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SOME INEQUALITIES OF JENSEN TYPE FOR OPERATOR CONVEX FUNCTIONS IN HILBERT SPACES

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Abstract. Some inequalities for operator convex functions of selfadjoint operators in Hilbert spaces that are related to the Jensen inequality are given. Natural examples for some fundamental operator convex functions are presented as well.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p}=(p_1,\dots,p_n)$ where $p_j\geq 0$, $j\in\{1,...,n\}$ with $P_n:=\sum_{j=1}^n p_j>0$ and $\mathbf{x}=(x_1,\dots,x_n)\in C^n$, then

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\leq\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

In order to extend this inequality for operator convex functions of selfdjoint bounded linear operators on complex Hilbert spaces we need the following preliminary facts.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $Sp(A), Sp(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$.

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The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

We also have the following Jensen type inequality for operator convex functions $f: I\to\mathbb{R}$.

Let A_j be selfadjoint operators with $Sp(A_j)\subseteq I$, $j\in\{1,...,n\}$. If $p_j\geq 0$, $j\in\{1,...,n\}$ with $P_n>0$ and f is an operator convex function on I then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_iA_j\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_if(A_j), \tag{1.2}$$

in the operator order.

This is a well known result and can be proved easily by mathematical induction over $n\geq 2$. The details are left to the reader.

For recent results related to the Jensen inequality for selfadjoint operators in Hilbert spaces see the papers [1]-[5], [8]-[14], [15] and the monograph [6].

In this paper we consider two functionals associated with the Jensen inequality for operator convex functions (1.2) and provide some refinements and reverse inequalities of interest. They will be ilustrated for some particular operator convex functions such as the power and logarithmic functions mentioned above.

2. A Functional of Weights

We consider the functional

$$J_n(\mathbf{p};\mathbf{A},f,I):=\sum_{j=1}^{n}p_jf(A_j)-P_nf\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) \tag{2.1}$$

where $\mathbf{p}=(p_1,...,p_n)$, $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n>0$, $\mathbf{A}=(A_1,...,A_n)$ is an n-tuple of selfadjoint operators with $Sp(A_j)\subseteq I$ for $j\in\{1,...,n\}$ and $f: I\to\mathbb{R}$ is a operator convex function defined on the interval I.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p}=(p_1,...,p_n)$, $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n>0$. For $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we denote $\mathbf{p}\geq\mathbf{q}$ if $p_j\geq q_j$ for any $j\in\{1,...,n\}$.

Theorem 1. *Assume that $f: I\to\mathbb{R}$ is an operator convex function and $\mathbf{A}=(A_1,...,A_n)$ an n-tuple of selfadjoint operators with $Sp(A_j)\subseteq I$, then for any $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I)\geq J_n(\mathbf{p};\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)\geq 0, \tag{2.2}$$

i.e., $J_n(\cdot;\mathbf{A},f,I)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$, then also

$$J_n(\mathbf{p};\mathbf{A},f,I)\geq J_n(\mathbf{q};\mathbf{A},f,I)\geq 0, \tag{2.3}$$

i.e., $J_n(\cdot;\mathbf{A},f,I)$ is a monotonic functional in the operator order.

Proof. We have

$$
\begin{aligned}
(2.4)\qquad & J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I) \\
&= \sum_{j=1}^{n}(p_j+q_j)f(A_j)-(P_n+Q_n)f\left(\frac{1}{P_n+Q_n}\sum_{j=1}^{n}(p_j+q_j)A_j\right) \\
&= \sum_{j=1}^{n}(p_j+q_j)f(A_j) \\
&- (P_n+Q_n)f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)}{P_n+Q_n}\right).
\end{aligned}
$$

Now, consider the operators

$$A:=\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j \text{ and } B:=\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j.$$

Then $Sp(A), Sp(B)\subseteq I$.

Applying the inequality (OC) for A and B given above and $\lambda=\frac{Q_n}{P_n+Q_n}$ we have

$$
\begin{aligned}
(2.5)\qquad & f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)}{P_n+Q_n}\right) \\
&\le \frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+\frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)
\end{aligned}
$$

in the operator order.

