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ON STRONGLY φ_h-CONVEX FUNCTIONS IN INNER PRODUCT SPACES

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Abstract. In this paper, we introduce the notion of strongly φ_h-convex functions with respect to $c>0$ and present some properties and representation of such functions. We obtain a characterization of inner product spaces involving the notion of strongly φ_h-convex functions. Finaly, a version of Hermite Hadamard-type inequalities for strongly φ_h-convex functions are established.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[4],[8, p.137]). These inequalities state that if $f:I\rightarrow\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([3]-[10]) and the references cited therein.

Let I be an interval in $\mathbb{R}$ and $h:(0,1)\rightarrow(0,\infty)$ be a given function. A function $f:I\rightarrow[0,\infty)$ is said to be h-convex if

$$f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y) \tag{1.2}$$

for all $x,y\in I$ and $t\in(0,1)$ [20]. This notion unifies and generalizes the known classes of functions, s-convex functions, Gudunova-Levin functions and P-functions, which are obtained by putting in (1.2), $h(t)=t$, $h(t)=t^s$, $h(t)=\frac{1}{t}$, and $h(t)=1$, respectively. Many properties of them can be found, for instance, in [6],[7],[14],[16],[17],[19],[20].

Let us consider a function $\varphi:[a,b]\rightarrow[a,b]$ where $[a,b]\subset\mathbb{R}$. Youness have defined the φ-convex functions in [15]:

Definition 1. *A function $f:[a,b]\rightarrow\mathbb{R}$ is said to be φ- convex on $[a,b]$ if for every two points $x\in[a,b],y\in[a,b]$ and $t\in[0,1]$ the following inequality holds:*

$$f(t\varphi(x)+(1-t)\varphi(y))\leq tf(\varphi(x))+(1-t)f(\varphi(y)).$$

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Obviously, if function φ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the φ-convex functions can be found, for instance, in [1], [2],[15],[18],[19].

Recall also that a function $f : I \to \mathbb{R}$ is called strongly convex with modulus $c > 0$, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all $x, y \in I$ and $t \in (0,1)$. Strongly convex functions have been introduced by Polyak in [11] and they play an important role in optimization theory and mathematical economics. Various properties and applicatins of them can be found in the literature see ([11]-[14]) and the references cited therein.

In this paper, we introduce the notion of strongly φ_h-convex functions defined in normed spaces and present some properties of them. In particular, we obtain a representation of strongly φ_h-convex functions in inner product spaces and, using the methods of [13],[14] and [18], we give a characterization of inner product spaces, among normed spaces, that involves the notion of strongly φ_h-convex function. Finally, a version of Hermite–Hadamard-type inequalities for strongly φ_h-convex functions is presented. This result generalizes the Hermite–Hadamard-type inequalities obtained by Sarikaya in [18] for strongly φ-convex functions, and for $c = 0$, coincides with the Hermite–Hadamard inequalities for φ_h-convex functions proved by Sarikaya in [19].

2. Main Results

In what follows $(X, \|.\|)$ denotes a real normed space, D stands for a convex subset of X, $\varphi : D \to D$ is a given function and c is a positive constant. Let $h : (0,1) \to (0,\infty)$ be a given function. We say that a function $f : D \to [0,\infty)$ is strongly φ_h-convex with modulus c if

$$\begin{aligned} & f(t\varphi(x) + (1-t)\varphi(y)) \\ \leq \; & h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 \end{aligned} \tag{2.1}$$

for all $x, y \in D$ and $t \in (0,1)$. In particular, if f satisfies (2.1) with $h(t) = t$, $h(t) = t^s$ ($s \in (0,1)$), $h(t) = \frac{1}{t}$, and $h(t) = 1$, then f is said to be strongly φ-convex, strongly φ_s-convex, strongly φ-Gudunova-Levin function and strongly φ-P-function, respectively. The notion of φ_h-convex function corresponds to the case $c = 0$. We start with the following lemma which give some relationships between strongly φ_h-convex functions and φ_h-convex functions in the case where X is a real inner product space (that is, the norm $\|.\|$ is induced by an inner product: $\|.\| := < x|x >$).

