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POWER SERIES INEQUALITIES VIA YOUNG'S INEQUALITY WITH APPLICATIONS

Alawiah Ibrahim[1,3], Sever S. Dragomir[1,2], and Maslina Darus[3]

Abstract. In this paper, we establish some inequalities for power series with real coefficients by utilizing the Young's inequality for sequences of complex numbers. Some applications for special functions such as polylogarithm, hypergeometric and Bessel functions are also presented.

1. Introduction

Let $a_k, b_k \in \mathbb{C}$, $k \in \{1, 2, \dots, n\}$, $p > 1$ and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, the classical Hölder's inequality [**4**, p. 19-21] states that

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}} \tag{1.1}$$

with equality holds if and only if the sequences $\{|a_k|^p\}$ and $\{|b_k|^q\}$ for $k \in \{1, 2, \dots, n\}$ are proportional and the $\arg(a_k b_k)$ is independent of k. The inequality (1.1) is reversed if $p < 1$.

The weighted version of the Hölder's inequality also holds, namely

$$\left| \sum_{k=1}^{n} p_k a_k b_k \right| \leq \left(\sum_{k=1}^{n} p_k |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} p_k |b_k|^q \right)^{\frac{1}{q}} \tag{1.2}$$

where $p_k \geq 0$, $a_k, b_k \in \mathbb{C}$, $k \in \{1, 2, \dots, n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Tolsted in [**22**] (see also [**17**, p. 457], [**21**, p. 63-64]) showed that the Hölder inequality (also known in the literature as Rogers inequality) can be easily proved by using the Young's inequality [**27**], namely

$$\frac{1}{q} x^q + \frac{1}{p} y^p \geq xy \tag{1.3}$$

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for any positive numbers x, y and $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Equality holds in (1.3) if and only if $x^q = y^p$. For other applications and extensions of Young's inequality, see [**6**], [**13**], [**18**, p. 379-389] and references therein.

It is well-known that the Hölder's inequality is one of the most important inequalities in real and complex analysis. For example, the celebrated Cauchy-Bunyakovsky-Schwarz (CBS) inequality (see [**10**, p. 16], [**18**, p. 83]) is a special case of Hölder's inequality (1.1) for $p=q=2$. Some other inequalities, such as the Minkowski's inequality can be proved by using the Hölder's inequality.

Various extensions, generalizations, refinements, etc. of the Hölder's inequality have been obtained by several authors (see [**1**], [**5**], [**7**], [**11**], [**15**], [**19**], [**16**], [**20**], [**24**], [**25**], [**26**] and references therein). For instance, it comes to our attention that an interesting generalizations of the Hölder's inequality (1.1) by utilizing the Young's inequality (1.3), which was established by Dragomir and Sándor in [**8**] (see also [**9**, p. 10-16]), is as follows

$$\begin{aligned}(1.4)\qquad &\sum_{k=1}^{n} p_k |x_k y_k| \sum_{k=1}^{n} q_k |x_k y_k| \\ &\le \frac{1}{p}\sum_{k=1}^{n} p_k |x_k|^p \sum_{k=1}^{n} q_k |y_k|^p + \frac{1}{q}\sum_{k=1}^{n} q_k |x_k|^q \sum_{k=1}^{n} p_k |y_k|^q\end{aligned}$$

for $x_k, y_k \in \mathbb{C}$, $p_k, q_k \ge 0, k \in \{1,2,\ldots,n\}$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If now, we consider an analytic function defined by power series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \tag{1.5}$$

with real coefficients and convergent on the disk $D(0,R), R>0$ and apply the weighted version of the Hölder's inequality (1.2), then we can state that

$$|f(xy)| \le \left(\sum_{n=0}^{\infty} |a_n| |x|^{pn}\right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} |a_n| |y|^{qn}\right)^{\frac{1}{q}} = f_A^{\frac{1}{p}}(|x|^p) f_A^{\frac{1}{q}}(|y|^q) \tag{1.6}$$

for any $x, y \in \mathbb{C}$ with $xy, |x|^p, |y|^q \in D(0,R)$ and $f_A(z)$ is a new power series defined by $\sum_{n=0}^{\infty} |a_n| z^n$, where $a_n = |a_n| \, sgn(a_n)$ with $sgn(x)$ is the real signum function defined to be 1 if $x>0$, -1 if $x<0$ and 0 if $x=0$. The power series $f_A(z)$ have the same radius of convergence as the original power series $f(z)$.

Motivated by the above results (1.6), (1.4) and the results from [**8**], and utilizing the Young's inequality we established in this paper some inequalities for functions defined by power series with real coefficients. Particular examples that are related to some fundamental complex functions such as the exponential, logarithm, trigonometric and hyperbolic functions are presented. Applications for some special functions such as polylogarithm, hypergeometric and Bessel functions for the first kind are presented as well.

2. Some inequalities via Young's inequality

On utilizing the Young's inequality (1.3) for power series with real coefficients, we establish the following result.