Making use of (2.4) and (2.5) we have

$$
\begin{aligned}
(2.6)\qquad & J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I) \\
&\ge \sum_{j=1}^{n}(p_j+q_j)f(A_j)-(P_n+Q_n) \\
&\times\left[\frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+\frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)\right] \\
&= \sum_{j=1}^{n}p_jf(A_j)-P_nf\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) \\
&+ \sum_{j=1}^{n}q_jf(A_j)-Q_nf\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right) \\
&= J_n(\mathbf{p};\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)
\end{aligned}
$$

in the operator order, and the inequality (2.2) is proved.

Now, let $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$. Then by the super-additivity property (2.2) we have

$$J_n(\mathbf{p};\mathbf{A},f,I)=J_n((\mathbf{p}-\mathbf{q})+\mathbf{q};\mathbf{A},f,I)\geq J_n((\mathbf{p}-\mathbf{q});\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)\geq J_n(\mathbf{q};\mathbf{A},f,I)\tag{2.7}$$

in the operator order, and the monotonicity property (2.3) is proved. □

Corollary 1. *Assume that the function $f:I\to\mathbb{R}$ is operator convex and the n-tuple of selfadjoint operators $(A_1,...,A_n)$ satisfies the condition $Sp(A_j)\subseteq I$ for any $j\in\{1,...,n\}$. If $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants m,M such that*

$$m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q}\tag{2.8}$$

then

$$mJ_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq MJ_n(\mathbf{q};\mathbf{A},f,I)\tag{2.9}$$

in the operator order.

Proof. Observe that for $\alpha>0$ we have $J_n(\alpha\mathbf{p};\mathbf{A},f,I)=\alpha J_n(\mathbf{p};\mathbf{A},f,I)$.

Utilising the monotonicity property (2.3) we have

$$J_n(m\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq J_n(M\mathbf{q};\mathbf{A},f,I)$$

which imply the desired result (2.9). □

Remark 1. *We observe that if all $q_j>0$ then we have the inequality*

$$\min_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq\max_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I)\tag{2.10}$$

in the operator order.

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n},j\in\{1,...,n\}$, then we have the inequalities

$$n\min_{j\in\{1,...,n\}}\{p_j\}J_n(\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq n\max_{j\in\{1,...,n\}}\{p_j\}J_n(\mathbf{A},f,I)\tag{2.11}$$

where

$$J_n(\mathbf{A},f,I):=\frac{1}{n}\sum_{j=1}^{n}f(A_j)-f\left(\frac{1}{n}\sum_{j=1}^{n}A_j\right).\tag{2.12}$$

For $n=2$ and by choosing $p_1=\alpha,p_2=1-\alpha$ with $\alpha\in[0,1]$, we get from (2.11) the inequality

$$\begin{aligned}&2\min\{\alpha,1-\alpha\}\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right]\\&\leq(1-\alpha)f(A)+\alpha f(B)-f((1-\alpha)A+\alpha B)\\&\leq2\max\{\alpha,1-\alpha\}\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right],\end{aligned}\tag{2.13}$$

in the operator order, where $f:I\to\mathbb{R}$ is an operator convex function and A and B are two bounded selfadjoint operators on the complex Hilbert space H with $Sp(A),Sp(B)\subseteq I$.

We have some refinements of the power inequality as follows.

Remark 2. *Assume that* $(A_1, ..., A_n)$ *is an* n*-tuple of positive operators. If* $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ *and* $q_j > 0$ *for* $j \in \{1, ..., n\}$, *then for* $p \in [1, 2]$ *we have*

$$\min_{j\in\{1,\dots,n\}} \left\{ \frac{p_j}{q_j} \right\} \left(\sum_{j=1}^{n} q_j A_j^p - Q_n^{1-p} \left(\sum_{j=1}^{n} q_j A_j \right)^p \right) \tag{2.14}$$
$$\leq \sum_{j=1}^{n} p_j A_j^p - P_n^{1-p} \left(\sum_{j=1}^{n} p_j A_j \right)^p$$
$$\leq \max_{j\in\{1,\dots,n\}} \left\{ \frac{p_j}{q_j} \right\} \left(\sum_{j=1}^{n} q_j A_j^p - Q_n^{1-p} \left(\sum_{j=1}^{n} q_j A_j \right)^p \right)$$

in the operator order.

