

**SOME INEQUALITIES OF HERMITE-HADAMARD TYPE FOR
CONVEX FUNCTIONS OF COMMUTING SELFADJOINT
OPERATORS IN HILBERT SPACES**

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ABSTRACT. Some operator inequalities for convex functions of commuting self-adjoint operators that are related to the Hermite-Hadamard inequality are given. Natural examples for some fundamental convex functions are presented as well.

1. INTRODUCTION

If $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I , then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$(HH) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}.$$

This remarkable result is well known in the literature as the *Hermite-Hadamard inequality* [23].

For various generalizations, extensions, reverses and related inequalities, see [1], [2], [3], [13], [15], [16], [17], [18], [19], [23] the monograph [11] and the references therein.

The Hermite-Hadamard inequality (HH) can be extended to selfadjoint operators as follows, see [7].

Let A and B selfadjoint operators on the Hilbert space H and assume that $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then

$$(1.1) \quad \begin{aligned} f\left(\frac{\langle Ax, x \rangle + \langle By, y \rangle}{2}\right) &\leq \int_0^1 f((1-t)\langle Ax, x \rangle + t\langle By, y \rangle) dt \\ &\leq \left\langle \left[\int_0^1 f((1-t)A + t\langle By, y \rangle 1_H) dt \right] x, x \right\rangle \\ &\leq \frac{1}{2} [\langle f(A)x, x \rangle + \langle f(B)y, y \rangle] \\ &\leq \frac{1}{2} [\langle f(A)x, x \rangle + \langle f(B)y, y \rangle] \end{aligned}$$

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and

$$(1.2) \quad f\left(\frac{\langle Ax, x \rangle + \langle By, y \rangle}{2}\right) \leq \left\langle f\left(\frac{A + \langle By, y \rangle 1_H}{2}\right) x, x \right\rangle \\ \leq \left\langle \left[\int_0^1 f((1-t)A + t\langle By, y \rangle 1_H) dt \right] x, x \right\rangle$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

It is important to remark that, from the inequalities (1.1) and (1.2) we have the following Hermite-Hadamard's type results in the operator order of $B(H)$ and for the convex function $f : [m, M] \rightarrow \mathbb{R}$

$$(1.3) \quad f\left(\frac{A + \langle By, y \rangle 1_H}{2}\right) \leq \int_0^1 f((1-t)A + t\langle By, y \rangle 1_H) dt \\ \leq \frac{1}{2} [f(A) + f(\langle By, y \rangle 1_H)]$$

for any $y \in H$ with $\|y\| = 1$ and any selfadjoint operators A, B with spectra in $[m, M]$.

In particular, we have from (1.3)

$$(1.4) \quad f\left(\frac{A + \langle Ay, y \rangle 1_H}{2}\right) \leq \int_0^1 f((1-t)A + t\langle Ay, y \rangle 1_H) dt \\ \leq \frac{1}{2} [f(A) + f(\langle Ay, y \rangle 1_H)]$$

for any $y \in H$ with $\|y\| = 1$ and

$$(1.5) \quad f\left(\frac{A + s1_H}{2}\right) \leq \int_0^1 f((1-t)A + ts1_H) dt \leq \frac{1}{2} [f(A) + f(s)1_H]$$

for any $s \in [m, M]$.

As a particular case of the above results we have the following refinement of the Jensen's inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then

$$(1.6) \quad f(\langle Ax, x \rangle) \leq \left\langle f\left(\frac{A + \langle Ax, x \rangle 1_H}{2}\right) x, x \right\rangle \\ \leq \left\langle \left[\int_0^1 f((1-t)A + t\langle Ax, x \rangle 1_H) dt \right] x, x \right\rangle \\ \leq \frac{1}{2} [\langle f(A)x, x \rangle + f(\langle Ax, x \rangle)] \leq \langle f(A)x, x \rangle.$$

A real valued continuous function f on an interval I is said to be *operator convex* (*operator concave*) if

$$(OC) \quad f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B)$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I . Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $Sp(A), Sp(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [14] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I . Then for any selfadjoint operators A and B with spectra in I we have the inequality in the operator order [8]

$$(1.7) \quad f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB) dt \leq \frac{f(A) + f(B)}{2}.$$

With the above assumptions for f, A and B we also have the inequality [8]

$$(1.8) \quad \begin{aligned} (0 \leq) \int_0^1 f((1-t)A + tB) dt - f\left(\frac{A+B}{2}\right) \\ \leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) dt \end{aligned}$$

in the operator order.

For other inequalities of Hermite-Hadamard type for operator convex functions, see [8].

Motivated by the above results, we investigate in this paper the corresponding Hermite-Hadamard inequality for two commuting operators and show that, in fact, the inequality (1.7) remains valid in this case for convex functions. Other related results, refinements, reverse inequalities and some applications for particular functions of interest are provided as well.

