for $\varphi_s$-convex functions.

**Remark 3.** *If function $\varphi$ is the identity in (2.4), then it reduces to the Hermite-Hadamard type inequalities for s-convex functions proved by Dragomir and Fitzpatrik* [6].

**Corollary 2.** *Under the assumptions of Theorem 1 with $h(t)=\frac{1}{t}$, $t\in(0,1)$, we have*

$$\frac{1}{4}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)dx\ \ (\leq\infty). \tag{2.5}$$

This inequality is associated Hermite-Hadamard type inequalities for $\varphi-$Godunova–Levin functions.

**Remark 4.** *If function $\varphi$ is the identity in (2.5), then it reduces to the Hermite-Hadamard type inequalities for Godunova–Levin functions obtained by Dragomir, Pecaric and Persson* [7].

**Corollary 3.** *Under the assumptions of Theorem 1 with $h(t) = 1,\ t \in (0,1)$, we have*

$$\frac{1}{2} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq f(\varphi(a)) + f(\varphi(b)). \tag{2.6}$$

These inequalities are associated Hermite-Hadamard type inequalities for $\varphi$-$P$-convex functions.

**Remark 5.** *If function $\varphi$ is the identity in (2.6), then it reduces to the Hermite-Hadamard type inequalities for $P$-convex functions obtained by Dragomir, Pecaric and Persson* [7].

**Theorem 2.** *Let $h : (0,1) \to (0,\infty)$ be a given function. If $f : I \to [0,\infty)$ is Lebesgue integrable and $\varphi_h$-convex for continuous function $\varphi : [a,b] \to [a,b]$, then the following inequalities hold:*

$$\begin{aligned} & \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) f(a+b-x) dx \\ \leq\ & \left[f^2(\varphi(a)) + f^2(\varphi(b))\right] \int_0^1 h(t)h(1-t)dt + 2f(\varphi(a))f(\varphi(b)) \int_0^1 h^2(t)dt. \end{aligned} \tag{2.7}$$

*Proof.* Since $f$ is $\varphi_h$-convex function, we have that for all $t \in (0,1)$

$$f(t\varphi(a) + (1-t)\varphi(b)) \leq h(t)f(\varphi(a)) + h(1-t)f(\varphi(b)) \tag{2.8}$$

and

$$f((1-t)\varphi(a) + t\varphi(b)) \leq h(1-t)f(\varphi(a)) + h(t)f(\varphi(b)). \tag{2.9}$$

Multiplying both sides of (2.8) by (2.9), it follows that

$$\begin{aligned} & f(t\varphi(a) + (1-t)\varphi(b)) f((1-t)\varphi(a) + t\varphi(b)) \\ \leq\ & h(t)h(1-t)\left[f^2(\varphi(a)) + f^2(\varphi(b))\right] + \left(h^2(t) + h^2(1-t)\right) f(\varphi(a))f(\varphi(b)). \end{aligned}$$

Integrating the above inequality with respect to $t$ over $(0,1)$, we obtain

$$\begin{aligned} & \int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) f((1-t)\varphi(a) + t\varphi(b)) dt \\ \leq\ & \left[f^2(\varphi(a)) + f^2(\varphi(b))\right] \int_0^1 h(t)h(1-t)dt + 2f(\varphi(a))f(\varphi(b)) \int_0^1 h^2(t)dt. \end{aligned}$$

If we change the variable $x := t\varphi(a) + (1-t)\varphi(b)$, $t \in (0,1)$, we get the required inequality in (2.5). This proves the theorem. $\square$

**Theorem 3.** *Let $h : (0,1) \to (0,\infty)$ be a given function. If $f, g : I \to [0,\infty)$ are Lebesgue integrable and $\varphi_h$-convex for continuous function $\varphi : [a,b] \to [a,b]$, then the following inequalities hold:*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq M(a,b)\int_0^1 h^2(t)dt + N(a,b)\int_0^1 h(t)h(1-t)dt \tag{2.10}$$

*where*

$$M(a,b) = f(\varphi(a))g(\varphi(a)) + f(\varphi(b))g(\varphi(b))$$
$$N(a,b) = f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a)).$$

*Proof.* Since $f, g : I \to \mathbb{R}$ is $\varphi_h$- convex functions, we have

$$f\left(t\varphi(a) + (1-t)\varphi(b)\right) \leq h(t)f\left(\varphi(a)\right) + h(1-t)f\left(\varphi(b)\right) \tag{2.11}$$

$$g\left(t\varphi(a) + (1-t)\varphi(b)\right) \leq h(t)g\left(\varphi(a)\right) + h(1-t)g\left(\varphi(b)\right). \tag{2.12}$$

Multiplying both sides of (2.11) by (2.12), it follows that

$$\begin{aligned} & f\left(t\varphi(a) + (1-t)\varphi(b)\right) g\left(t\varphi(a) + (1-t)\varphi(b)\right) \\ \leq \; & h^2(t)f\left(\varphi(a)\right) g\left(\varphi(a)\right) + h^2(1-t)f(\varphi(b))f(\varphi(b)) \\ & + h(t)h(1-t)\left[f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))\right]. \end{aligned}$$

Integrating both sides of the above inequality over $(0,1)$, we obtain

$$\begin{aligned} & \int_0^1 f\left(t\varphi(a) + (1-t)\varphi(b)\right) g\left(t\varphi(a) + (1-t)\varphi(b)\right) dt \\ \leq \; & \left[f\left(\varphi(a)\right) g\left(\varphi(a)\right) + f(\varphi(b))f(\varphi(b))\right]\int_0^1 h^2(t)dt \\ & + \left[f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))\right]\int_0^1 h(t)h(1-t)dt. \end{aligned}$$

In the first integral, we substitute $x = t\varphi(a) + (1-t)\varphi(b)$ and simple integrals calculated, we obtain the required inequality in (2.10). $\square$

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