If $(A_1, ..., A_n)$ *is an* n*-tuple of positive definite operators then for* $p \in [-1, 0)$ *the inequality (2.14) also holds.*

If $q \in (0, 1]$ *then we have the reverse inequalities*

$$\min_{j\in\{1,\dots,n\}} \left\{ \frac{p_j}{q_j} \right\} \left(Q_n^{1-q} \left(\sum_{j=1}^{n} q_j A_j \right)^q - \sum_{j=1}^{n} q_j A_j^q \right) \tag{2.15}$$
$$\leq P_n^{1-q} \left(\sum_{j=1}^{n} p_j A_j \right)^q - \sum_{j=1}^{n} p_j A_j^q$$
$$\leq \max_{j\in\{1,\dots,n\}} \left\{ \frac{p_j}{q_j} \right\} \left(Q_n^{1-q} \left(\sum_{j=1}^{n} q_j A_j \right)^q - \sum_{j=1}^{n} q_j A_j^q \right).$$

When $q_j = \frac{1}{n}, j \in \{1, ..., n\}$ we get from (2.14) the inequality

$$n \min_{j\in\{1,\dots,n\}} \{p_j\} \left(\frac{1}{n} \sum_{j=1}^{n} A_j^p - \frac{1}{n^p} \left(\sum_{j=1}^{n} A_j \right)^p \right) \tag{2.16}$$
$$\leq \sum_{j=1}^{n} p_j A_j^p - P_n^{1-p} \left(\sum_{j=1}^{n} p_j A_j \right)^p$$
$$\leq n \max_{j\in\{1,\dots,n\}} \{p_j\} \left(\frac{1}{n} \sum_{j=1}^{n} A_j^p - \frac{1}{n^p} \left(\sum_{j=1}^{n} A_j \right)^p \right).$$

The case for two operators is as follows:

$$2 \min \{\alpha, 1-\alpha\} \left[\frac{A^p + B^p}{2} - \left(\frac{A+B}{2} \right)^p \right] \tag{2.17}$$
$$\leq (1-\alpha) A^p + \alpha B^p - ((1-\alpha) A + \alpha B)^p$$
$$\leq 2 \max \{\alpha, 1-\alpha\} \left[\frac{A^p + B^p}{2} - \left(\frac{A+B}{2} \right)^p \right],$$

where A and B are positive and $p \in [1, 2]$, or positive definite and $p \in [-1, 0]$.

We have some logarithmic inequalities as follows:

Remark 3. *Assume that* $(A_1,...,A_n)$ *is an* n*-tuple of positive definite operators. If* $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ *and* $q_j>0$ *for* $j\in\{1,...,n\}$*, then*

$$\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left[Q_n\ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_jA_j\right)-\sum_{j=1}^n q_j\ln A_j\right] \tag{2.18}$$
$$\le P_n\ln\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)-\sum_{j=1}^n p_j\ln A_j$$
$$\le \max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left[Q_n\ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_jA_j\right)-\sum_{j=1}^n q_j\ln A_j\right].$$

In particular, we have

$$\min_{j\in\{1,\dots,n\}}\{p_j\}\left[\ln\left(\frac{1}{n}\sum_{j=1}^n A_j\right)-\frac{1}{n}\sum_{j=1}^n\ln A_j\right] \tag{2.19}$$
$$\le P_n\ln\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)-\sum_{j=1}^n p_j\ln A_j$$
$$\le \max_{j\in\{1,\dots,n\}}\{p_j\}\left[\ln\left(\frac{1}{n}\sum_{j=1}^n A_j\right)-\frac{1}{n}\sum_{j=1}^n\ln A_j\right].$$

The case of two operators is as follows:

$$2\min\{\alpha,1-\alpha\}\left[\ln\left(\frac{A+B}{2}\right)-\frac{\ln A+\ln B}{2}\right] \tag{2.20}$$
$$\le \ln\left((1-\alpha)A+\alpha B\right)-(1-\alpha)\ln A-\ln\alpha B$$
$$\le 2\max\{\alpha,1-\alpha\}\left[\ln\left(\frac{A+B}{2}\right)-\frac{\ln A+\ln B}{2}\right],$$

where A and B are positive definite operators.