2. SOME GENERAL RESULTS

It is known, see for instance [28, p. 356-358], that if A and B are two *commuting bounded selfadjoint operators* on the complex Hilbert space H , then there exists a bounded selfadjoint operator S on H and two bounded functions φ and ψ such that $A = \varphi(S)$ and $B = \psi(S)$. Moreover, if $\{E_t\}$ is the spectral family over the closed interval $[0, 1]$ for the selfadjoint operator S , then $S = \int_{0-}^1 t dE_t$, where the integral is taken in the Riemann-Stieltjes sense, the functions φ and ψ are summable with respect with $\{E_t\}$ on $[0, 1]$ and

$$(2.1) \quad A = \varphi(S) = \int_{0-}^1 \varphi(t) dE_t \text{ and } B = \psi(S) = \int_{0-}^1 \psi(t) dE_t.$$

Now, if A and B are as above with $Sp(A), Sp(B) \subseteq J$ an interval of real numbers, then for any continuous functions $f, g : J \rightarrow \mathbb{C}$ we have the representations

$$(2.2) \quad f(A) = \int_{0-}^1 (f \circ \varphi)(t) dE_t \text{ and } g(B) = \int_{0-}^1 (g \circ \psi)(t) dE_t.$$

For some applications of these facts to synchronous functions and Čebyšev type inequalities, see [12].

Now, if the function $f : J \rightarrow \mathbb{R}$ is continuous convex and if A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$, then utilising the representations (2.1) we have in the operator order

$$\begin{aligned}
 (2.3) \quad f((1-\lambda)A + \lambda B) &= \int_{0-}^1 f[(1-\lambda)\varphi(t) + \lambda\psi(t)] dE_t \\
 &\leq (1-\lambda) \int_{0-}^1 f(\varphi(t)) dE_t + \lambda \int_{0-}^1 f(\psi(t)) dE_t \\
 &= (1-\lambda)f(A) + \lambda f(B)
 \end{aligned}$$

for any $\lambda \in [0, 1]$.

This shows that the usual convexity is preserved for the operator order when commutativity of the operators A and B is assumed.

Therefore we can state the following particular inequalities of interest:

Let A and B be two commuting positive operators and $p \geq 1$ ($p \in (0, 1)$), then

$$(2.4) \quad ((1-\lambda)A + \lambda B)^p \leq (\geq) (1-\lambda)A^p + \lambda B^p, \lambda \in [0, 1].$$

Moreover, if A and B are positive definite then the inequality with " \leq " also holds for $p < 0$.

If A and B are two commuting selfadjoint operators and $p \geq 1$ then we have the modulus inequality

$$(2.5) \quad |(1-\lambda)A + \lambda B|^p \leq (1-\lambda)|A|^p + \lambda|B|^p, \lambda \in [0, 1].$$

In particular, we have the triangle inequality

$$(2.6) \quad |A + B| \leq |A| + |B|.$$

For A and B positive definite and commuting, we also have the logarithmic inequalities

$$(2.7) \quad \ln((1-\lambda)A + \lambda B) \geq (1-\lambda)\ln A + \lambda\ln B, \lambda \in [0, 1]$$

and

$$\begin{aligned}
 (2.8) \quad &((1-\lambda)A + \lambda B) \ln((1-\lambda)A + \lambda B) \\
 &\leq (1-\lambda)A \ln A + \lambda B \ln B, \lambda \in [0, 1].
 \end{aligned}$$

Theorem 1. *If the function $f : J \rightarrow \mathbb{R}$ is continuous convex on the interval J and if A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$, then*

$$(2.9) \quad f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-\lambda)A + \lambda B) d\lambda \leq \frac{f(A) + f(B)}{2}.$$

Proof. Since for any $\lambda \in [0, 1]$ the operators $(1-\lambda)A + \lambda B$ and $\lambda A + (1-\lambda)B$ are commutative and $Sp((1-\lambda)A + \lambda B), Sp(\lambda A + (1-\lambda)B) \subseteq J$, then by (2.3) we have

$$f\left(\frac{A+B}{2}\right) \leq \frac{1}{2} [f((1-\lambda)A + \lambda B) + f(\lambda A + (1-\lambda)B)]$$

for any $\lambda \in [0, 1]$.

Integrating this inequality on $[0, 1]$ and taking into account that

$$\int_0^1 f((1-\lambda)A + \lambda B) d\lambda = \int_0^1 f(\lambda A + (1-\lambda)B) d\lambda$$

we obtain the first inequality in (2.9).