Theorem 1. *Let* $f(z)=\sum_{n=0}^{\infty}p_n z^n$ *and* $g(z)=\sum_{n=0}^{\infty}q_n z^n$ *be two power series with real coefficients and convergent on the open disk* $D(0,R)$, $R>0$. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $x,y\in\mathbb{C}$ *so that* xy, $|x|^p$, $|x|^q$, $|y|^p$, $|y|^q\in D(0,R), |x|,|y|\neq 0$, *then*

$$\frac{1}{p}g_A\left(|x|^p\right)f_A\left(|y|^p\right)+\frac{1}{q}f_A\left(|x|^q\right)g_A\left(|y|^q\right)\geq |f(xy)\,g(xy)| \tag{2.1}$$

and

$$\frac{1}{p}g_A\left(|x|^p\right)f_A\left(|y|^q\right)+\frac{1}{q}f_A\left(|x|^q\right)g_A\left(|y|^p\right)\geq \left|f\left(x\,|y|^{q-1}\right)g\left(x\,|y|^{p-1}\right)\right|. \tag{2.2}$$

Proof. If we choose $x=|x|^j|y|^k$, $y=|x|^k|y|^j$, $j,k\in\{0,1,2,\dots,n\}$ in (1.3), then we have

$$p\,|x|^{qj}\,|y|^{qk}+q\,|x|^{pk}\,|y|^{pj}\geq pq\,|xy|^j\,|xy|^k \tag{2.3}$$

for any $j,k\in\{0,1,2,\dots,n\}$.

Now, if we multiply this inequality (2.3) with positive quantities $|p_j|\,|q_k|$ and summing over j and k from 0 to n, then we derive

$$\begin{aligned}p\sum_{j=0}^{n}|p_j|\,|x|^{qj}\sum_{k=0}^{n}|q_k|\,|y|^{qk}+q\sum_{k=0}^{n}|q_k|\,|x|^{pk}\sum_{j=0}^{n}|p_j|\,|y|^{pj}\\ \geq pq\left|\sum_{j=0}^{n}p_j\,(xy)^j\sum_{k=0}^{n}q_k\,(xy)^k\right|.\end{aligned} \tag{2.4}$$

Since all the series whose partial sums are involved in inequality (2.4) are convergent on the disk $D(0,R)$ and taking the limit as $n\to\infty$ in (2.4), we deduce the desired result (2.1).

Further, if we choose in (1.3), $x=\frac{|x|^j}{|y|^j}$, $y=\frac{|x|^k}{|y|^k}$, then we get

$$p\left(\frac{|x|^j}{|y|^j}\right)^q+q\left(\frac{|x|^k}{|y|^k}\right)^p\geq pq\frac{|x|^j}{|y|^j}\frac{|x|^k}{|y|^k} \tag{2.5}$$

for any $|y|^j$, $|y|^k\neq 0$, $j,k\in\{0,1,2,\dots,n\}$.

Simplifying (2.5), we obtain that

$$\begin{aligned}p\,|x|^{qj}\,|y|^{pk}+q\,|x|^{pk}\,|y|^{qj} &\geq pq\,|x|^j\,|y|^{(q-1)j}\,|x|^k\,|y|^{(p-1)k}\\ &= pq\left|\left(x\,|y|^{q-1}\right)^j\left(x\,|y|^{p-1}\right)^k\right|\end{aligned} \tag{2.6}$$

for any $j,k\in\{0,1,2,\dots,n\}$.

Multiplying (2.6) by $|p_j|\,|q_k|\geq 0$, $j,k\in\{0,1,2,\dots,n\}$ and summing over j and k from 0 to n, we get

$$\begin{aligned}p\sum_{j=0}^{n}|p_j|\,|x|^{qj}\sum_{k=0}^{n}|q_k|\,|y|^{pk}+q\sum_{k=0}^{n}|q_k|\,|x|^{pk}\sum_{j=0}^{n}|p_j|\,|y|^{qj}\\ \geq pq\left|\sum_{j=0}^{n}p_j\left(x\,|y|^{q-1}\right)^j\sum_{k=0}^{n}q_k\left(x\,|y|^{p-1}\right)^k\right|.\end{aligned} \tag{2.7}$$

Since all the series whose partial sums are involved in inequality (2.7) are convergent on the disk $D(0,R)$ and letting $n\to\infty$ in (2.7), we deduce the desired inequality (2.2). □

The following particular case is of interest.

Corollary 1. *If $g(z)=f(z)$ in (2.1) and (2.2), then*

$$\frac{1}{p}f_A\left(|x|^p\right)f_A\left(|y|^p\right)+\frac{1}{q}f_A\left(|x|^q\right)f_A\left(|y|^q\right)\geq\left|f\left(xy\right)\right|^2 \tag{2.8}$$

and

$$\frac{1}{p}f_A\left(|x|^p\right)f_A\left(|y|^q\right)+\frac{1}{q}f_A\left(|x|^q\right)f_A\left(|y|^p\right)\geq\left|f\left(x\left|y\right|^{q-1}\right)f\left(x\left|y\right|^{p-1}\right)\right| \tag{2.9}$$

