Received 08/05/12

SOME INEQUALITIES OF JENSEN TYPE FOR CONVEX FUNCTIONS OF COMMUTING SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR[1,2]

Abstract. Some operator inequalities for convex functions of commuting selfadjoint operators in Hilbert spaces that are related to the Jensen inequality are given. Natural examples for some fundamental convex functions are presented as well.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p}=(p_1,\dots,p_n)$ where $p_j\geq 0,\ j\in\{1,...,n\}$ with $P_n:=\sum_{j=1}^n p_j>0$ and $\mathbf{x}=(x_1,\dots,x_n)\in C^n$, then

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\leq\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i), \tag{1.1}$$

is well known in the literature as *Jensen's inequality*.

The following result that provides an operator version for the Jensen inequality holds, see for instance [17]:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq I$, an interval of real numbers. If f is a convex function on I, then

$$f(\langle Ax,x\rangle)\leq\langle f(A)x,x\rangle \tag{MP}$$

for each $x\in H$ with $\|x\|=1$.

Let A_j be selfadjoint operators with $Sp(A_j)\subseteq I,\ j\in\{1,...,n\}$. If $p_j\geq 0$, $j\in\{1,...,n\}$ with $P_n>0$ and f is a convex function on I then (see for instance [6]):

$$f\left(\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_j x,x\rangle\right)\leq\frac{1}{P_n}\sum_{j=1}^n p_j\langle f(A_j)x,x\rangle, \tag{1.2}$$

for any $x\in H$ with $\|x\|=1$.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint bounded linear operators, Functions of operators, Power series.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f\left((1-\lambda)A+\lambda B\right)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $Sp(A), Sp(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [9] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function$f(t)=e^t$ is neither operator convex nor operator monotone.

We also have the following Jensen type inequality for operator convex functions $f:I\to\mathbb{R}$.

Let A_j be selfadjoint operators with $Sp(A_j)\subseteq I$, $j\in\{1,...,n\}$. If $p_j\geq 0$, $j\in\{1,...,n\}$ with $P_n>0$ and f is an operator convex function on I then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_iA_j\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_if(A_j), \tag{1.3}$$

in the operator order.

For recent results related to the Jensen inequality for selfadjoint operators in Hilbert spaces see the papers [1]-[5], [10]-[16], [19] and the monograph [8].

Motivated by the above results, we investigate in this paper the corresponding Jensen inequality for two commuting operators and show that, in fact, the inequality (1.3) remains valid in this case for convex functions. Other related results, refinements, reverse inequalities and some applications for particular functions of interest are provided as well.

2. Some General Results

It is known, see for instance [18, p. 356-358], that if A and B are two *commuting bounded selfadjoint operators* on the complex Hilbert space H, then there exists a bounded selfadjoint operator S on H and two bounded functions φ and ψ such that $A=\varphi(S)$ and $B=\psi(S)$. Moreover, if $\{E_t\}$ is the spectral family over the closed interval $[0,1]$ for the selfadjoint operator S, then $S=\int_{0-}^{1}tdE_t$, where the integral is taken in the Riemann-Stieltjes sense, the functions φ and ψ are summable with respect with $\{E_t\}$ on $[0,1]$ and

$$A=\varphi(S)=\int_{0-}^{1}\varphi(t)\,dE_t \text{ and } B=\psi(S)=\int_{0-}^{1}\psi(t)\,dE_t. \tag{2.1}$$

Now, if A and B are as above with $Sp(A), Sp(B) \subseteq J$ an interval of real numbers, then for any continuous functions $f, g : J \to \mathbb{C}$ we have the representations

$$f(A) = \int_{0-}^{1} (f \circ \varphi)(t)\, dE_t \text{ and } g(B) = \int_{0-}^{1} (g \circ \psi)(t)\, dE_t. \tag{2.2}$$

For some applications of these facts to synchronous functions and Čebyšev type inequalities, see **??**.