By (2.3) we also have

$$\frac{1}{2} [f((1-\lambda)A + \lambda B) + f(\lambda A + (1-\lambda)B)] \leq \frac{f(A) + f(B)}{2}$$

for any $\lambda \in [0, 1]$, which by integration produces the second part of (2.9). \square

Corollary 1. *With the assumptions of Theorem 1 we have*

$$(2.10) \quad \begin{aligned} & \frac{1}{2} \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right) \right] \\ & \leq \int_0^1 f((1-\lambda)A + \lambda B) d\lambda \\ & \leq \frac{1}{2} \left[f\left(\frac{A+B}{2}\right) + \frac{f(A) + f(B)}{2} \right]. \end{aligned}$$

Proof. On making use of the change of variable $u = 2\lambda$ we have

$$\int_0^{1/2} f((1-\lambda)A + \lambda B) d\lambda = \frac{1}{2} \int_0^1 f\left((1-u)A + u\frac{A+B}{2}\right) du$$

and by the change of variable $u = 2\lambda - 1$ we have

$$\int_{1/2}^1 f((1-\lambda)A + \lambda B) d\lambda = \frac{1}{2} \int_0^1 f\left((1-u)\frac{A+B}{2} + uB\right) du.$$

Utilising the Hermite-Hadamard inequality (2.9) we can write

$$f\left(\frac{3A+B}{4}\right) \leq \int_0^1 f\left((1-u)A + u\frac{A+B}{2}\right) du \leq \frac{1}{2} \left[f(A) + f\left(\frac{A+B}{2}\right) \right]$$

and

$$f\left(\frac{A+3B}{4}\right) \leq \int_0^1 f\left((1-u)\frac{A+B}{2} + uB\right) du \leq \frac{1}{2} \left[f(A) + f\left(\frac{A+B}{2}\right) \right],$$

which by summation and division by two produces the desired result (2.10). \square

Remark 1. *The second inequality in (2.10) is equivalent with*

$$(2.11) \quad \begin{aligned} 0 & \leq \int_0^1 f((1-\lambda)A + \lambda B) d\lambda - f\left(\frac{A+B}{2}\right) \\ & \leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-\lambda)A + \lambda B) d\lambda, \end{aligned}$$

which shows that $\int_0^1 f((1-\lambda)A + \lambda B) d\lambda$ is closed in the operator order from $f\left(\frac{A+B}{2}\right)$ than from $\frac{f(A)+f(B)}{2}$.

Remark 2. Assume that A and B are two commuting positive operators and such that $A - B$ is invertible. Then for $p > 0$, by utilising the representation (2.1) and Fubini's theorem we have

$$\begin{aligned} \int_0^1 ((1-\lambda)A + \lambda B)^p d\lambda &= \int_0^1 \left(\int_{0-}^1 [(1-\lambda)\varphi(t) + \lambda\psi(t)]^p dE_t \right) d\lambda \\ &= \int_{0-}^1 \left(\int_0^1 [(1-\lambda)\varphi(t) + \lambda\psi(t)]^p d\lambda \right) dE_t \\ &= \frac{1}{p+1} \int_{0-}^1 \frac{\varphi^{p+1}(t) - \psi^{p+1}(t)}{\varphi(t) - \psi(t)} dE_t \\ &= \frac{1}{p+1} (A - B)^{-1} (A^{p+1} - B^{p+1}). \end{aligned}$$

Similarly, if A and B are two commuting positive definite operators and such that $A - B$ is invertible, then for $p \in (-\infty, 0) \setminus \{-1\}$ we also have

$$\int_0^1 ((1-\lambda)A + \lambda B)^p d\lambda = \frac{1}{p+1} (A - B)^{-1} (A^{p+1} - B^{p+1}).$$

Also, if A and B are two commuting positive definite operators and such that $A - B$ is invertible, then

$$\int_0^1 ((1-\lambda)A + \lambda B)^{-1} d\lambda = (A - B)^{-1} (\ln A - \ln B).$$

Utilising the Hermite-Hadamard inequality (2.9) we then have for $p \geq 1$ ($p \in (0, 1)$)

$$(2.12) \quad \left(\frac{A+B}{2} \right)^p \leq (\geq) \frac{1}{p+1} (A - B)^{-1} (A^{p+1} - B^{p+1}) \leq (\geq) \frac{A^p + B^p}{2},$$

where A and B are two commuting positive operators and such that $A - B$ is invertible.

If $p \in (-\infty, 0) \setminus \{-1\}$ the inequality (2.12) also holds for “ \geq ” provided A and B are two commuting positive definite operators and such that $A - B$ is invertible.

With the last assumptions for A and B we also have

$$(2.13) \quad \left(\frac{A+B}{2} \right)^{-1} \leq (A - B)^{-1} (\ln A - \ln B) \leq \frac{A^{-1} + B^{-1}}{2}.$$

We also have that

Theorem 2. If the function $f : J \rightarrow \mathbb{R}$ is continuous convex on the interval J and if A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$, then

$$\begin{aligned} (2.14) \quad 2 \min \{t, 1-t\} &\left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) \right] \\ &\leq (1-\lambda)f(A) + \lambda f(B) - f((1-\lambda)A + \lambda B) \\ &\leq 2 \max \{t, 1-t\} \left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) \right] \end{aligned}$$

in the operator order.

Proof. First of all, we recall the following result obtained by the author in [6] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$(2.15) \quad \begin{aligned} & n \min_{i \in \{1, \dots, n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^n f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right] \\ & \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \\ & \leq n \max_{i \in \{1, \dots, n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^n f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right], \end{aligned}$$

where $f : C \rightarrow \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X , $\{x_i\}_{i \in \{1, \dots, n\}}$ are vectors in C and $\{p_i\}_{i \in \{1, \dots, n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^n p_i > 0$.