3. A Functional of Indicies

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $\mathcal{A}(H)$ the linear space of all sequences of selfadjoint operators defined on the complex Hilbert space, i.e.,

$$\mathcal{A}(H)=\left\{\mathbf{A}=(A_k)_{k\in\mathbb{N}}\mid A_k\text{ are selfadjoint operators on }H\text{ for all }k\in\mathbb{N}\right\}$$

and $\mathcal{S}_+(\mathbb{R})$ the family of nonnegative real sequences.

We consider the functional

$$J(K,\mathbf{p};\mathbf{A},f,I):=\sum_{k\in K}p_kf(A_k)-P_Kf\left(\frac{1}{P_K}\sum_{k\in K}p_kA_k\right) \tag{3.1}$$

where $K\in\mathcal{P}_f(\mathbb{N})$, $\mathbf{p}\in\mathcal{S}_+(\mathbb{R})$, $\mathbf{A}\in\mathcal{A}(H)$ with $P_K:=\sum_{k\in K}p_k>0$ and $f:I\to\mathbb{R}$ is a operator convex function on the interval I.

Theorem 2. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\mathbf{p} \in \mathcal{S}_+(\mathbb{R}), \mathbf{A} \in \mathcal{A}(H)$. Assume that $Sp(A_k) \subseteq I$ for any $k \in \mathbb{N}$.*

If $K, L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$ and $P_K, P_L > 0$, then we have the inequality

$$J(K \cup L, \mathbf{p}; \mathbf{A}, f, I) \geq J(K, \mathbf{p}; \mathbf{A}, f, I) + J(L, \mathbf{p}; \mathbf{A}, f, I) \geq 0, \tag{3.2}$$

i.e., $J(\cdot, \mathbf{p}; \mathbf{A}, f, I)$ is super-additive as an index set functional in the operator order.

If $\emptyset \neq K \subset L$ then we have

$$J(L, \mathbf{p}; \mathbf{A}, f, I) \geq J(K, \mathbf{p}; \mathbf{A}, f, I) \geq 0, \tag{3.3}$$

i.e., $J(\cdot, \mathbf{p}; \mathbf{A}, f, I)$ is monotonic as an index set functional in the operator order.

Proof. If $K, L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$ and $P_K, P_L > 0$, then we have the equality

$$\begin{aligned} &J(K \cup L, \mathbf{p}; \mathbf{A}, f, I) \\ &= \sum_{k \in K \cup L} p_k f(A_k) - P_{K \cup L} f\left(\frac{1}{P_{K \cup L}} \sum_{k \in K \cup L} p_k A_k\right) \\ &= \sum_{k \in K} p_k f(A_k) + \sum_{k \in L} p_k f(A_k) \\ &- (P_K + P_L) f\left(\frac{P_K \cdot \frac{1}{P_K} \sum_{k \in K} p_k A_k + P_L \cdot \frac{1}{P_L} \sum_{k \in L} p_k A_k}{P_K + P_L}\right). \end{aligned} \tag{3.4}$$

Consider the operators

$$A = \frac{1}{P_K} \sum_{k \in K} p_k A_k \text{ and } B = \frac{1}{P_L} \sum_{k \in L} p_k A_k.$$

We have that $Sp(A), Sp(B) \subseteq I$.

Utilising the inequality (OC) for the operators A and B as above and $\lambda = \frac{P_L}{P_K + P_L w}$ we have

$$\begin{aligned} &\frac{P_K}{P_K + P_L} f\left(\frac{1}{P_K} \sum_{k \in K} p_k A_k\right) + \frac{P_L}{P_K + P_L} f\left(\frac{1}{P_L} \sum_{k \in L} p_k A_k\right) \\ &\geq f\left(\frac{P_K \cdot \frac{1}{P_K} \sum_{k \in K} p_k A_k + P_L \cdot \frac{1}{P_L} \sum_{k \in L} p_k A_k}{P_K + P_L}\right). \end{aligned} \tag{3.5}$$