Now, if the function $f : J \to \mathbb{R}$ is continuous convex and if A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$, then utilising the representations (2.1) we have in the operator order

$$\begin{aligned} f((1-\lambda)A + \lambda B) &= \int_{0-}^{1} f[(1-\lambda)\varphi(t) + \lambda\psi(t)]\, dE_t \\ &\le (1-\lambda)\int_{0-}^{1} f(\varphi(t))\, dE_t + \lambda \int_{0-}^{1} f(\psi(t))\, dE_t \\ &= (1-\lambda) f(A) + \lambda f(B) \end{aligned} \tag{2.3}$$

for any $\lambda \in [0,1]$.

This shows that *the usual convexity is preserved for the operator order when commutativity of the operators A and B is assumed.*

We say that the n-tuple of operators $(A_1, ..., A_n)$ is *mutually commutative* if $A_i A_j = A_j A_i$ for any $i, j \in \{1, ..., n\}$.

Theorem 1. *If the function $f : J \to \mathbb{R}$ is continuous convex and if the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative and with $Sp(A_j) \subseteq J$ for any $j \in \{1, ..., n\}$, then for any $p_j \ge 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^{n} p_j > 0$ we have*

$$f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \le \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) \tag{2.4}$$

in the operator order.

Proof. We prove the inequality by induction.

For $n = 2$ we have

$$f\left(\frac{p_1 A_1 + p_2 A_2}{p_1 + p_2}\right) \le \frac{p_1 f(A_1) + p_2 f(A_2)}{p_1 + p_2},$$

which follows from (2.3) for the commuting operators $A = A_1$, $B = A_2$ and $\lambda = \frac{p_2}{p_1 + p_2}$.

Now, assume that the inequality (2.4) holds for the n-tuple of operators $(A_1, ..., A_n)$ that is mutually commutative and with $Sp(A_j) \subseteq J$ for any $j \in \{1, ..., n\}$.

Assume that $(A_1, ..., A_n, A_{n+1})$ is a $(n+1)$-tuple of operators that is mutually commutative and with $Sp(A_j) \subseteq J$ for any $j \in \{1, ..., n+1\}$.

Let $p_j \ge 0$ with $j \in \{1, ..., n+1\}$ and $P_{n+1} := \sum_{j=1}^{n+1} p_j > 0$.

Assume that $P_n > 0$ and consider the operators

$$A := \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \text{ and } B = A_{n+1}.$$

We observe that $Sp(A) \subseteq J$.

Since A_{n+1} commutes with all $A_1, ..., A_n$ then it also commutes with A and applying the inequality (2.3) for $\lambda = \frac{p_n}{P_{n+1}}$ we have

$$f\left(\frac{1}{P_{n+1}}\sum_{j=1}^{n+1} p_j A_j\right) = f\left(\frac{P_n}{P_{n+1}} \cdot \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j + \frac{p_{n+1}}{P_{n+1}} A_{n+1}\right) \tag{2.5}$$
$$\leq \frac{P_n}{P_{n+1}} f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + \frac{p_{n+1}}{P_{n+1}} f\left(A_{n+1}\right)$$

in the operator order.

By the induction hypothesis we also have

$$\frac{P_n}{P_{n+1}} f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + \frac{p_{n+1}}{P_{n+1}} f\left(A_{n+1}\right) \tag{2.6}$$
$$\leq \frac{P_n}{P_{n+1}}\left[\frac{1}{P_n}\sum_{j=1}^{n} p_j f\left(A_j\right)\right] + \frac{p_{n+1}}{P_{n+1}} f\left(A_{n+1}\right)$$
$$= \frac{1}{P_{n+1}}\sum_{j=1}^{n+1} p_j f\left(A_j\right),$$

in the operator order.

By (2.5) and (2.6) we get that

$$f\left(\frac{1}{P_{n+1}}\sum_{j=1}^{n+1} p_j A_j\right) \leq \frac{1}{P_{n+1}}\sum_{j=1}^{n+1} p_j f\left(A_j\right) \tag{2.7}$$

in the operator order, which proves the statement.