For $n = 2$ we deduce from (4.12) that

$$(2.16) \quad \begin{aligned} & 2 \min \{\lambda, 1 - \lambda\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right] \\ & \leq \lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda)y) \\ & \leq 2 \max \{\lambda, 1 - \lambda\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right] \end{aligned}$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

Now, let φ and ψ the functions that represent the commuting operators A and B as in the equation (2.1). Then by (2.16) we have

$$(2.17) \quad \begin{aligned} & 2 \min \{\lambda, 1 - \lambda\} \left[\frac{f(\varphi(t)) + f(\psi(t))}{2} - f\left(\frac{\varphi(t) + \psi(t)}{2}\right) \right] \\ & \leq \lambda f(\varphi(t)) + (1 - \lambda) f(\psi(t)) - f(\lambda \varphi(t) + (1 - \lambda)\psi(t)) \\ & \leq 2 \max \{\lambda, 1 - \lambda\} \left[\frac{f(\varphi(t)) + f(\psi(t))}{2} - f\left(\frac{\varphi(t) + \psi(t)}{2}\right) \right] \end{aligned}$$

for any $\lambda \in [0, 1]$ and $t \in [0, 1]$.

Integrating over dE_t we deduce from (2.17) the following inequality in the operator order

$$(2.18) \quad \begin{aligned} & 2 \min \{\lambda, 1 - \lambda\} \\ & \times \left[\frac{\int_{0-}^1 f(\varphi(t)) dE_t + \int_{0-}^1 f(\psi(t)) dE_t}{2} - \int_{0-}^1 f\left(\frac{\varphi(t) + \psi(t)}{2}\right) dE_t \right] \\ & \leq \lambda \int_{0-}^1 f(\varphi(t)) dE_t + (1 - \lambda) \int_{0-}^1 f(\psi(t)) dE_t \\ & \quad - \int_{0-}^1 f(\lambda \varphi(t) + (1 - \lambda)\psi(t)) dE_t \\ & \leq 2 \max \{\lambda, 1 - \lambda\} \\ & \times \left[\frac{\int_{0-}^1 f(\varphi(t)) dE_t + \int_{0-}^1 f(\psi(t)) dE_t}{2} - \int_{0-}^1 f\left(\frac{\varphi(t) + \psi(t)}{2}\right) dE_t \right]. \end{aligned}$$

Now, taking into account the integral representation of continuous functions of selfadjoint operators (2.2) we deduce the desired result (2.14). \square

Remark 3. *If A and B are two commuting selfadjoint operators and $p \geq 1$ then we have the modulus inequality*

$$(2.19) \quad \begin{aligned} & 2 \min \{ \lambda, 1 - \lambda \} \left[\frac{|A|^p + |B|^p}{2} - \left| \frac{A + B}{2} \right|^p \right] \\ & \leq (1 - \lambda) |A|^p + \lambda |B|^p - |(1 - \lambda) A + \lambda B|^p \\ & \leq 2 \max \{ \lambda, 1 - \lambda \} \left[\frac{|A|^p + |B|^p}{2} - \left| \frac{A + B}{2} \right|^p \right] \end{aligned}$$

where $\lambda \in [0, 1]$.

In particular, we have

$$(2.20) \quad \begin{aligned} & \min \{ \lambda, 1 - \lambda \} [|A| + |B| - |A + B|] \\ & \leq (1 - \lambda) |A| + \lambda |B| - |(1 - \lambda) A + \lambda B| \\ & \leq 2 \max \{ \lambda, 1 - \lambda \} [|A| + |B| - |A + B|] \end{aligned}$$

for any $\lambda \in [0, 1]$.

If A and B are two commuting positive definite operators, then we have the logarithmic inequalities

$$(2.21) \quad \begin{aligned} & 2 \min \{ \lambda, 1 - \lambda \} \left[\ln \left(\frac{A + B}{2} \right) - \frac{\ln A + \ln B}{2} \right] \\ & \leq \ln ((1 - \lambda) A + \lambda B) - (1 - \lambda) \ln A - \lambda \ln B \\ & \leq 2 \max \{ \lambda, 1 - \lambda \} \left[\left[\ln \left(\frac{A + B}{2} \right) - \frac{\ln A + \ln B}{2} \right] \right] \end{aligned}$$

for any $\lambda \in [0, 1]$.

3. SOME RESULTS FOR DIFFERENTIABLE FUNCTIONS

The following reverse of the inequality (2.3) for differentiable functions holds:

Theorem 3. *Let $f : J \rightarrow \mathbb{R}$ be a convex continuously differentiable function on the interior \mathring{J} of J and A and B two commuting selfadjoint operators with $Sp(A), Sp(B) \subset \mathring{J}$. Then we have in the operator order*

$$(3.1) \quad \begin{aligned} & 0 \leq (1 - \lambda) f(A) + \lambda f(B) - f((1 - \lambda) A + \lambda B) \\ & \leq (1 - \lambda) \lambda (A - B) [f'(A) - f'(B)], \end{aligned}$$

for any $\lambda \in [0, 1]$.