Remark 1. *Let $h : (0,1) \to (0,\infty)$ be a given function such that $h(t) \geq t$ for all $t \in (0,1)$. If f is strongly φ-convex on I, then for $x, y \in I$ and $t \in (0,1)$*

$$\begin{aligned} f(t\varphi(x) + (1-t)\varphi(y)) &\leq tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 \\ &\leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 \end{aligned}$$

i.e $f : I \to [0,\infty)$ is strongly φ_h-convex.

Lemma 1. *Let $h_1, h_2 : (0,1) \to (0,\infty)$ be a given functions such that $h_2(t) \leq h_1(t)$ for all $t \in (0,1)$. If f is strongly φ_{h_2}-convex on I, then for $x, y \in I$, f is strongly φ_{h_1}-convex on I.*

Proof. Since f is strongly φ_{h_2}-convex on I, thus for $x, y \in I$ and $t \in (0,1)$, we have

$$\begin{aligned} f(t\varphi(x)+(1-t)\varphi(y)) &\leq h_2(t)f(\varphi(x))+h_2(1-t)f(\varphi(y))-ct(1-t)\left\|\varphi(x)-\varphi(y)\right\|^2 \\ &\leq h_1(t)f(\varphi(x))+h_1(1-t)f(\varphi(y))-ct(1-t)\left\|\varphi(x)-\varphi(y)\right\|^2. \end{aligned}$$

□

Lemma 2. *Let $h:(0,1)\to(0,\infty)$ be a given functions. If $f,g:I\to[0,\infty)$ are strongly φ_h-convex function on I and $\alpha>0$, then for all $t\in(0,1)$. $f+g$ and αf are strongly φ_h-convex on I.*

Proof. By definition of strongly φ_h-convexity, the proof is obvious. □

Lemma 3. *Let $(X,\|.\|)$ be a real inner product space, D be a convex subset of X and c be a positive constant and $\varphi:D\to D$. Assume that $h:(0,1)\to(0,\infty)$ be a given function.*

i) If $h(t)\leq t,\ t\in(0,1)$ and a function $f:D\to[0,\infty)$ is strongly φ_h-convex with modulus c, then the function $g=f-c\|.\|^2$ is φ_h-convex.

ii) If $h(t)\leq t,\ t\in(0,1)$ and the function $g=f-c\|.\|^2$ is φ_h-convex, then the function $f:D\to[0,\infty)$ is strongly φ-convex with modulus c.

iii) If $h(t)\geq t,\ t\in(0,1)$ and a function $f:D\to[0,\infty)$ is strongly φ_h-convex with modulus c, then the function $g=f-c\|.\|^2$ is φ_h-convex.

Proof. i) Assume that f is strongly φ_h-convex with modulus c. Using properties of the inner product and assumption $h(t)\leq t,\ t\in(0,1)$, we obtain

$$\begin{aligned} & g(t\varphi(x)+(1-t)\varphi(y)) \\ = {} & f(t\varphi(x)+(1-t)\varphi(y))-c\left\|t\varphi(x)+(1-t)\varphi(y)\right\|^2 \\ \leq {} & h(t)f(\varphi(x))+h(1-t)f(\varphi(y))-ct(1-t)\left\|\varphi(x)-\varphi(y)\right\|^2-c\left\|t\varphi(x)+(1-t)\varphi(y)\right\|^2 \\ \leq {} & h(t)f(\varphi(x))+h(1-t)f(\varphi(y))-c\Big(t(1-t)\left[\left\|\varphi(x)\right\|^2-2<\varphi(x)|\varphi(y)>+\left\|\varphi(y)\right\|^2\right] \\ & -\left[t^2\left\|\varphi(x)\right\|^2+2t(1-t)<\varphi(x)|\varphi(y)>+(1-t)\left\|\varphi(y)\right\|^2\right]\Big) \\ = {} & h(t)f(\varphi(x))+h(1-t)f(\varphi(y))-ct\left\|\varphi(x)\right\|^2-c(1-t)\left\|\varphi(y)\right\|^2 \\ \leq {} & h(t)f(\varphi(x))+h(1-t)f(\varphi(y))-ch(t)\left\|\varphi(x)\right\|^2-ch(1-t)\left\|\varphi(y)\right\|^2 \\ = {} & h(t)g(\varphi(x))+h(1-t)g(\varphi(y)) \end{aligned}$$

which gives that g is φ_h-convex function.