If $P_n = 0$ then all $p_1, ..., p_n = 0$ and the inequality (2.7) holds with equality. □

Corollary 1. *With the assumptions from Theorem 1 for $(A_1, ..., A_n)$ we have the inequality*

$$f\left(\frac{1}{n}\sum_{j=1}^{n} A_j\right) \leq \frac{1}{n}\sum_{j=1}^{n} f\left(A_j\right). \tag{2.8}$$

We have some power operator inequalities as follows:

Remark 1. *If the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative then for any $p \geq 1$ we have in the operator order*

$$\left|\sum_{j=1}^{n} p_j A_j\right|^p \leq P_n^{p-1}\sum_{j=1}^{n} p_j \left|A_j\right|^p \tag{2.9}$$

for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^{n} p_j > 0$.

In particular we have

$$\left|\sum_{j=1}^{n} A_j\right|^p \leq n^{p-1}\sum_{j=1}^{n} \left|A_j\right|^p. \tag{2.10}$$

We also have the generalized triangle inequality

$$\left|\sum_{j=1}^{n} A_j\right| \leq \sum_{j=1}^{n} |A_j| . \tag{2.11}$$

We have also some exponential operator inequalities as follows:

Remark 2. *If the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^{n} p_j > 0$ we have*

$$\exp\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j \exp(A_j) . \tag{2.12}$$

We consider the functional

$$J_n(\mathbf{p}; \mathbf{A}, f, J) := \sum_{j=1}^{n} p_j f(A_j) - P_n f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \tag{2.13}$$

where $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$, $\mathbf{A} = (A_1, ..., A_n)$ is a mutually commutative n-tuple of operators with $Sp(A_j) \subseteq J$ for $j \in \{1, ..., n\}$ and $f : J \to \mathbb{R}$ is a convex function defined on the interval J.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$. For $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1, ..., n\}$.

Theorem 2. *With the assumptions of Theorem 1 for $\mathbf{A}, f$ and J, then for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, f, J) \geq J_n(\mathbf{p}; \mathbf{A}, f, J) + J_n(\mathbf{q}; \mathbf{A}, f, J) \geq 0, \tag{2.14}$$

i.e., $J_n(\cdot; \mathbf{A}, f, J)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$ then also

$$J_n(\mathbf{p}; \mathbf{A}, f, J) \geq J_n(\mathbf{q}; \mathbf{A}, f, J) \geq 0, \tag{2.15}$$

i.e., $J_n(\cdot; \mathbf{A}, f, J)$ is a monotonic functional in the operator order.

Proof. We have the equality

$$\begin{aligned} &J_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, f, J) \\ &= \sum_{j=1}^{n} (p_j + q_j) f(A_j) - (P_n + Q_n) f\left(\frac{1}{P_n + Q_n}\sum_{j=1}^{n} (p_j + q_j) A_j\right) \\ &= \sum_{j=1}^{n} (p_j + q_j) f(A_j) \\ &\quad - (P_n + Q_n) f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n} q_j A_j\right)}{P_n + Q_n}\right). \end{aligned} \tag{2.16}$$

Now, consider the operators

$$A := \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \text{ and } B := \frac{1}{Q_n}\sum_{j=1}^{n} q_j A_j.$$

Since $(A_1, ..., A_n)$ is a mutually commutative n-tuple of operators with $Sp(A_j) \subseteq J$ for $j \in \{1, ..., n\}$, then A and B defined above are commutative and $Sp(A), Sp(B) \subseteq J$.

Applying the inequality (2.3) for A and B given above and $\lambda = \frac{Q_n}{P_n+Q_n}$ we have

$$f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+Q_n\left(\frac{1}{Q_n}\sum_{j=1}^n q_jA_j\right)}{P_n+Q_n}\right) \leq \frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+\frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^n q_jA_j\right) \tag{2.17}$$

in the operator order.

Making use of (2.16) and (2.17) we have

$$\begin{aligned} J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,J) &\geq \sum_{j=1}^n (p_j+q_j) f(A_j) - (P_n+Q_n) \\ &\times\left[\frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+\frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^n q_jA_j\right)\right] \\ &= \sum_{j=1}^n p_j f(A_j) - P_n f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right) \\ &+ \sum_{j=1}^n q_j f(A_j) - Q_n f\left(\frac{1}{Q_n}\sum_{j=1}^n q_jA_j\right) \\ &= J_n(\mathbf{p};\mathbf{A},f,J)+J_n(\mathbf{q};\mathbf{A},f,J) \end{aligned} \tag{2.18}$$

in the operator order, and the inequality (2.14) is proved.