In particular, we have

$$(3.2) \quad 0 \leq \frac{f(A) + f(B)}{2} - f\left(\frac{A + B}{2}\right) \leq \frac{1}{4} (A - B) [f'(A) - f'(B)].$$

Proof. We have the gradient inequality

$$(3.3) \quad f(u) - f(v) \geq f'(v)(u - v)$$

for any $u, v \in \mathring{J}$.

Utilising the gradient inequality (3.3) we have

$$(3.4) \quad f(\lambda x + (1 - \lambda) y) - f(x) \geq (1 - \lambda) f'(x)(y - x)$$

and

$$(3.5) \quad f(\lambda x + (1 - \lambda)y) - f(y) \geq -\lambda f'(y)(y - x),$$

for any $\lambda \in [0, 1]$ and $x, y \in \mathring{J}$.

If we multiply (3.4) with λ and (3.5) with $1 - \lambda$ and add the resultant inequalities we obtain

$$\begin{aligned} & f(\lambda x + (1 - \lambda)y) - \lambda f(x) - (1 - \lambda)f(y) \\ & \geq (1 - \lambda)\lambda f'(x)(y - x) - \lambda(1 - \lambda)f'(y)(y - x) \end{aligned}$$

which is clearly equivalent with

$$(3.6) \quad \begin{aligned} 0 & \leq \lambda f(x)(1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y) \\ & \leq (1 - \lambda)\lambda(x - y)[f'(x) - f'(y)] \end{aligned}$$

for any $\lambda \in [0, 1]$ and $x, y \in \mathring{J}$.

Now, let φ and ψ the functions that represent the commuting operators A and B as in the equation (2.1). Then by (3.6) we have

$$(3.7) \quad \begin{aligned} 0 & \leq \lambda f(\varphi(t)) + (1 - \lambda)f(\psi(t)) - f(\lambda\varphi(t) + (1 - \lambda)\psi(t)) \\ & \leq (1 - \lambda)\lambda(\varphi(t) - \psi(t))[f'(\varphi(t)) - f'(\psi(t))] \end{aligned}$$

for any $\lambda \in [0, 1]$ and $t \in [0, 1]$.

Integrating over dE_t we deduce from (3.7) the following inequality in the operator order

$$(3.8) \quad \begin{aligned} 0 & \leq \lambda \int_{0-}^1 f(\varphi(t)) dE_t + (1 - \lambda) \int_{0-}^1 f(\psi(t)) dE_t \\ & \quad - \int_{0-}^1 f(\lambda\varphi(t) + (1 - \lambda)\psi(t)) dE_t \\ & \leq (1 - \lambda)\lambda \int_{0-}^1 (\varphi(t) - \psi(t)) [f'(\varphi(t)) - f'(\psi(t))] dE_t \end{aligned}$$

for any $\lambda \in [0, 1]$.

Utilising the representation (2.2) for continuous functions of selfadjoint operators, we deduce the desired result (3.1). \square

Remark 4. For A and B positive definite and commuting operators, we have the following reverse of (2.7)

$$(3.9) \quad \begin{aligned} 0 & \leq \ln((1 - \lambda)A + \lambda B) - (1 - \lambda)\ln A - \lambda\ln B \\ & \leq (1 - \lambda)\lambda(A - B)(B^{-1} - A^{-1}) \end{aligned}$$

and the following reverse of (2.8)

$$(3.10) \quad \begin{aligned} & (1 - \lambda)A \ln A + \lambda B \ln B - ((1 - \lambda)A + \lambda B) \ln((1 - \lambda)A + \lambda B) \\ & \leq (1 - \lambda)\lambda(A - B)(\ln A - \ln B), \end{aligned}$$

where $\lambda \in [0, 1]$.

We use now the following result whose proof in a slightly more general form can be found in [4] and [5].

Lemma 1. Let $f : J \rightarrow \mathbb{R}$ be a convex differentiable function on the interior \mathring{J} . Then for any $a, b \in \mathring{J}$ we have the inequalities

$$(3.11) \quad 0 \leq \frac{f(a) + f(b)}{2} - \int_0^1 f((1-t)a + tb) dt \leq \frac{1}{8} (a-b) (f'(a) - f'(b))$$

and

$$(3.12) \quad 0 \leq \int_0^1 f((1-t)a + tb) dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8} (a-b) (f'(a) - f'(b)).$$

The constant $\frac{1}{8}$ is best possible in both inequalities.

For extensions of these inequalities for functions defined on segments of vectors in linear spaces and applications for semi-inner products, see [4] and [5].