respectively, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and xy, $|x|^p$, $|x|^q$, $|y|^p$, $|y|^q\in D(0,R)$, $x,y\neq 0$. In addition, if $y=x$ in (2.8) and (2.9), then we have

$$\frac{1}{p}f_A\left(|x|^p\right)f_A\left(|x|^p\right)+\frac{1}{q}f_A\left(|x|^q\right)f_A\left(|x|^q\right)\geq\left|f\left(x^2\right)\right|^2$$

and

$$\frac{1}{p}f_A\left(|x|^p\right)f_A\left(|x|^q\right)+\frac{1}{q}f_A\left(|x|^q\right)f_A\left(|x|^p\right)\geq\left|f\left(sgn\left(x\right)|x|^q\right)f\left(sgn\left(x\right)|x|^p\right)\right|$$

for any $x\in\mathbb{C}$ with x^2, $|x|^p$, $|x|^q\in D(0,R)$, $|x|\neq 0$ and $sgn(x)$ is the complex signum function defined to be $\frac{x}{|x|}$ if $x\neq 0$ and 0 if $x=0$.

Remark 1. *In particular case $p=q=2$ in (2.8) and (2.9), we get the inequalities*

$$f_A\left(|x|^2\right)f_A\left(|y|^2\right)\geq\left|f\left(xy\right)\right|^2 \tag{2.10}$$

and

$$f_A\left(|x|^2\right)f_A\left(|y|^2\right)\geq\left|f\left(x\left|y\right|\right)\right|^2 \tag{2.11}$$

respectively, for $x,y\in\mathbb{C}$ with xy, $|x|^2$, $|y|^2\in D(0,R)$.

Some applications for particular functions of interest are as follows.

(1) If we apply the inequalities (2.8) and (2.9) for the function $f(z)=\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$, $z\in D(0,1)$, then we get

$$\frac{1}{p\left(1-|x|^p\right)\left(1-|y|^p\right)}+\frac{1}{q\left(1-|x|^q\right)\left(1-|y|^q\right)}\geq\frac{1}{\left|1-xy\right|^2} \tag{2.12}$$

and

$$\begin{aligned}&\frac{1}{p\left(1-|x|^p\right)\left(1-|y|^q\right)}+\frac{1}{q\left(1-|x|^q\right)\left(1-|y|^p\right)}\\ &\geq\frac{1}{\left|1-x\left|y\right|^{q-1}\right|\left|1-x\left|y\right|^{p-1}\right|}\end{aligned} \tag{2.13}$$

respectively, for any $x,y\in\mathbb{C}$ with $x,y\neq 0$, xy, $|x|^p$, $|x|^q$, $|y|^p$, $|y|^q\in D(0,1)$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

(2) If we apply the inequalities (2.8) and (2.9) for the function $f(z)=\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z\in\mathbb{C}$, then we can state that

$$\frac{1}{p}\exp\left(|x|^p+|y|^p\right)+\frac{1}{q}\exp\left(|x|^q+|y|^q\right)\geq\left|\exp\left(xy\right)\right|^2$$

and

$$\frac{1}{p}\exp\left(|x|^p+|y|^q\right)+\frac{1}{q}\exp\left(|x|^q+|y|^p\right)\geq\left|\exp\left(x\left|y\right|^{q-1}+x\left|y\right|^{p-1}\right)\right|$$

respectively, for any $x,\ y\in\mathbb{C}$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

(3) If we apply the function $f(z)=\ln\left(\frac{1}{1-z}\right)=\sum_{n=0}^{\infty}\frac{1}{n}z^n$, $z\in D(0,1)$, then from (2.8) and (2.9) we have

$$\frac{1}{p}\ln\left(1-|x|^p\right)\ln\left(1-|y|^p\right)+\frac{1}{q}\ln\left(1-|x|^q\right)\ln\left(1-|y|^q\right)\geq\left|\ln\left(1-xy\right)\right|^2 \tag{2.14}$$

and

$$\begin{aligned}&\frac{1}{p}\ln\left(1-|x|^p\right)\ln\left(1-|y|^q\right)+\frac{1}{q}\ln\left(1-|x|^q\right)\ln\left(1-|y|^p\right)\\&\geq\left|\ln\left(1-x\left|y\right|^{q-1}\right)\ln\left(1-x\left|y\right|^{p-1}\right)\right|\end{aligned} \tag{2.15}$$

respectively, for any $x,y\in\mathbb{C}$ with $x,y\neq0$, $|x|^p,\ |x|^q,\ |y|^p,\ |y|^q\in D(0,1)$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

(4) Also, if we consider the function $f(z)=\sin(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n+1}$, $z\in\mathbb{C}$, then obviously we have $f_A(z)=\sinh(z)$, $z\in\mathbb{C}$. Applying the inequalities (2.8) and (2.9) for this function, we get

$$\frac{1}{p}\sinh(|x|^p)\sinh(|y|^p)+\frac{1}{q}\sinh(|x|^q)\sinh(|y|^q)\geq\left|\sin(xy)\right|^2$$

and

$$\begin{aligned}&\frac{1}{p}\sinh(|x|^p)\sinh(|y|^q)+\frac{1}{q}\sinh(|x|^q)\sinh(|y|^p)\\&\geq\left|\sin\left(x\left|y\right|^{q-1}\right)\sin\left(x\left|y\right|^{p-1}\right)\right|\end{aligned}$$

respectively, for $x,\ y\in\mathbb{C}$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Similar results can be obtained for $\cosh(x)$ as well.
The following result also holds.