Now, let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$. Then by the super-additivity property (2.14) we have

$$\begin{aligned} J_n(\mathbf{p};\mathbf{A},f,J) &= J_n((\mathbf{p}-\mathbf{q})+\mathbf{q};\mathbf{A},f,J) \\ &\geq J_n((\mathbf{p}-\mathbf{q});\mathbf{A},f,J)+J_n(\mathbf{q};\mathbf{A},f,J) \geq J_n(\mathbf{q};\mathbf{A},f,J) \end{aligned} \tag{2.19}$$

in the operator order, and the monotonicity property (2.15) is proved. □

Corollary 2. *Assume that the function $f : J \to \mathbb{R}$ is continuous convex and the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative and with $Sp(A_j) \subseteq J$ for any $j \in \{1, ..., n\}$. If $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and there exists the positive constants m, M such that*

$$m\mathbf{q} \leq \mathbf{p} \leq M\mathbf{q} \tag{2.20}$$

then

$$mJ_n(\mathbf{q};\mathbf{A},f,J) \leq J_n(\mathbf{p};\mathbf{A},f,J) \leq MJ_n(\mathbf{q};\mathbf{A},f,J) \tag{2.21}$$

in the operator order.

Proof. Observe that for $\alpha > 0$ we have $J_n(\alpha \mathbf{p}; \mathbf{A}, f, J) = \alpha J_n(\mathbf{p}; \mathbf{A}, f, J)$.

Utilising the monotonicity property (2.15) we have

$$J_n(m\mathbf{q}; \mathbf{A}, f, J) \le J_n(\mathbf{p}; \mathbf{A}, f, J) \le J_n(M\mathbf{q}; \mathbf{A}, f, J)$$

which imply the desired result (2.21). □

Remark 3. *We observe that if all $q_j > 0$ then we have the inequality*

$$\min_{j\in\{1,...,n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, J) \le J_n(\mathbf{p}; \mathbf{A}, f, J) \le \max_{j\in\{1,...,n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, J) \tag{2.22}$$

in the operator order.

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j = \frac{1}{n}, j \in \{1, ..., n\}$, then we have the inequalities

$$n \min_{j\in\{1,...,n\}} \{p_j\} J_n(\mathbf{A}, f, J) \le J_n(\mathbf{p}; \mathbf{A}, f, J) \le n \max_{j\in\{1,...,n\}} \{p_j\} J_n(\mathbf{A}, f, J) \tag{2.23}$$

where

$$J_n(\mathbf{A}, f, J) := \frac{1}{n} \sum_{j=1}^{n} f(A_j) - f\left(\frac{1}{n} \sum_{j=1}^{n} A_j \right). \tag{2.24}$$

For $n = 2$ and by choosing $p_1 = \alpha, p_2 = 1 - \alpha$ with $\alpha \in [0, 1]$, we get from (2.23) the inequality

$$\begin{aligned} &2 \min\{\alpha, 1-\alpha\} \left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2} \right) \right] \\ &\le (1-\alpha) f(A) + \alpha f(B) - f((1-\alpha) A + \alpha B) \\ &\le 2 \max\{\alpha, 1-\alpha\} \left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2} \right) \right], \end{aligned} \tag{2.25}$$

in the operator order, where $f : J \to \mathbb{R}$ is continuous convex and A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$.

We have some refinements of the generalized triangle inequality as follows.

Remark 4. *Assume that the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative. If $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and $q_j > 0$ for $j \in \{1, ..., n\}$, then for $p \ge 1$ we have*

$$\begin{aligned} &\min_{j\in\{1,...,n\}} \left\{ \frac{p_j}{q_j} \right\} \left(\sum_{j=1}^{n} q_j |A_j|^p - Q_n^{1-p} \left| \sum_{j=1}^{n} q_j A_j \right|^p \right) \\ &\le \sum_{j=1}^{n} p_j |A_j|^p - P_n^{1-p} \left| \sum_{j=1}^{n} p_j A_j \right|^p \\ &\le \max_{j\in\{1,...,n\}} \left\{ \frac{p_j}{q_j} \right\} \left(\sum_{j=1}^{n} q_j |A_j|^p - Q_n^{1-p} \left| \sum_{j=1}^{n} q_j A_j \right|^p \right) \end{aligned} \tag{2.26}$$