Theorem 4. Let $f : J \rightarrow \mathbb{R}$ be a convex continuously differentiable function on the interior \mathring{J} of J and A and B two commuting selfadjoint operators with $Sp(A), Sp(B) \subset \mathring{J}$. Then we have in the operator order

$$(3.13) \quad \begin{aligned} 0 &\leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-\lambda)A + \lambda B) d\lambda \\ &\leq \frac{1}{8} (A - B) (f'(A) - f'(B)) \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} 0 &\leq \int_0^1 f((1-\lambda)A + \lambda B) d\lambda - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{8} (A - B) (f'(A) - f'(B)). \end{aligned}$$

Proof. Let φ and ψ the functions that represent the commuting operators A and B as in the equation (2.1). Then by (3.11) we have

$$(3.15) \quad \begin{aligned} 0 &\leq \frac{f(\varphi(t)) + f(\psi(t))}{2} - \int_0^1 f((1-\lambda)\varphi(t) + \lambda\psi(t)) d\lambda \\ &\leq \frac{1}{8} (\varphi(t) - \psi(t)) (f'(\varphi(t)) - f'(\psi(t))) \end{aligned}$$

for any $t \in [0, 1]$.

Integrating over dE_t we deduce from (3.15) the following inequality in the operator order

$$(3.16) \quad \begin{aligned} 0 &\leq \frac{\int_{0-}^1 f(\varphi(t)) dE_t + \int_{0-}^1 f(\psi(t)) dE_t}{2} \\ &\quad - \int_{0-}^1 \left(\int_0^1 f((1-\lambda)\varphi(t) + \lambda\psi(t)) d\lambda \right) dE_t \\ &\leq \frac{1}{8} \int_{0-}^1 (\varphi(t) - \psi(t)) (f'(\varphi(t)) - f'(\psi(t))) dE_t. \end{aligned}$$

By Fubini's theorem we have

$$\begin{aligned} &\int_{0-}^1 \left(\int_0^1 f((1-\lambda)\varphi(t) + \lambda\psi(t)) d\lambda \right) dE_t \\ &= \int_0^1 \left(\int_{0-}^1 f((1-\lambda)\varphi(t) + \lambda\psi(t)) dE_t \right) d\lambda \end{aligned}$$

and employing the representation of continuous functions of commuting operators (2.2) we deduce the desired result (3.13).

The inequality (3.14) can be obtained in a similar manner and the details are omitted. \square

Remark 5. *If A and B are two commuting positive operators and such that $A - B$ is invertible, then for $p \geq 1$ we have the inequalities*

$$(3.17) \quad \begin{aligned} 0 &\leq \frac{A^p + B^p}{2} - \frac{1}{p+1} (A - B)^{-1} (A^{p+1} - B^{p+1}) \\ &\leq \frac{1}{8} p (A - B) (A^{p-1} - B^{p-1}) \end{aligned}$$

and

$$(3.18) \quad \begin{aligned} 0 &\leq \frac{1}{p+1} (A - B)^{-1} (A^{p+1} - B^{p+1}) - \left(\frac{A + B}{2} \right)^p \\ &\leq \frac{1}{8} p (A - B) (A^{p-1} - B^{p-1}). \end{aligned}$$

If A and B are two commuting positive definite operators and such that $A - B$ is invertible, then these two inequalities also hold for $p \in (-\infty, 0) \setminus \{-1\}$.

If $p \in (0, 1)$ and A and B are two commuting positive definite operators and such that $A - B$ is invertible, then we have

$$(3.19) \quad \begin{aligned} 0 &\leq \frac{1}{p+1} (A - B)^{-1} (A^{p+1} - B^{p+1}) - \frac{A^p + B^p}{2} \\ &\leq \frac{1}{8} p (A - B) (B^{p-1} - A^{p-1}) \end{aligned}$$

and

$$(3.20) \quad \begin{aligned} 0 &\leq \left(\frac{A + B}{2} \right)^p - \frac{1}{p+1} (A - B)^{-1} (A^{p+1} - B^{p+1}) \\ &\leq \frac{1}{8} p (A - B) (B^{p-1} - A^{p-1}). \end{aligned}$$

Moreover, if A and B are two commuting positive definite operators and such that $A - B$ is invertible, then we also have

$$(3.21) \quad \begin{aligned} 0 &\leq (A - B)^{-1} (\ln A - \ln B) - \left(\frac{A + B}{2} \right)^{-1} \\ &\leq \frac{1}{8} (A - B) (B^{-2} - A^{-2}) \end{aligned}$$

and

$$(3.22) \quad \begin{aligned} 0 &\leq \frac{A^{-1} + B^{-1}}{2} - (A - B)^{-1} (\ln A - \ln B) \\ &\leq \frac{1}{8} (A - B) (B^{-2} - A^{-2}). \end{aligned}$$

4. MORE RESULTS FOR TWICE DIFFERENTIABLE FUNCTIONS

The following result for twice differentiable functions holds.