Theorem 2. *Let $f(z)$ and $g(z)$ be as in Theorem 1. Then one has the inequality*

$$\frac{1}{p}g_A\left(|x|^p\right)f_A\left(|y|^q\right)+\frac{1}{q}f_A\left(|x|^p\right)g_A\left(|y|^q\right)\geq\left|f\left(|x|^{p-1}\left|y\right|^{q-1}\right)g\left(xy\right)\right| \tag{2.16}$$

and

$$\frac{1}{p}f_A\left(|x|^p\right)g_A\left(|y|^2\right)+\frac{1}{q}g_A\left(|x|^2\right)f_A\left(|y|^q\right)\geq\left|f\left(xy\right)g\left(|x|^{\frac{2}{q}}\left|y\right|^{\frac{2}{p}}\right)\right|. \tag{2.17}$$

PROOF. If we choose in (1.3), $x = \frac{|y|^k}{|y|^j}$, $y = \frac{|x|^k}{|x|^j}$, $|x|^j,\ |y|^j \neq 0$, $j,k \in \{0,1,2,\cdots,n\}$, we have

$$
\begin{aligned}
p\,|y|^{qk}\,|x|^{pj} + q\,|x|^{pk}\,|y|^{qj} &\geq pq\,|x|^{(p-1)j}\,|y|^{(q-1)j}\,|xy|^k \\
&= pq\left|\left(|x|^{p-1}\,|y|^{q-1}\right)^j (xy)^k\right| \tag{2.18}
\end{aligned}
$$

for any $j,k \in \{0,1,2,\cdots,n\}$.

Multiplying (2.18) with $|p_j|\,|q_k| \geq 0$ and summing over j and k from 0 to n, we obtain that

$$
\begin{aligned}
&p\sum_{k=0}^{n}|q_k|\,|y|^{qk}\sum_{j=0}^{n}|p_j|\,|x|^{pj} + q\sum_{k=0}^{n}|q_k|\,|x|^{pk}\sum_{j=0}^{n}|p_j|\,|y|^{qj} \tag{2.19}\\
&\geq pq\left|\sum_{j=0}^{n}p_j\left(|x|^{p-1}\,|y|^{q-1}\right)^j\sum_{k=0}^{n}q_k\,(xy)^k\right|.
\end{aligned}
$$

From (1.3), we also have the inequality

$$
\begin{aligned}
&p\sum_{k=1}^{n}|q_k|\,|x|^{2k}\sum_{j=1}^{n}|p_j|\,|y|^{qj} + q\sum_{j=1}^{n}|p_j|\,|x|^{pj}\sum_{k=1}^{n}|q_k|\,|y|^{2k} \tag{2.20}\\
&\geq pq\left|\sum_{j=1}^{n}p_j\,(xy)^j\sum_{k=1}^{n}q_k\left(|x|^{\frac{2}{q}}\,|y|^{\frac{2}{p}}\right)^k\right|
\end{aligned}
$$

for any $x,y \in C, p>1, \frac{1}{p}+\frac{1}{q}=1$, which was obtained by choosing $x = |x|^{\frac{2}{q}k}\,|y|^j$, $y = |x|^j\,|y|^{\frac{2}{p}k}$ and repeating the same method as above.

Now, since all the series whose partial sums are involved in inequalities (2.19) and (2.20) are convergent on the disk $D(0,R)$, by letting $n \to \infty$ in (2.19) and (2.20) respectively, we deduce the desired inequalities, i.e., (2.16) and (2.17). □

COROLLARY 2. *If $g(z) = f(z)$in (2.16) and (2.17), then*

$$
f_A\left(|x|^p\right) f_A\left(|y|^q\right) \geq \left|f(xy)\, f\left(|x|^{p-1}\,|y|^{q-1}\right)\right| \tag{2.21}
$$

and

$$
\frac{1}{p}f_A\left(|x|^p\right) f_A\left(|y|^2\right) + \frac{1}{q}f_A\left(|x|^2\right) f_A\left(|y|^q\right) \geq \left|f(xy)\, f\left(|x|^{\frac{2}{q}}\,|y|^{\frac{2}{p}}\right)\right| \tag{2.22}
$$

respectively, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and xy, $|x|^2$, $|x|^p$, $|x|^{\frac{2}{q}}$, $|y|^2$, $|y|^q$, $|y|^{\frac{2}{p}} \in D(0,R)$, $x,y \neq 0$. In addition, if $y = x$ in (2.21) and (2.22), then we have

$$
f_A\left(|x|^p\right) f_A\left(|x|^q\right) \geq \left|f\left(x^2\right) f\left(|x|^{pq-2}\right)\right|
$$

and

$$
f_A\left(|x|^2\right)\left[\frac{1}{p}f_A\left(|x|^p\right) + \frac{1}{q}f_A\left(|x|^q\right)\right] \geq \left|f\left(x^2\right) f\left(|x|^2\right)\right|
$$

for x^2, $|x|^2$, $|x|^p$, $|x|^q$, $\in D(0,R)$, $x \neq 0$.