in the operator order. For $p=1$ we get

$$\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left(\sum_{j=1}^{n}q_j\left|A_j\right|-\left|\sum_{j=1}^{n}q_jA_j\right|\right)\tag{2.27}$$
$$\leq\sum_{j=1}^{n}p_j\left|A_j\right|-\left|\sum_{j=1}^{n}p_jA_j\right|$$
$$\leq\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left(\sum_{j=1}^{n}q_j\left|A_j\right|-\left|\sum_{j=1}^{n}q_jA_j\right|\right).$$

We also have the bounds in terms of unweighted sums:

$$\min_{j\in\{1,\dots,n\}}\{p_j\}\left(\sum_{j=1}^{n}\left|A_j\right|-\left|\sum_{j=1}^{n}A_j\right|\right)\tag{2.28}$$
$$\leq\sum_{j=1}^{n}p_j\left|A_j\right|-\left|\sum_{j=1}^{n}p_jA_j\right|$$
$$\leq\max_{j\in\{1,\dots,n\}}\{p_j\}\left(\sum_{j=1}^{n}\left|A_j\right|-\left|\sum_{j=1}^{n}A_j\right|\right).$$

The case for two operators is as follows:

$$2\min\{\alpha,1-\alpha\}\left[\frac{|A|^p+|B|^p}{2}-\left|\frac{A+B}{2}\right|^p\right]\tag{2.29}$$
$$\leq(1-\alpha)|A|^p+\alpha|B|^p-|(1-\alpha)A+\alpha B|^p$$
$$\leq2\max\{\alpha,1-\alpha\}\left[\frac{|A|^p+|B|^p}{2}-\left|\frac{A+B}{2}\right|^p\right],$$

where A and B are commutative selfadjoint operators.

We have some exponential inequalities as follows:

Remark 5. *Assume that the n-tuple of selfadjoint operators $(A_1,\dots,A_n)$ is mutually commutative. If $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and $q_j>0$ for $j\in\{1,\dots,n\}$, then*

$$\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left[\sum_{j=1}^{n}q_j\exp(A_j)-Q_n\exp\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)\right]\tag{2.30}$$
$$\leq\sum_{j=1}^{n}p_j\exp(A_j)-P_n\exp\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)$$
$$\leq\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}\left[\sum_{j=1}^{n}q_j\exp(A_j)-Q_n\exp\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)\right].$$

In particular, we have

$$\begin{aligned}(2.31)\qquad & \min_{j\in\{1,...,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\exp(A_j)-\exp\left(\frac{1}{n}\sum_{j=1}^{n}A_j\right)\right]\\ &\leq\sum_{j=1}^{n}p_j\exp(A_j)-P_n\exp\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)\\ &\leq\max_{j\in\{1,...,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\exp(A_j)-\exp\left(\frac{1}{n}\sum_{j=1}^{n}A_j\right)\right].\end{aligned}$$

The case of two operators is as follows:

$$\begin{aligned}(2.32)\qquad & 2\min\{\alpha,1-\alpha\}\left[\frac{\exp(A)+\exp(B)}{2}-\exp\left(\frac{A+B}{2}\right)\right]\\ &\leq(1-\alpha)\exp(A)+\alpha\exp(B)-\exp((1-\alpha)A+\alpha B)\\ &\leq 2\max\{\alpha,1-\alpha\}\left[\frac{\exp(A)+\exp(B)}{2}-\exp\left(\frac{A+B}{2}\right)\right],\end{aligned}$$

where A and B are commutative positive definite operators.

3. A Reverse Inequality

The following result also holds:

Theorem 3. *If the function $f:[m,M]\to\mathbb{R}$ is continuous convex and if the n-tuple of selfadjoint operators $(A_1,...,A_n)$ is mutually commutative and with $Sp(A_j)\subseteq[m,M]$ for any $j\in\{1,...,n\}$, then for any $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n:=\sum_{j=1}^{n}p_j>0$ we have*

$$\begin{aligned}(3.1)\qquad 0 &\leq\frac{1}{P_n}\sum_{j=1}^{n}p_jf(A_j)-f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)\\ &\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ &\times\left(\frac{1}{2}(M-m)1_H+\left|\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-\frac{m+M}{2}1_H\right|\right)\\ &\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H\end{aligned}$$

in the operator order.