Theorem 5. Let $f : J \rightarrow \mathbb{R}$ be a twice differentiable function on the interior \mathring{J} and k, K two real numbers such that

$$(4.1) \quad k \leq f''(\lambda) \leq K \quad \text{for any } \lambda \in \mathring{J}.$$

If A and B are two commuting selfadjoint operators with $Sp(A), Sp(B) \subset \mathring{J}$, then we have in the operator order

$$(4.2) \quad \begin{aligned} & \frac{1}{2}k\lambda(1-\lambda)(A-B)^2 \\ & \leq (1-\lambda)f(A) + \lambda f(B) - f((1-\lambda)A + \lambda B) \\ & \leq \frac{1}{2}K\lambda(1-\lambda)(A-B)^2 \end{aligned}$$

and

$$(4.3) \quad \begin{aligned} & \frac{1}{2}k \left(\lambda - \frac{1}{2} \right)^2 (A-B)^2 \\ & \leq \frac{f((1-\lambda)A + \lambda B) + f((1-\lambda)B + \lambda A) - f\left(\frac{A+B}{2}\right)}{2} \\ & \leq \frac{1}{2}K \left(\lambda - \frac{1}{2} \right)^2 (A-B)^2 \end{aligned}$$

for any $\lambda \in [0, 1]$.

Proof. Consider the auxiliary function $g_k : J \rightarrow \mathbb{R}$ given by $g_k(x) = f(x) - \frac{1}{2}kx^2$. This function is twice differentiable on \mathring{J} and $g_k''(x) = f''(x) - k \geq 0$ which shows that g_k is convex on J .

By the definition of convexity we have

$$\begin{aligned} 0 & \leq (1-\lambda)g_k(a) + \lambda g_k(b) - g_k((1-\lambda)a + \lambda b) \\ & = (1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \\ & \quad - \frac{1}{2}k \left[(1-\lambda)a^2 + \lambda b^2 - ((1-\lambda)a + \lambda b)^2 \right] \\ & = (1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \\ & \quad - \frac{1}{2}k\lambda(1-\lambda)(a-b)^2 \end{aligned}$$

for any $a, b \in \mathring{J}$ for any $\lambda \in [0, 1]$.

This implies the inequality

$$(4.4) \quad \begin{aligned} & \frac{1}{2}k\lambda(1-\lambda)(a-b)^2 \\ & \leq (1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \end{aligned}$$

for any $a, b \in \mathring{J}$ and for any $\lambda \in [0, 1]$.

If we consider the auxiliary function $g_K : J \rightarrow \mathbb{R}$ given by $g_K(x) = \frac{1}{2}Kx^2 - f(x)$ and apply a similar argument, we deduce the opposite inequality

$$(4.5) \quad \begin{aligned} & (1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \\ & \leq \frac{1}{2}K\lambda(1-\lambda)(a-b)^2 \end{aligned}$$

for any $a, b \in \mathring{J}$ and for any $\lambda \in [0, 1]$.

Now, utilising the spectral representation (2.1) for the commuting selfadjoint operators A and B , the scalar inequalities (4.4), (4.5) and a similar argument to that in the proof of Theorem 2, we deduce the desired result (4.2).

From (4.2) for $\lambda = \frac{1}{2}$ we get in the operator order

$$(4.6) \quad \frac{1}{8}k(C-D)^2 \leq \frac{f(C)+f(D)}{2} - f\left(\frac{C+D}{2}\right) \leq \frac{1}{8}K(C-D)^2,$$

for any commuting selfadjoint operators C and D with $Sp(C), Sp(D) \subset \mathring{J}$.

If we take $C = (1-\lambda)A + \lambda B$ and $D = (1-\lambda)B + \lambda A$, then C and D are commuting operators with $Sp(C), Sp(D) \subset \mathring{J}$ and by (4.6) we deduce the desired result (4.3). \square

Remark 6. *The above operator inequalities have some particular cases of interest. For instance, for $p \geq 2$, if A and B are two commuting selfadjoint operators such that $Sp(A), Sp(B) \subset [m, M] \subset (0, \infty)$, then we have in the operator order*

$$(4.7) \quad \begin{aligned} & \frac{1}{2}p(p-1)m^{p-2}\lambda(1-\lambda)(A-B)^2 \\ & \leq (1-\lambda)A^p + \lambda B^p - ((1-\lambda)A + \lambda B)^p \\ & \leq \frac{1}{2}p(p-1)M^{p-2}\lambda(1-\lambda)(A-B)^2 \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} & \frac{1}{2}p(p-1)m^{p-2}\left(\lambda - \frac{1}{2}\right)^2(A-B)^2 \\ & \leq \frac{((1-\lambda)A + \lambda B)^p + ((1-\lambda)B + \lambda A)^p}{2} - \left(\frac{A+B}{2}\right)^p \\ & \leq \frac{1}{2}p(p-1)M^{p-2}\left(\lambda - \frac{1}{2}\right)^2(A-B)^2 \end{aligned}$$

for any $\lambda \in [0, 1]$.