Remark 2. *In particular case $p = q = 2$ in (2.21) or (2.22), we get the inequality*

$$f_A\left(|x|^2\right) f_A\left(|y|^2\right) \geq |f(xy) f(|xy|)|$$

for any $x, y \in \mathbb{C}$ with $xy, |xy|, |x|^2, |y|^2 \in D(0, R)$.

In what follows, we provide some applications of inequalities (2.21) and (2.22) for particular functions of interest.

(1) If we apply the inequalities (2.21) and (2.22) the function $f(z) = \frac{1}{1-z}$, $z \in D(0,1)$, then we get

$$|1 - xy| \left|1 - |x|^{p-1} |y|^{q-1}\right| \geq (1 - |x|^p)(1 - |y|^q) \tag{2.23}$$

and

$$\frac{1}{p(1 - |x|^p)\left(1 - |y|^2\right)} + \frac{1}{q\left(1 - |x|^2\right)(1 - |y|^q)} \geq \frac{1}{|(1 - xy)| \left|\left(1 - |x|^{\frac{2}{q}} |y|^{\frac{2}{p}}\right)\right|} \tag{2.24}$$

respectively, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and xy, $|x|^2$, $|x|^p$, $|x|^{\frac{1}{q}}$, $|y|^2$, $|y|^q$, $|y|^{\frac{1}{p}} \in D(0,1)$, $x, y \neq 0$.

(2) If we apply the inequalities (2.21) and (2.22) for the function $f(z) = \exp(z)$, $z \in \mathbb{C}$, then we can state that

$$\exp(|x|^p + |y|^q) \geq \left|\exp\left(xy + |x|^{p-1} |y|^{q-1}\right)\right| \tag{2.25}$$

and

$$\frac{1}{p}\exp\left(|x|^p + |y|^2\right) + \frac{1}{q}\exp\left(|x|^2 + |y|^q\right) \geq \left|\exp\left(xy + |x|^{\frac{2}{q}} |y|^{\frac{2}{p}}\right)\right| \tag{2.26}$$

respectively, for any x, $y \in \mathbb{C}$, $x, y \neq 0$.

(3) If we take the function $f(z) = \ln\left(\frac{1}{1-z}\right)$, $z \in D(0,1)$, then from (2.21) and (2.22) we have

$$\ln(1 - |x|^p) \ln(1 - |y|^q) \geq \left|\ln(1 - xy) \ln\left(1 - |x|^{p-1} |y|^{q-1}\right)\right| \tag{2.27}$$

and

$$\frac{1}{p}\ln(1 - |x|^p) \ln\left(1 - |y|^2\right) + \frac{1}{q}\ln\left(1 - |x|^2\right)\ln(1 - |y|^q) \geq \left|\ln(1 - xy)\ln\left(1 - |x|^{\frac{2}{q}} |y|^{\frac{2}{p}}\right)\right|$$

respectively, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and xy, $|x|^2$, $|x|^p$, $|x|^{\frac{1}{q}}$, $|y|^2$, $|y|^q$, $|y|^{\frac{1}{p}} \in D(0,1)$, $x, y \neq 0$.

(4) If we consider the function $f(z)=\sin(z)$, $z\in\mathbb{C}$, then we have $f_A(z)=\sinh(z)$, $z\in\mathbb{C}$. Applying the inequalities (2.21) and (2.22) for this function, we get

$$\sinh\left(|x|^p\right)\sinh\left(|y|^q\right)\geq\left|\sin(xy)\sin\left(|x|^{p-1}|y|^{q-1}\right)\right|$$

and

$$\begin{aligned}&\frac{1}{p}\sinh\left(|x|^p\right)\sinh\left(|y|^2\right)+\frac{1}{q}\sinh\left(|x|^2\right)\sinh\left(|y|^q\right)\\&\geq\left|\sin(xy)\sin\left(|x|^{\frac{2}{q}}|y|^{\frac{2}{p}}\right)\right|\end{aligned}$$

respectively, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and x, $y\in\mathbb{C}$, x, $y\neq 0$.

Similar result can be obtained for $\cosh(x)$ as well.