On making use of (3.4) and (3.5) we have

$$\begin{aligned} & J\left(K\cup L,\mathbf{p};\mathbf{A},f,I\right) \\ & =\sum_{k\in K\cup L}p_{k}f\left(A_{k}\right)-P_{K\cup L}f\left(\frac{1}{P_{K\cup L}}\sum_{k\in K\cup L}p_{k}A_{k}\right) \\ & \geq\sum_{k\in K}p_{k}f\left(A_{k}\right)+\sum_{k\in L}p_{k}f\left(A_{k}\right)-\left(P_{K}+P_{L}\right) \\ & \times\left[\frac{P_{K}}{P_{K}+P_{L}}f\left(\frac{1}{P_{K}}\sum_{k\in K}p_{k}A_{k}\right)+\frac{P_{L}}{P_{K}+P_{L}}f\left(\frac{1}{P_{L}}\sum_{k\in L}p_{k}A_{k}\right)\right] \\ & =\sum_{k\in K}p_{k}f\left(A_{k}\right)-P_{K}f\left(\frac{1}{P_{K}}\sum_{k\in K}p_{k}A_{k}\right) \\ & +\sum_{k\in L}p_{k}f\left(A_{k}\right)-P_{L}f\left(\frac{1}{P_{L}}\sum_{k\in L}p_{k}A_{k}\right) \\ & =J\left(K,\mathbf{p};\mathbf{A},f,I\right)+J\left(L,\mathbf{p};\mathbf{A},f,I\right) \end{aligned} \tag{3.6}$$

and the inequality (3.2) is proved.

If $\emptyset\neq K\subset L$ with $L\diagdown K\neq\emptyset$ then we have by (3.2)

$$\begin{aligned} J\left(L,\mathbf{p};\mathbf{A},f,I\right) & =J\left(K\cup\left(L\diagdown K\right),\mathbf{p};\mathbf{A},f,I\right) \\ & \geq J\left(K,\mathbf{p};\mathbf{A},f,I\right)+J\left(L\diagdown K,\mathbf{p};\mathbf{A},f,I\right)\geq J\left(K,\mathbf{p};\mathbf{A},f,I\right) \end{aligned}$$

and the inequality (3.3) is proved. □

We consider the functionals:

$$O_{n}\left(\mathbf{p};\mathbf{A},f,I\right):=\sum_{j=1}^{n}p_{2j-1}f\left(A_{2j-1}\right)-\sum_{j=1}^{n}p_{2j-1}f\left(\frac{1}{\sum_{j=1}^{n}p_{2j-1}}\sum_{j=1}^{n}p_{2j-1}A_{2j-1}\right)$$

and

$$E_{n}\left(\mathbf{p};\mathbf{A},f,I\right):=\sum_{j=1}^{n}p_{2j}f\left(A_{2j}\right)-\sum_{j=1}^{n}p_{2j}f\left(\frac{1}{\sum_{j=1}^{n}p_{2j}}\sum_{j=1}^{n}p_{2j}A_{2j}\right).$$

We can state the following corollary.

Corollary 2. *Let $f:I\rightarrow\mathbb{R}$ be an operator convex function on the interval I and $\mathbf{p}=\left(p_{1},...,p_{2n}\right),\mathbf{A}=\left(A_{1},...,A_{2n}\right)$ with $p_{k}>0$, A_{k} selfadjoint operators and such that $Sp\left(A_{k}\right)\subseteq I$ for any $k\in\left\{1,...,2n\right\},n\geq1$. The we have the inequality*

$$J_{2n}\left(\mathbf{p};\mathbf{A},f,I\right)\geq O_{n}\left(\mathbf{p};\mathbf{A},f,I\right)+E_{n}\left(\mathbf{p};\mathbf{A},f,I\right)\geq0 \tag{3.7}$$

in the operator order, where, as in (2.1)

$$J_{2n}\left(\mathbf{p};\mathbf{A},f,I\right)=J_{n}\left(\mathbf{p};\mathbf{A},f,I\right):=\sum_{j=1}^{2n}p_{j}f\left(A_{j}\right)-P_{2n}f\left(\frac{1}{P_{2n}}\sum_{j=1}^{2n}p_{j}A_{j}\right).$$