Proof. Since the function $f:[m,M]\to\mathbb{R}$ is continuous convex, then we have the inequality

$$\begin{aligned}f(t) &= f\left(\frac{(M-t)m+(t-m)M}{M-m}\right)\\ &\leq\frac{(M-t)f(m)+(t-m)f(M)}{M-m}\end{aligned}$$

for any $t\in[m,M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator A with spectrum $Sp(A) \subseteq [m, M]$, we have in the operator order

$$f(A_j) \leq \frac{f(m)(M1_H - A_j) + f(M)(A_j - m1_H)}{M - m} \tag{3.2}$$

for any $j \in \{1, ..., n\}$.

If we multiply the inequality (3.2) by p_j and sum over j from 1 to n we get

$$\begin{aligned} &\frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) \\ &\leq \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M - m} \end{aligned} \tag{3.3}$$

in the operator order.

Therefore we have

$$\begin{aligned} 0 &\leq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \\ &\leq \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M - m} \\ &\quad - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \end{aligned} \tag{3.4}$$

in the operator order, which is a reverse of Jensen's inequality that is of interest in itself.

Now, from the scalar version of (2.25) we have

$$\begin{aligned} 0 &\leq (1-t) f(m) + t f(M) - f((1-t)m + tM) \\ &\leq 2\max\{t, 1-t\}\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &= 2\left(\frac{1}{2} + \left|t - \frac{1}{2}\right|\right)\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \end{aligned} \tag{3.5}$$

for any $t \in [m, M]$, where $f : [m, M] \to \mathbb{R}$ is a continuous convex function on $[m, M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator T with $0 \leq T \leq 1_H$ we have from (3.5) that

$$\begin{aligned} 0 &\leq f(m)(1_H - T) + f(M)T - f((1_H - T)m + TM) \\ &\leq 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2} + \left|T - \frac{1}{2}1_H\right|\right) \end{aligned} \tag{3.6}$$

in the operator order.

Writing the inequality (3.6) for the operator

$$0 \leq T = \frac{\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H}{M - m} \leq 1_H$$

we have

$$\begin{aligned}(3.7)\qquad & \frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m}\\ & -f\left[\frac{m\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+M\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m}\right]\\ & =\frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m}\\ & -f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)\\ & \le\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\ & \times\left(\frac{1}{2}(M-m)+\left|\frac{1}{P_n}\sum_{j=1}^n p_jA_j-\frac{m+M}{2}\right|\right)\end{aligned}$$

in the operator order.

Utilising (3.4) and (3.7) we deduce the first inequality in (3.1).

The last part is obvious. □

Remark 6. *Assume that the n-tuple of selfadjoint operators $(A_1,...,A_n)$ is mutually commutative and with $Sp(A_j)\subseteq[m,M]$ for any $j\in\{1,...,n\}$. Then for any $p\ge 1$ we have the inequality*

$$\begin{aligned}(3.8)\qquad 0 & \le\frac{1}{P_n}\sum_{j=1}^n p_j|A_j|^p-\left|\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right|^p\\ & \le\frac{2}{M-m}\left[\frac{|m|^p+|M|^p}{2}-\left|\frac{m+M}{2}\right|^p\right]\\ & \times\left(\frac{1}{2}(M-m)1_H+\left|\frac{1}{P_n}\sum_{j=1}^n p_jA_j-\frac{m+M}{2}1_H\right|\right)\\ & \le\frac{2}{M-m}\left[\frac{|m|^p+|M|^p}{2}-\left|\frac{m+M}{2}\right|^p\right]1_H.\end{aligned}$$

We also have the exponential inequality

$$
\begin{aligned}
0 &\leq \frac{1}{P_n}\sum_{j=1}^{n} p_j \exp(A_j) - \exp\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \\
&\leq \frac{2}{M-m}\left[\frac{\exp(m)+\exp(M)}{2} - \exp\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}1_H\right|\right) \\
&\leq \frac{2}{M-m}\left[\frac{\exp(m)+\exp(M)}{2} - \exp\left(\frac{m+M}{2}\right)\right]1_H.
\end{aligned} \tag{3.9}
$$

4. A Refinement of Jensen Inequality

The following result provides an additive refinement of Jensen inequality (2.4).