ii) Since g is φ_h-convex function and by using assumption $h(t) \leq t,\ t \in (0,1)$, then we get

$$\begin{aligned} f(t\varphi(x)+(1-t)\varphi(y)) &= g(t\varphi(x)+(1-t)\varphi(y))+c\left\|t\varphi(x)+(1-t)\varphi(y)\right\|^2 \\ &\leq h(t)g(\varphi(x))+h(1-t)g(\varphi(y))+c\left\|t\varphi(x)+(1-t)\varphi(y)\right\|^2 \\ &\leq t\left[f(\varphi(x))+c\left\|\varphi(x)\right\|^2\right]+(1-t)\left[f(\varphi(y))+c\left\|\varphi(y)\right\|^2\right] \\ &\quad -ct(1-t)\left[\left\|\varphi(x)\right\|^2-2<\varphi(x)|\varphi(y)>+\left\|\varphi(y)\right\|^2\right] \\ &= tf(\varphi(x))+(1-t)f(\varphi(y))-ct(1-t)\left\|\varphi(x)-\varphi(y)\right\|^2 \\ &\leq tf(\varphi(x))+(1-t)f(\varphi(y))-ct(1-t)\left\|\varphi(x)-\varphi(y)\right\|^2 \end{aligned}$$

which shows that f is strongly φ-convex with modulus c.

iii) In a similar way we can prove it. This completes to proof. □

The following example shows that the assumption that X is an inner product space is essentials in the above lemma.

Example. Let $X = \mathbb{R}^2$ and $h(t) = t,\ t \in (0,1)$. Let us consider a function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\varphi(x) = x$ for every $x \in \mathbb{R}^2$ and $\|x\| = \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2)$. Take $f = \|.\|^2$. Then $g = f - \|.\|^2$ is φ_h-convex being the zero function. However, f is not strongly φ_h-convex with modulus 1. Indeed, for $x = (1,0)$ and $y = (0,1)$, we have

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2} \geq \frac{3}{4} = \frac{f(x)+f(y)}{2} - \frac{1}{4}\left\|x-y\right\|^2$$

which this contradicts (2.1).

The assumption that X is an inner product space in Lemma 3 is essential. Moreover, it appears that the fact that for every φ_h-convex function $g : X \to \mathbb{R}$ the function $f = g + c\|.\|^2$ is strongly φ_h-convex characterizes inner product spaces among normed spaces. Similar characterizations of inner product spaces by strongly convex, strongly h-convex and strongly φ-convex functions are presented in [13], [14] and [18], respectively.

Theorem 1. *Let $(X, \|.\|)$ be a real normed space, D be a convex subset of X and $\varphi : D \to D$. Assume that $h : (0,1) \to (0,\infty)$ and $h(\frac{1}{2}) = \frac{1}{2}$. Then the following conditions are equivalent:*

i) $(X, \|.\|)$ be a real inner product;

ii) For every $c > 0$, $f : D \to [0,\infty)$ defined on a convex subset D of X, the function $f = g + c\|.\|^2$ is strongly φ_h-convex with modulus c;

iii) $\|.\|^2 : X \to [0,\infty)$ is strongly φ_h-convex with modulus 1.

Proof. The implication i)⇒ii) follows by Lemma 3. To see that ii)⇒iii) take $g = 0$. Clearly, g is φ_h-convex function, whence $f = c\|.\|^2$ is strongly φ_h-convex with modulus c. Consequently, $\|.\|^2$ is strongly φ_h-convex with modulus 1. Finaly, to prove iii)$\Rightarrow$i) observe that by the strongly φ_h-convexity of $\|.\|^2$ and assumption $h(\frac{1}{2}) = \frac{1}{2}$, we obtain

$$\left\|\frac{\varphi(x)+\varphi(y)}{2}\right\|^2 \leq \frac{\|\varphi(x)\|^2}{2} + \frac{\|\varphi(y)\|^2}{2} - \frac{1}{4}\|\varphi(x)+\varphi(y)\|^2$$

and hence

$$\|\varphi(x)+\varphi(y)\|^2 \leq 2\|\varphi(x)\|^2 + 2\|\varphi(y)\|^2 \tag{2.2}$$

for all $x, y \in X$. Now, putting $u = \varphi(x)+\varphi(y)$ and $v = \varphi(x)-\varphi(y)$ in (2.2), we have

$$2\|u\|^2 + 2\|v\|^2 \leq \|u+v\|^2 + \|u-v\|^2 \tag{2.3}$$

for all $u, v \in X$.