The proof follows by (3.2) on choosing $K=\left\{2,...,2n\right\}$ and $L=\left\{1,...,2n-1\right\}.$

Corollary 3. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\mathbf{p} = (p_1, ..., p_n)$, $\mathbf{A} = (A_1, ..., A_n)$ with $p_k > 0$, A_k selfadjoint operators and such that $Sp(A_k) \subseteq I$ for any $k \in \{1, ..., n\}$, $n \geq 2$. Then we have the inequality*

$$J_k(\mathbf{p}; \mathbf{A}, f, I) \geq J_{k-1}(\mathbf{p}; \mathbf{A}, f, I) \geq 0 \tag{3.8}$$

for any $k \in \{1, ..., n\}$ with $n \geq k \geq 2$.

We also have that

$$\begin{aligned} &J_n(\mathbf{p}; \mathbf{A}, f, I) \\ \geq &\max_{k,j \in \{1,...,n\}} \left[p_j f(A_j) + p_k f(A_k) - (p_j + p_k) f\left(\frac{p_j A_j + p_k A_k}{p_j + p_k} \right) \right] \geq 0 \end{aligned} \tag{3.9}$$

in the operator order.

The proof follows by the monotonicity property (3.3).

Remark 4. *Utilising the inequality for the operator convex function $f(t) = t^p$, $p \in [1, 2]$ we have the inequality*

$$\begin{aligned} &\sum_{j=1}^{n} p_j A_j^p - P_n^{1-p} \left(\sum_{j=1}^{n} p_j A_j \right)^p \\ &\geq \max_{k,j \in \{1,...,n\}} \left[p_j A_j^p + p_k A_k^p - (p_j + p_k) \left(\frac{p_j A_j + p_k A_k}{p_j + p_k} \right)^p \right] \geq 0, \end{aligned} \tag{3.10}$$

for the positive selfadjoint operators $(A_1, ..., A_n)$.

In particular, we have the inequality

$$\sum_{j=1}^{n} p_j A_j^2 - P_n^{-1} \left(\sum_{j=1}^{n} p_j A_j \right)^2 \geq \max_{k,j \in \{1,...,n\}} \left\{ \frac{p_i p_j}{p_j + p_k} (A_j - A_k)^2 \right\} \geq 0. \tag{3.11}$$

If $(A_1, ..., A_n)$ are positive definite operators, then we have

$$\begin{aligned} &P_n \ln \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j A_j \right) - \sum_{j=1}^{n} p_j \ln A_j \\ &\geq \max_{k,j \in \{1,...,n\}} \left[(p_j + p_k) \ln \left(\frac{p_j A_j + p_k A_k}{p_j + p_k} \right) - p_j \ln A_j - p_k \ln A_k \right] \geq 0. \end{aligned} \tag{3.12}$$

4. A Reverse Inequality

The following result also holds:

Theorem 3. *If the function $f : [m, M] \to \mathbb{R}$ is operator convex and if the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ has the property that $Sp(A_j) \subseteq [m, M]$ for any $j \in \{1, ..., n\}$, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^{n} p_j > 0$ we* *have*

$$0 \le \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \tag{4.1}$$
$$\le \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]$$
$$\times \left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}1_H\right|\right)$$
$$\le \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]1_H$$

in the operator order.

Proof. Since the function $f:[m,M]\to\mathbb{R}$ is operator convex, then it is convex and we have the inequality

$$f(t) = f\left(\frac{(M-t)m+(t-m)M}{M-m}\right) \le \frac{(M-t)f(m)+(t-m)f(M)}{M-m}$$

for any $t\in[m,M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator A with spectrum $Sp(A)\subseteq[m,M]$, we have in the operator order

$$f(A_j) \le \frac{f(m)(M1_H - A_j) + f(M)(A_j - m1_H)}{M-m} \tag{4.2}$$

for any $j\in\{1,...,n\}$.

If we multiply the inequality (4.2) by p_j and sum over j from 1 to n we get

$$\frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) \tag{4.3}$$
$$\le \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m}$$

in the operator order.