Theorem 4. *If the function $f : J \to \mathbb{R}$ is continuous convex and if the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative and with $Sp(A_j) \subseteq J$ for any $j \in \{1, ..., n\}$, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_k := \sum_{j=1}^{k} p_j > 0$, $\overline{P}_k := P_n - P_k > 0$, with $k \in \{1, ..., n-1\}$ we have*

$$
\begin{aligned}
0 &\leq \max_{k\in\{1,...,n-1\}}\left\{\left(1 - \frac{\left|P_k - \overline{P}_k\right|}{P_n}\right)\right. \\
&\times\left[\frac{f\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j\right) + f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j\right)}{2}\right. \\
&\left.\left.-f\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j + \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j}{2}\right)\right]\right\} \\
&\leq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)
\end{aligned} \tag{4.1}
$$

in the operator order.

Proof. Since the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative, then the operators

$$
A = \frac{1}{P_k}\sum_{j=1}^{k} p_j A_j \text{ and } B = \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j
$$

are commutative and $Sp(A), Sp(B) \subseteq J$ for $k \in \{1, ..., n-1\}$.

Applying the first inequality in (2.25) for A and B as above and $\alpha = \frac{\overline{P}_k}{P_n}$, for $k \in \{1, ..., n-1\}$, we have

$$
\begin{aligned}
(4.2)\qquad & 2\min\left\{\frac{\overline{P}_k}{P_n}, \frac{P_k}{P_n}\right\} \\
& \times\left[\frac{f\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j\right) + f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j\right)}{2}\right. \\
& \left. - f\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j + \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j}{2}\right)\right] \\
& \le \frac{P_k}{P_n} f\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j\right) + \frac{\overline{P}_k}{P_n} f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j\right) \\
& - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right),
\end{aligned}
$$

in the operator order.

By Jensen's inequality (2.4) we have

$$
f\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j A\right) \le \frac{1}{P_k}\sum_{j=1}^{k} p_j f(A_j)
$$

and

$$
f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j\right) \le \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j f(A_j)
$$

which imply that

$$
\begin{aligned}
(4.3)\qquad & \frac{P_k}{P_n} f\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j\right) + \frac{\overline{P}_k}{P_n} f\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j\right) \\
& \le \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j)
\end{aligned}
$$

in the operator order.

Since

$$
\min\left\{\frac{\overline{P}_k}{P_n}, \frac{P_k}{P_n}\right\} = \frac{1}{2} - \frac{1}{2P_n}\left|P_k - \overline{P}_k\right|
$$

then we get from (4.2) and (4.3) the desired result (4.1). □

Remark 7. *If the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ is mutually commutative, then for any $p_j \ge 0$ with $j \in \{1, ..., n\}$ and $P_k := \sum_{j=1}^{k} p_j > 0$,*

$\overline{P}_k := P_n - P_k > 0$, *with* $k \in \{1, ..., n-1\}$ *we have*

$$0 \leq \max_{k\in\{1,\ldots,n-1\}} \left\{ \left(1 - \frac{\left| P_k - \overline{P}_k \right|}{P_n} \right) \times \left[\frac{\frac{1}{P_k^p} \left| \sum_{j=1}^k p_j A_j \right|^p + \frac{1}{\overline{P^p}_k} \left| \sum_{j=k+1}^n p_j A_j \right|^p}{2} - \left| \frac{\frac{1}{P_k} \sum_{j=1}^k p_j A_j + \frac{1}{\overline{P}_k} \sum_{j=k+1}^n p_j A_j}{2} \right|^p \right] \right\} \leq \frac{1}{P_n} \sum_{j=1}^n p_j \left| A_j \right|^p - \frac{1}{P_n^p} \left| \sum_{j=1}^n p_j A_j \right|^p \tag{4.4}$$

for any $p \geq 1$.