THEOREM 3. *Let $f(z)$ and $g(z)$ be as in Theorem 1. Then one has the inequality*

$$\begin{aligned}&\frac{1}{p}g_A\left(|x|^2\right)f_A\left(|y|^q\right)+\frac{1}{q}f_A\left(|x|^p\right)g_A\left(|y|^2\right)\\&\geq\left|f\left(|x|^{p-1}|y|^{q-1}\right)g\left(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\right)\right|\end{aligned}\tag{2.28}$$

and

$$\frac{1}{p}g_A\left(|x|^2\right)f_A\left(|y|^p\right)+\frac{1}{q}f_A\left(|x|^2\right)g_A\left(|y|^q\right)\geq\left|f\left(|x|^{\frac{2}{q}}y\right)g\left(|x|^{\frac{2}{p}}y\right)\right|.\tag{2.29}$$

PROOF. Follows from the inequality (1.3) on choosing $x=|x|^{\frac{2}{q}k}|y|^j$, $y=|x|^j|y|^{\frac{2}{p}k}$ and $x=|x|^{\frac{2}{q}j}|y|^k$, $y=|x|^{\frac{2}{p}k}|y|^j$. That is, for any $i,j\in\{0,1,2,\dots n\}$, we have the following inequalities

$$\begin{aligned}p|x|^{pj}|y|^{2k}+q|x|^{2k}|y|^{qj}&\geq pq|x|^{(p-1)j}|y|^{(q-1)j}|x|^{\frac{2}{p}k}|y|^{\frac{2}{q}k}\\&=pq\left|\left(|x|^{(p-1)}|y|^{(q-1)}\right)^j\left(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\right)^k\right|\end{aligned}\tag{2.30}$$

and

$$\begin{aligned}p|x|^{2j}|y|^{qk}+q|x|^{2k}|y|^{pj}&\geq pq|x|^{\frac{2}{q}j}|y|^j|x|^{\frac{2}{p}k}|y|^k\\&=pq\left|\left(|x|^{\frac{2}{q}}y\right)^j\left(|x|^{\frac{2}{p}}y\right)^k\right|\end{aligned}\tag{2.31}$$

respectively.

Repeating the same method as in Theorem 1 for (2.30) and (2.31), we deduce the desired results, i.e., (2.28) and (2.29). □

As a particular case of interest we can state the following corollary:

COROLLARY 3. *If $g(z)=f(z)$ in (2.28) and (2.29), then*

$$\begin{aligned}&\frac{1}{p}f_A\left(|x|^2\right)f_A\left(|y|^q\right)+\frac{1}{q}f_A\left(|x|^p\right)f_A\left(|y|^2\right)\\&\geq\left|f\left(|x|^{p-1}|y|^{q-1}\right)f\left(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\right)\right|\end{aligned}\tag{2.32}$$

and

$$f_A\left(|x|^2\right)\left[\frac{1}{p}f_A\left(|y|^p\right)+\frac{1}{q}f_A\left(|y|^q\right)\right]\geq\left|f\left(|x|^{\frac{2}{q}}y\right)f\left(|x|^{\frac{2}{p}}y\right)\right| \tag{2.33}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $|x|^2$, $|x|^p$, $|x|^q$,$|y|^2$, $|y|^p$, $|y|^q\in D(0,R)$, $x,y\neq 0$. *In addition, if* $y=x$ *in (2.32) and (2.33), then we have*

$$f_A\left(|x|^2\right)\left[\frac{1}{p}f_A\left(|x|^q\right)+\frac{1}{q}f_A\left(|x|^p\right)\right]\geq\left|f\left(|x|^2\right)f\left(|x|^{pq-2}\right)\right|$$

and

$$f_A\left(|x|^2\right)\left[\frac{1}{p}f_A\left(|x|^p\right)+\frac{1}{q}f_A\left(|x|^q\right)\right]\geq\left|f\left(|x|^{\frac{2}{q}}x\right)f\left(|x|^{\frac{2}{p}}x\right)\right|$$

for $|x|^2$, $|x|^p$, $|x|^q\in D(0,R)$, $x\neq 0$.

The inequalities (2.32) and (2.33) are also valuable sources of particular inequalities for complex numbers as will be outlined in the following.

(1) If we apply the inequalities (2.32) and (2.33) for the function $f(z)=\frac{1}{1-z}$, $z\in D(0,1)$, then we get

$$\begin{aligned}&\frac{1}{p\left(1-|x|^2\right)\left(1-|y|^q\right)}+\frac{1}{q\left(1-|x|^p\right)\left(1-|y|^2\right)}\\&\geq\frac{1}{\left|\left(1-|x|^{p-1}|y|^{q-1}\right)\left(1-|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\right)\right|}\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}&\frac{1}{1-|x|^2}\left(\frac{1}{p\left(1-|y|^p\right)}+\frac{1}{q\left(1-|y|^q\right)}\right)\\&\geq\frac{1}{\left|\left(1-|x|^{\frac{2}{q}}y\right)\left(1-|x|^{\frac{2}{p}}y\right)\right|}\end{aligned} \tag{2.35}$$

respectively, for any $x,y\in\mathbb{C}$ with $|x|^2,|y|^2,|x|^p,|y|^q,|x|^{\frac{1}{p}},|x|^{\frac{1}{q}},|y|^{\frac{1}{p}}$, $|y|^{\frac{1}{q}}\in D(0,1)$, $x,y\neq 0$.