Therefore we have

$$0 \le \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \tag{4.4}$$
$$\le \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m}$$
$$- f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)$$

in the operator order, which is a reverse of Jensen's inequality that is of interest in itself.

Now, from the scalar version of (2.13) we have

$$0 \le (1-t) f(m) + t f(M) - f((1-t)m + tM) \le 2\max\{t, 1-t\}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] = 2\left(\frac{1}{2} + \left|t - \frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \tag{4.5}$$

for any $t \in [m, M]$, where $f : [m, M] \to \mathbb{R}$ is a continuous convex function on $[m, M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator T with $0 \le T \le 1_H$ we have from (4.5) that

$$0 \le f(m)(1_H - T) + f(M) T - f((1_H - T)m + TM) \le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2} + \left|T - \frac{1}{2}1_H\right|\right) \tag{4.6}$$

in the operator order.

Writing the inequality (4.6) for the operator

$$0 \le T = \frac{\frac{1}{P_n}\sum_{j=1}^n p_j A_j - m 1_H}{M - m} \le 1_H$$

we have

$$\begin{aligned} &\frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j - m1_H\right)}{M-m} \\ &- f\left[\frac{m\left(M1_H - \frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) + M\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j - m1_H\right)}{M-m}\right] \\ &= \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j - m1_H\right)}{M-m} \\ &- f\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) \\ &\le \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^n p_j A_j - \frac{m+M}{2}1_H\right|\right) \end{aligned} \tag{4.7}$$

in the operator order.

The last part is obvious since

$$\left|\frac{1}{P_n}\sum_{j=1}^n p_j A_j - \frac{m+M}{2}1_H\right| \le \frac{1}{2}(M-m)1_H.$$

□

Remark 5. *Assume that $(A_1,...,A_n)$ are positive and with $Sp(A_j) \subseteq [m,M]$ for any $j \in \{1,...,n\}$. Then for any $p \in [1,2]$ we have the inequality*

$$
\begin{aligned}
(4.8)\qquad 0 &\le \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j^p - \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)^p \\
&\le \frac{2}{M-m}\left[\frac{m^p+M^p}{2} - \left(\frac{m+M}{2}\right)^p\right] \\
&\quad \times \left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}1_H\right|\right) \\
&\le \frac{2}{M-m}\left[\frac{m^p+M^p}{2} - \left(\frac{m+M}{2}\right)^p\right]1_H.
\end{aligned}
$$

If the operators $(A_1,...,A_n)$ are positive definite, i.e., $m>0$, then

$$
\begin{aligned}
(4.9)\qquad 0 &\le \ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) - \frac{1}{P_n}\sum_{j=1}^{n} p_j \ln A_j \\
&\le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} \\
&\quad \times \left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}1_H\right|\right) \\
&\le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} 1_H.
\end{aligned}
$$

5. A Refinement of Jensen Inequality

The following result provides an additive refinement of Jensen inequality (1.2).

Theorem 4. *If the function $f: I \to \mathbb{R}$ is operator convex and if the n-tuple of selfadjoint operators $(A_1,...,A_n)$ is such that $Sp(A_j) \subseteq I$ for any $j \in \{1,...,n\}$, then for any $p_j \ge 0$ with $j \in \{1,...,n\}$ and $P_k := \sum_{j=1}^{k} p_j > 0$, $\overline{P}_k := P_n - P_k > 0$, with $k \in \{1,...,n-1\}$ we have*

$$
\begin{aligned}
(5.1)\qquad 0 &\le \max_{k\in\{1,...,n-1\}}\left\{\left(1 - \frac{\left|P_k - \overline{P}_k\right|}{P_n}\right)\right. \\
&\quad \times \left[\frac{f\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j\right) + f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j\right)}{2}\right. \\
&\quad \left.\left. - f\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j + \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j}{2}\right)\right]\right\} \\
&\le \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)
\end{aligned}
$$

in the operator order.

Proof. Consider the n-tuple of selfadjoint operators $(A_1,...,A_n)$ and define the operators

$$A=\frac{1}{P_k}\sum_{j=1}^{k}p_jA_j \text{ and } B=\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_jA_j.$$

Then $Sp(A), Sp(B)\subseteq I$ for $k\in\{1,...,n-1\}$.