Conditions (2.2) and (2.3) mean that the norm $\|.\|^2$ satisfies the parallelogram law, which implies, by the classical Jordan-Von Neumann theorem, that $(X, \|.\|)$ is an inner product space. This completes to proof. $\square$

Now, we give a new Hermite–Hadamard-type inequalities for strongly φ_h-convex functions with modulus c as follows:

Theorem 2. *Let $h : (0,1) \to (0,\infty)$ be a given function. If a function $f : I \to [0,\infty)$ is Lebesgue integrable and strongly φ_h- convex with modulus $c > 0$ for the continuous function $\varphi : [a,b] \to [a,b]$, then*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{24h(\frac{1}{2})}\left(\varphi(a)-\varphi(b)\right)^2 \\ \leq\ & \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \\ \leq\ & [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t)dt - \frac{c}{6}\left(\varphi(a)-\varphi(b)\right)^2 . \end{aligned} \tag{2.4}$$

Proof. From the strongly φ_h- convexity of f, we have

$$\begin{aligned} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &= f\left(\frac{t\varphi(a)+(1-t)\varphi(b)}{2} + \frac{(1-t)\varphi(a)+t\varphi(b)}{2}\right) \\ &\leq h(\frac{1}{2})\left[f\left(t\varphi(a)+(1-t)\varphi(b)\right) + f\left((1-t)\varphi(a)+t\varphi(b)\right)\right] \\ &\quad -\frac{c}{4}(1-2t)^2\left(\varphi(a)-\varphi(b)\right)^2 . \end{aligned}$$

Integrating the above inequality over the interval $(0,1)$, we obtain

$$\begin{aligned} &f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{12}\left(\varphi(a)-\varphi(b)\right)^2 \\ \leq\ & h(\frac{1}{2})\left[\int_0^1 f\left(t\varphi(a)+(1-t)\varphi(b)\right)dt + \int_0^1 f\left((1-t)\varphi(a)+t\varphi(b)\right)dt\right]. \end{aligned}$$

In the first integral, we substitute $x = t\varphi(a)+(1-t)\varphi(b)$. Meanwhile, in the second integral we also use the substitution $x = (1-t)\varphi(a) + t\varphi(b)$, we obtain

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{12}\left(\varphi(a)-\varphi(b)\right)^2 \le \frac{2h(\frac{1}{2})}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx.$$

In order to prove the second inequality, we start from the strongly φ_h- convexity of f meaning that for every $t \in (0,1)$ one has

$$f(t\varphi(a) + (1-t)\varphi(b)) \le h(t)f(\varphi(a)) + h(1-t)f(\varphi(b)) - ct(1-t)\left(\varphi(a)-\varphi(b)\right)^2.$$

Integrating the above inequality over the interval $(0,1)$, we get

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))dt \le [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t)dt - c\left(\varphi(a)-\varphi(b)\right)^2 \int_0^1 t(1-t)dt.$$

The previous substitution in the first side of this inequality leads to

$$\frac{1}{(\varphi(a)-\varphi(b))} \int_{\varphi(b)}^{\varphi(a)} f(x)\,dx \le [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t)dt - \frac{c}{6}\left(\varphi(a)-\varphi(b)\right)^2$$

which gives the second inequality of (2.4). This completes to proof. □

Remark 2. *If $h(t) = t,\ t \in (0,1)$, then the inequalities (2.4) coincide with the Hermite-Hadamard type inequalities for strongly φ- convex functions proved by Sarikaya in* [18].

Corollary 1. *Under the assumptions of Theorem 2 with $h(t) = t^s$ $(s \in (0,1))$, $t \in (0,1)$, we have*

$$\begin{aligned} & 2^{s-1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c2^s}{24}\left(\varphi(a)-\varphi(b)\right)^2 \\ \le\ & \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \\ \le\ & \frac{f(\varphi(a)) + f(\varphi(b))}{s+1} - \frac{c}{6}\left(\varphi(a)-\varphi(b)\right)^2. \end{aligned}$$

These inequalities are associated Hermite-Hadamard type inequalities for strongly φ_s-convex functions.