(2) If we apply the inequalities (2.32) and (2.33) for the function $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we can state that

$$\begin{aligned}&\frac{1}{p}\exp\left(|x|^2+|y|^q\right)+\frac{1}{q}\exp\left(|x|^p+|y|^2\right)\\&\geq\left|\exp\left(|x|^{p-1}|y|^{q-1}+|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\right)\right|\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}&\frac{1}{p}\exp\left(|x|^2+|y|^p\right)+\frac{1}{q}\exp\left(|x|^2+|y|^q\right)\\&\geq\left|\exp\left(|x|^{\frac{2}{q}}y+|x|^{\frac{2}{p}}y\right)\right|\end{aligned} \tag{2.37}$$

respectively, for x, $y\in\mathbb{C}$, x, $y\neq 0$.

(3) If we apply the function $f(z) = \ln\left(\frac{1}{1-z}\right)$, $z \in D(0,1)$, then from (2.32) and (2.33) we have

$$\frac{1}{p}\ln\left(1-|x|^2\right)\ln\left(1-|y|^q\right)+\frac{1}{q}\ln\left(1-|x|^p\right)\ln\left(1-|y|^2\right) \geq \left|\ln\left(1-|x|^{p-1}|y|^{q-1}\right)\ln\left(1-|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\right)\right| \tag{2.38}$$

and

$$\ln\left(1-|x|^2\right)\left[\frac{1}{p}\ln\left(1-|y|^p\right)+\frac{1}{q}\ln\left(1-|y|^q\right)\right] \geq \left|\ln\left(1-|x|^{\frac{2}{q}}y\right)\ln\left(1-|x|^{\frac{2}{p}}y\right)\right| \tag{2.39}$$

respectively, for any $x, y \in \mathbb{C}$ with $|x|^2, |y|^2, |x|^p, |y|^q, |x|^{\frac{1}{p}}, |x|^{\frac{1}{q}}, |y|^{\frac{1}{p}}, |y|^{\frac{1}{q}} \in D(0,1)$, x, $y \neq 0$.

(4) If we consider the function $f(z) = \sin(z)$, $z \in \mathbb{C}$, then we have $f_A(z) = \sinh(z)$, $z \in \mathbb{C}$. Applying the inequalities (2.32) and (2.33) for this function, we get

$$\frac{1}{p}\sinh\left(|x|^2\right)\sinh\left(|y|^q\right)+\frac{1}{q}\sinh\left(|x|^p\right)\sinh\left(|y|^2\right) \geq \left|\sin\left(|x|^{p-1}|y|^{q-1}\right)\sin\left(|x|^{\frac{2}{p}}|y|^{\frac{2}{q}}\right)\right| \tag{2.40}$$

and

$$\sinh\left(|x|^2\right)\left[\frac{1}{p}\sinh\left(|y|^p\right)+\frac{1}{q}\sinh\left(|y|^q\right)\right] \geq \left|\sin\left(|x|^{\frac{2}{q}}y\right)\sin\left(|x|^{\frac{2}{p}}y\right)\right| \tag{2.41}$$

respectively, for any x, $y \in \mathbb{C}$, x, $y \neq 0$.

Similar result can be obtained for $\cosh(x)$ as well.

3. Applications to special functions

In this section, we give some inequalities for some special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. Before that, we state here some basic concepts and definitions of those functions.

The *polylogarithm* $Li_n(z)$ is a function defined by the power series

$$Li_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n} \tag{3.1}$$

which converges absolutely for all complex values of the order n and the argument z where $|z| < 1$. It is also also known in the literature as the *Jonquiére's function.* The special cases $z = -1, 1$ reduce to $Li_n(1) = \zeta(n)$ and $Li_n(-1) = -\eta(n)$, where ζ and η are the *Riemann zeta function* and *Dirichlet eta function* respectively. When $n = 1$, the first polylogarithm involves the ordinary logarithm, i.e., $Li_1(z) = -\ln(1-z)$ while the second,

$$Li_2 = \sum_{k=1}^{\infty}\frac{z^k}{k^2} \tag{3.2}$$

is called the *dilogarithm* or *Spence's function.*

For other integer values of order n, the polylogarithm reduces to the ratio of a polynomial in z, for instance

$$Li_0 = \frac{z}{1-z}, \qquad Li_{-1} = \frac{z}{(1-z)^2},$$
$$Li_{-2} = \frac{z(z+1)}{(1-z)^3}, \quad Li_{-3} = \frac{z\left(1+4z+z^2\right)}{(1-z)^4}.$$

The *Hypergeometric function* ${}_2F_1(a,b;c;z)$ is defined for all $|z|<1$ by the series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{3.3}$$

where $a,b,c \in \mathbb{R}$ with $c \neq 0,-1,-2,\ldots$ and the $(t)_n$, $n \in \{0,1,2,\ldots\}$ is a *Pochhammer symbol* which is defined by

$$(t)_n = \begin{cases} 1, & \text{if } n=0, \\ t(t+1)\ldots(t+n-1), & \text{if } n>0. \end{cases}$$