Applying the first inequality in (2.13) for A and B as above and $\alpha=\frac{\overline{P}_k}{P_n}$, for $k\in\{1,...,n-1\}$, we have

$$\begin{aligned}(5.2)\qquad & 2\min\left\{\frac{\overline{P}_k}{P_n},\frac{P_k}{P_n}\right\}\\ &\times\left[\frac{f\left(\frac{1}{P_k}\sum_{j=1}^{k}p_jA_j\right)+f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_jA_j\right)}{2}\right.\\ &\left.-f\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k}p_jA_j+\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_jA_j}{2}\right)\right]\\ &\le\frac{P_k}{P_n}f\left(\frac{1}{P_k}\sum_{j=1}^{k}p_jA_j\right)+\frac{\overline{P}_k}{P_n}f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_jA_j\right)\\ &-f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right),\end{aligned}$$

in the operator order.

By Jensen's inequality (1.2) we have

$$f\left(\frac{1}{P_k}\sum_{j=1}^{k}p_jA\right)\le\frac{1}{P_k}\sum_{j=1}^{k}p_jf(A_j)$$

and

$$f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_jA_j\right)\le\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_jf(A_j)$$

which imply that

$$\begin{aligned}(5.3)\qquad & \frac{P_k}{P_n}f\left(\frac{1}{P_k}\sum_{j=1}^{k}p_jA_j\right)+\frac{\overline{P}_k}{P_n}f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n}p_jA_j\right)\\ &\le\frac{1}{P_n}\sum_{j=1}^{n}p_jf(A_j)\end{aligned}$$

in the operator order.

Since

$$\min\left\{\frac{\overline{P}_k}{P_n},\frac{P_k}{P_n}\right\}=\frac{1}{2}-\frac{1}{2P_n}\left|P_k-\overline{P}_k\right|$$

then we get from (5.2) and (5.3) the desired result (5.1). □

Remark 6. *If the operators* $(A_1,...,A_n)$ *are positive, then for any* $p_j \geq 0$ *with* $j \in \{1,...,n\}$ *and* $P_k := \sum_{j=1}^{k} p_j > 0$, $\overline{P}_k := P_n - P_k > 0$, *with* $k \in \{1,...,n-1\}$ *we have*

$$0 \leq \max_{k\in\{1,...,n-1\}} \left\{ \left(1 - \frac{\left|P_k - \overline{P}_k\right|}{P_n}\right) \times \left[\frac{\frac{1}{P_k^p}\left(\sum_{j=1}^{k} p_j A_j\right)^p + \frac{1}{\left(\overline{P}_k\right)^p}\left(\sum_{j=k+1}^{n} p_j A_j\right)^p}{2} - \left(\frac{\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j + \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j}{2}\right)^p \right] \right\} \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j^p - \frac{1}{P_n^p}\left(\sum_{j=1}^{n} p_j A_j\right)^p \tag{5.4}$$

for any $p \in [1,2]$. *If* $(A_1,...,A_n)$ *are positive definite, then (5.4) also holds for* $p \in [-1,0]$.

If $(A_1,...,A_n)$ *are positive definite, then we also have the logarithmic inequality*

$$0 \leq \max_{k\in\{1,...,n-1\}} \left\{ \left(1 - \frac{\left|P_k - \overline{P}_k\right|}{P_n}\right) \times \left[\ln\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j + \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j}{2}\right) - \frac{\ln\left(\frac{\sum_{j=1}^{k} p_j A_j}{P_k}\right) + \ln\left(\frac{\sum_{j=k+1}^{n} p_j A_j}{\overline{P}_k}\right)}{2} \right] \right\} \leq \ln\left(\frac{\sum_{j=1}^{n} p_j A_j}{P_n}\right) - \frac{1}{P_n}\sum_{j=1}^{n} p_j \ln(A_j). \tag{5.5}$$

Remark 7. *The interested reader may obtain some other inequalities of interest in the operator order by using, for instance, the operator convex function* $f(t) = t\ln t$ *on the interval* $(0,\infty)$. *The details are omitted.*

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