We have also the logarithmic inequalities

$$(4.9) \quad \begin{aligned} & \frac{1}{2}M^{-2}\lambda(1-\lambda)(A-B)^2 \\ & \leq \ln((1-\lambda)A + \lambda B) - (1-\lambda)\ln(A) - \lambda\ln(B) \\ & \leq \frac{1}{2}m^{-2}\lambda(1-\lambda)(A-B)^2 \end{aligned}$$

and

$$(4.10) \quad \begin{aligned} & \frac{1}{2}M^{-2}\left(\lambda - \frac{1}{2}\right)^2(A-B)^2 \\ & \leq \frac{\ln((1-\lambda)A + \lambda B) + \ln((1-\lambda)B + \lambda A)}{2} - \ln\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2}m^{-2}\left(\lambda - \frac{1}{2}\right)^2(A-B)^2 \end{aligned}$$

for any $\lambda \in [0, 1]$.

We use the following inequalities for twice differentiable functions that are related to the Hermite-Hadamard inequality:

Lemma 2. Let $f : J \rightarrow \mathbb{R}$ be a twice differentiable function on the interior \mathring{J} and k, K two real numbers such that (4.1) is true. Then for any $a, b \in \mathring{J}$ we have the inequalities

$$(4.11) \quad \frac{1}{24}k(a-b)^2 \leq \int_0^1 f((1-\lambda)a + \lambda b) d\lambda - f\left(\frac{a+b}{2}\right) \leq \frac{1}{24}K(a-b)^2,$$

$$(4.12) \quad \frac{1}{12}k(a-b)^2 \leq \frac{f(a) + f(b)}{2} - \int_0^1 f((1-\lambda)a + \lambda b) d\lambda \leq \frac{1}{12}K(a-b)^2$$

and

$$(4.13) \quad \begin{aligned} \frac{1}{48}k(a-b)^2 &\leq \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \int_0^1 f((1-\lambda)a + \lambda b) d\lambda \\ &\leq \frac{1}{48}K(a-b)^2. \end{aligned}$$

Proof. The inequality (4.11) was obtained in [9] while (4.12) was obtained in [10]. The inequality (4.13) was established in [11, p. 44]. \square

Theorem 6. Let $f : J \rightarrow \mathbb{R}$ be a twice differentiable function on the interior \mathring{J} and satisfying the condition (4.1). If A and B are two commuting selfadjoint operators with $Sp(A), Sp(B) \subset \mathring{J}$, then we have in the operator order

$$(4.14) \quad \begin{aligned} \frac{1}{24}k(A-B)^2 &\leq \int_0^1 f((1-\lambda)A + \lambda B) d\lambda - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{24}K(A-B)^2, \end{aligned}$$

$$(4.15) \quad \frac{1}{12}k(A-B)^2 \leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-\lambda)A + \lambda B) d\lambda$$

$$(4.16) \quad \leq \frac{1}{12}K(A-B)^2$$

and

$$(4.17) \quad \begin{aligned} \frac{1}{48}k(A-B)^2 &\leq \frac{1}{2} \left[\frac{f(A) + f(B)}{2} + f\left(\frac{A+B}{2}\right) \right] \\ &\quad - \int_0^1 f((1-\lambda)A + \lambda B) d\lambda \leq \frac{1}{48}K(A-B)^2. \end{aligned}$$

The argument follows in a similar way to that in the proof of Theorem 4 by utilising Lemma 2 and the details are omitted.

Remark 7. The above operator inequalities have some particular cases of interest. For instance, for $p \geq 2$, if A and B are two commuting selfadjoint operators such that $A - B$ is invertible and $Sp(A), Sp(B) \subset [m, M] \subset (0, \infty)$, then we have in the operator order

$$(4.18) \quad \begin{aligned} \frac{p(p-1)}{24}m^{p-2}(A-B)^2 &\leq \frac{1}{p+1}(A-B)^{-1}(A^{p+1} - B^{p+1}) \\ &\quad - \left(\frac{A+B}{2}\right)^p \leq \frac{p(p-1)}{24}M^{p-2}(A-B)^2, \end{aligned}$$

$$(4.19) \quad \frac{p(p-1)}{12} m^{p-2} (A-B)^2 \leq \frac{A^p + B^p}{2} \\ - \frac{1}{p+1} (A-B)^{-1} (A^{p+1} - B^{p+1}) \leq \frac{p(p-1)}{12} M^{p-2} (A-B)^2$$

and

$$(4.20) \quad \frac{p(p-1)}{48} m^{p-2} (A-B)^2 \leq \frac{1}{2} \left[\frac{A^p + B^p}{2} + \left(\frac{A+B}{2} \right)^p \right] \\ - \frac{1}{p+1} (A-B)^{-1} (A^{p+1} - B^{p+1}) \leq \frac{p(p-1)}{48} M^{p-2} (A-B)^2.$$

If $p \in [1, 2)$, then the above inequalities (4.18)-(4.20) also hold by replacing M^{p-2} with m^{p-2} .

Similar results may be stated for $p \in (0, 1)$, $p \in (\infty, 0) \setminus \{-1\}$ and $p = -1$. However the details are not presented here.

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