Corollary 2. *Under the assumptions of Theorem 2 with $h(t) = \frac{1}{t}$, $t \in (0,1)$, we have*

$$\frac{1}{4} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{48}\left(\varphi(a)-\varphi(b)\right)^2 \le \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \ \ (\le \infty).$$

This inequality is associated Hermite-Hadamard type inequalities for strongly $\varphi-$Godunova–Levin functions.

Corollary 3. *Under the assumptions of Theorem 2 with* $h(t)=1,\ t\in(0,1)$, *we have*

$$\begin{aligned}&\frac{1}{2}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)+\frac{c}{24}\left(\varphi(a)-\varphi(b)\right)^2\\ \leq\ & \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)dx\\ \leq\ & f(\varphi(a))+f(\varphi(b))-\frac{c}{6}\left(\varphi(a)-\varphi(b)\right)^2.\end{aligned}$$

These inequalities are associated Hermite-Hadamard type inequalities for strongly φ-P-convex functions.

Theorem 3. *Let* $h:(0,1)\rightarrow(0,\infty)$ *be a given function. If* $f:I\rightarrow[0,\infty)$ *is Lebesgue integrable and strongly* φ_h*- convex with modulus* $c>0$ *for the continuous function* $\varphi:[a,b]\rightarrow[a,b]$, *then*

$$\begin{aligned}(2.5)\quad & \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f\left(x\right)f\left(a+b-x\right)dx\\ \leq\ & \left[f^2(\varphi(a))+f^2(\varphi(b))\right]\int_0^1 h(t)h(1-t)dt+2f(\varphi(a))f(\varphi(b))\int_0^1 h^2(t)dt\\ & -2c\left(\varphi(a)-\varphi(b)\right)^2\left[f(\varphi(a))+f(\varphi(b))\right]\int_0^1 t(1-t)h(t)dt+\frac{c^2}{30}\left(\varphi(a)-\varphi(b)\right)^4.\end{aligned}$$

Proof. Since f is strongly φ_h-convex with respect to $c>0$, we have that for all $t\in(0,1)$

$$(2.6)\quad f(t\varphi(a)+(1-t)\varphi(b))\leq h(t)f(\varphi(a))+h(1-t)f(\varphi(b))-ct(1-t)\left(\varphi(a)-\varphi(b)\right)^2$$

and

$$(2.7)\quad f((1-t)\varphi(a)+t\varphi(b))\leq h(1-t)f(\varphi(a))+h(t)f(\varphi(b))-ct(1-t)\left(\varphi(a)-\varphi(b)\right)^2.$$

Multiplying both sides of (2.6) by (2.7), it follows that

$$\begin{aligned}(2.8)\quad & f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+t\varphi(b))\\ \leq\ & h(t)h(1-t)\left[f^2(\varphi(a))+f^2(\varphi(b))\right]+\left(h^2(t)+h^2(1-t)\right)f(\varphi(a))f(\varphi(b))\\ & -ct(1-t)\left(\varphi(a)-\varphi(b)\right)^2\left[f(\varphi(a))+f(\varphi(b))\right]\left[h(t)+h(1-t)\right]\\ & +c^2t^2(1-t)^2\left(\varphi(a)-\varphi(b)\right)^4.\end{aligned}$$

Integrating the inequality (2.8) with respect to t over $(0,1)$, we obtain

$$\begin{aligned}
&\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+t\varphi(b))dt \\
\leq\;& \left[f^2(\varphi(a))+f^2(\varphi(b))\right]\int_0^1 h(t)h(1-t)dt+2f(\varphi(a))f(\varphi(b))\int_0^1 h^2(t)dt \\
&-2c\left(\varphi(a)-\varphi(b)\right)^2\left[f(\varphi(a))+f(\varphi(b))\right]\int_0^1 t(1-t)h(t)dt \\
&+\frac{c^2}{30}\left(\varphi(a)-\varphi(b)\right)^4.
\end{aligned}$$

If we change the variable $x:=t\varphi(a)+(1-t)\varphi(b)$, $t\in(0,1)$, we get the required inequality in (2.5). This proves the theorem. □