Hypergeometric function (3.3) with particular arguments of a,b and c reduce to elementary functions. For instance

$$\begin{aligned} {}_2F_1(1,1;1;z) &= {}_2F_1(1,2;2;z) = \frac{1}{1-z}, \\ {}_2F_1(1,2;1;z) &= \frac{1}{(1-z)^2}, \\ {}_2F_1(a,b;b;z) &= \frac{1}{(1-z)^a}, \\ {}_2F_1(1,1;2;z) &= \frac{1}{z}\ln\left(\frac{1}{1-z}\right), \\ {}_2F_1(1,1;2;-z) &= \frac{1}{z}\ln(1+x). \end{aligned}$$

Further, the *Bessel functions* of the first kind, denoted as $J_\alpha(z)$ are defined by the power series

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(\alpha+k)!} \left(\frac{z}{2}\right)^{2k+\alpha} \tag{3.4}$$

for $\alpha, z \in \mathbb{C}$ with $|z|<1$. If z is replaced by arguments $\pm iz$, then from (3.4) we have

$$I_\alpha(z) = i^{-\alpha} J_\alpha(iz) = \sum_{k=0}^{\infty} \frac{1}{k!\,(\alpha+k)!} \left(\frac{z}{2}\right)^{2k+\alpha} \tag{3.5}$$

for $\alpha, z \in \mathbb{C}$ with $|z|<1$. These functions (3.5) are called the *modified Bessel functions* of the first kind.

It is clearly seen that from (3.1), (3.3), (3.4) and (3.5), that is, $Li_n(z)$, ${}_2F_1(a,b;c;z)$, $J_\alpha(z)$ and $I_\alpha(z)$ are power series with real coefficients and convergent on the open disk $D(0,1)$. Therefore, all the results in the above section hold true. For instance, from (2.21) we have the following corollaries.

Corollary 4. *If $Li_n(z)$ is the polylogarithm function, then we have*

$$Li_n(|x|^p)\, Li_n(|y|^q) \geq \left| Li_n(xy)\, Li_n\left(|x|^{p-1}|y|^{q-1}\right)\right| \tag{3.6}$$

for any $x, y \in \mathbb{C}$ *with* $xy, |x|^p, |y|^q \in D(0,1), x, y \neq 0$ *and* $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$.

In particular, if $n = 0$ *in (3.6), then we have the following inequality*

$$|1 - xy| \left|1 - |x|^{p-1} |y|^{q-1}\right| \geq (1 - |x|^p)(1 - |y|^q)$$

for all $xy, |x|^p, |y|^q \in D(0,1), x, y \neq 0$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

If we take $n = 1$ *in (3.6), then we get the inequality (2.27) for all* $x, y \neq 0$ *with* $xy, |x|^p, |y|^q \in D(0,1)$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Also, if we choose in (3.6) $n = 2$*, then we obtain*

$$Li_2(|x|^p) Li_2(|y|^q) \geq \left|Li_2(xy) Li_2\left(|x|^{p-1} |y|^{q-1}\right)\right| \tag{3.7}$$

for any $xy, |x|^p, |y|^q \in D(0,1), x, y \neq 0,\ p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ *and* $Li_2(z)$ *is the dilogarithm function which is defined in (3.2).*

Corollary 5. *If* ${}_2F_1(a, b; c; z)$ *is a hypergeometric function, then for any* $a, b, c \in \mathbb{R}$ *we have*

$$\begin{aligned} &{}_2F_1(a, b; c; |x|^p)\ {}_2F_1(a, b; c; |y|^q) \\ &\geq \left|{}_2F_1(a, b; c; xy)\ {}_2F_1\left(a, b; c; |x|^{p-1} |y|^{q-1}\right)\right| \end{aligned} \tag{3.8}$$

where $xy, |x|^p, |y|^q \in D(0,1), x, y \neq 0$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if we choose $c = b$ *in (3.8), then we get the inequality (2.23) for any* $a \in \mathbb{R}$*. Also, if we choose* $a = b = 1, c = 2$*, then the inequality (3.8) reduces to (2.27).*

Corollary 6. *If* $J_\alpha(z)$ *and* $I_\alpha(z)$ *are the Bessel and modified Bessel function of the first kind respectively, then for any* $\alpha, x, y \in \mathbb{C}$*, we have*

$$I_\alpha(|x|^p) I_\alpha(|y|^q) \geq \left|J_\alpha(xy) J_\alpha\left(|x|^{p-1} |y|^{q-1}\right)\right| \tag{3.9}$$

where $xy, |x|^p, |y|^q \in D(0,1), x, y \neq 0$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if $\alpha = 0$ *in (3.9), then for* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, we get*

$$J_0(i|x|^p) J_0(i|y|^q) \geq \left|J_0(xy) J_0\left(|x|^{p-1} |y|^{q-1}\right)\right|$$

where $J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k}$.

Other inequalities involving the polylogarithm, hypergeometric, Bessel and modified Bessel functions can be found in the literature (see [**2**], [**3**], [**12**], [**14**], [**23**], [**28**] and references therein).

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[1]School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: alawiah.ibrahim@live.vu.edu.au

[2]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.
E-mail address: sever.dragomir@vu.edu.au
URL: http://www.staff.vu.edu.au/dragimir/

[3]School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia
E-mail address: maslina@ukm.my