Theorem 4. *Let $h:(0,1)\to(0,\infty)$ be a given function. If $f,g:I\to[0,\infty)$ is Lebesgue integrable and strongly φ_h- convex with modulus $c>0$ for the continuous function $\varphi:[a,b]\to[a,b]$, then*

$$\begin{aligned}
(2.9)\qquad &\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\leq M(a,b)\int_0^1 h^2(t)dt+N(a,b)\int_0^1 h(t)h(1-t)dt \\
&-c\left(\varphi(a)-\varphi(b)\right)^2 S(a,b)\int_0^1 t\left(1-t\right)h(t)dt+\frac{c^2}{30}\left(\varphi(a)-\varphi(b)\right)^4
\end{aligned}$$

where

$$M(a,b)=f(\varphi(a))g(\varphi(a))+f(\varphi(b))g(\varphi(b))$$

$$N(a,b)=f(\varphi(a))g(\varphi(b))+f(\varphi(b))g(\varphi(a))$$

$$S(a,b)=f\left(\varphi(a)\right)+f\left(\varphi(b)\right)+g\left(\varphi(a)\right)+g\left(\varphi(b)\right).$$

Proof. Since $f,g:I\to[0,\infty)$ is strongly φ_h- convex with modulus $c>0$, we have

$$(2.10)\qquad f\left(t\varphi(a)+(1-t)\varphi(b)\right)\leq h(t)f\left(\varphi(a)\right)+h(1-t)f\left(\varphi(b)\right)-ct\left(1-t\right)\left(\varphi(a)-\varphi(b)\right)^2$$

$$(2.11)\qquad g\left(t\varphi(a)+(1-t)\varphi(b)\right)\leq h(t)g\left(\varphi(a)\right)+h(1-t)g\left(\varphi(b)\right)-ct\left(1-t\right)\left(\varphi(a)-\varphi(b)\right)^2.$$

Multiplying both sides of (2.10) by (2.11), it follows that

$$
\begin{aligned}
& f\left(t\varphi(a)+(1-t)\varphi(b)\right) g\left(t\varphi(a)+(1-t)\varphi(b)\right) \\
\leq\ & h^2(t) f\left(\varphi(a)\right) g\left(\varphi(a)\right)+h^2(1-t) f(\varphi(b)) f(\varphi(b)) \\
& +h(t)h(1-t)\left[f(\varphi(a))g(\varphi(b))+f(\varphi(b))g(\varphi(a))\right] \\
& -ct\left(1-t\right)h(t)\left(\varphi(a)-\varphi(b)\right)^2\left[f\left(\varphi(a)\right)+g\left(\varphi(a)\right)\right] \\
& -ct\left(1-t\right)h(1-t)\left(\varphi(a)-\varphi(b)\right)^2\left[f\left(\varphi(b)\right)+g\left(\varphi(b)\right)\right] \\
& +c^2t^2\left(1-t\right)^2\left(\varphi(a)-\varphi(b)\right)^4.
\end{aligned}
$$

Integrating the above inequality over the interval $(0,1)$, we get

$$
\begin{aligned}
& \int_0^1 f\left(t\varphi(a)+(1-t)\varphi(b)\right) g\left(t\varphi(a)+(1-t)\varphi(b)\right) dt \\
\leq\ & \left[f\left(\varphi(a)\right) g\left(\varphi(a)\right)+f(\varphi(b)) f(\varphi(b))\right] \int_0^1 h^2(t)dt \\
& +\left[f(\varphi(a))g(\varphi(b))+f(\varphi(b))g(\varphi(a))\right] \int_0^1 h(t)h(1-t)dt \\
& -c\left(\varphi(a)-\varphi(b)\right)^2\left[f\left(\varphi(a)\right)+g\left(\varphi(a)\right)+f\left(\varphi(b)\right)+g\left(\varphi(b)\right)\right] \int_0^1 t\left(1-t\right)h(t)dt \\
& +c^2\left(\varphi(a)-\varphi(b)\right)^4 \int_0^1 t^2\left(1-t\right)^2 dt.
\end{aligned}
$$

In the first integral, we substitute $x=t\varphi(a)+(1-t)\varphi(b)$ and simple integrals calculated, we obtain the required inequality in (2.9). □

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