For $p = 1$ *we get*

$$0 \leq \frac{1}{2} \max_{k\in\{1,\ldots,n-1\}} \left\{ \left(P_n - \left| P_k - \overline{P}_k \right| \right) \times \left[\frac{1}{P_k} \left| \sum_{j=1}^k p_j A_j \right| + \frac{1}{\overline{P}_k} \left| \sum_{j=k+1}^n p_j A_j \right| - \left| \frac{1}{P_k} \sum_{j=1}^k p_j A_j + \frac{1}{\overline{P}_k} \sum_{j=k+1}^n p_j A_j \right| \right] \right\} \leq \sum_{j=1}^n p_j \left| A_j \right| - \left| \sum_{j=1}^n p_j A_j \right|. \tag{4.5}$$

The unweighted case is as follows:

$$0 \leq \frac{1}{2} \max_{k\in\{1,\ldots,n-1\}} \left\{ (n - |2k-n|) \left[\frac{1}{k} \left| \sum_{j=1}^k A_j \right| + \frac{1}{n-k} \left| \sum_{j=k+1}^n A_j \right| - \left| \frac{1}{k} \sum_{j=1}^k A_j + \frac{1}{n-k} \sum_{j=k+1}^n A_j \right| \right] \right\} \leq \sum_{j=1}^n \left| A_j \right| - \left| \sum_{j=1}^n A_j \right|. \tag{4.6}$$

We also have the exponential inequality

$$0 \le \max_{k\in\{1,\dots,n-1\}} \left\{ \left(1 - \frac{\left|P_k - \overline{P}_k\right|}{P_n}\right) \times \left[\frac{\exp\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j\right) + \exp\left(\frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j\right)}{2} - \exp\left(\frac{\frac{1}{P_k}\sum_{j=1}^{k} p_j A_j + \frac{1}{\overline{P}_k}\sum_{j=k+1}^{n} p_j A_j}{2}\right)\right]\right\} \le \frac{1}{P_n}\sum_{j=1}^{n} p_j \exp(A_j) - \exp\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right). \tag{4.7}$$

References

[1] R. P. Agarwal and S. S. Dragomir, A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces. *Comput. Math. Appl.* 5**9** (2010), no. 12, 3785–3812.

[2] J. Barić, A. Matković and J. Pečarić, A variant of the Jensen-Mercer operator inequality for superquadratic functions. *Math. Comput. Modelling* **51** (2010), no. 9-10, 1230–1239.

[3] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces. *J. Inequal. Appl.* **2010**, Art. ID 496821, 15 pp.

[4] S. S. Dragomir, Some Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces. *Bull. Malays. Math. Sci. Soc.* (2) **34** (2011), no. 3, 445–454.

[5] S. S. Dragomir, New Jensen's type inequalities for differentiable log-convex functions of selfadjoint operators in Hilbert spaces. *Sarajevo J. Math.* **7**(19) (2011), no. 1, 67–80.

[6] S.S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. . [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp].

[7] S.S. Dragomir and M. Uchiyama, Some inequalities of Čebyšev type for functions of operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **15**(2012), to appear.

[8] S.S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[10] T. Hayashi, A note on the Jensen inequality for self-adjoint operators. *J. Math. Soc. Japan* **62** (2010), no. 3, 949–961.

[11] S. Ivelić, A. Matković and J. E. Pečarić, On a Jensen-Mercer operator inequality. *Banach J. Math. Anal.* **5** (2011), no. 1, 19–28.

[12] M. Khosravi, J. S. Aujla, S. S. Dragomir and M. S. Moslehian, Refinements of Choi-Davis-Jensen's inequality. *Bull. Math. Anal. Appl.* **3** (2011), no. 2, 127–133.

[13] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), no. 2-3, 551–564.

[14] J. Mićić, Z. Pavić and J. Pečarić, Jensen type inequalities on quasi-arithmetic operator means. *Sci. Math. Jpn.* **73** (2011), no. 2-3, 183–192.

[15] J. Mićić, Z. Pavić and J. Pečarić, Extension of Jensen's inequality for operators without operator convexity. *Abstr. Appl. Anal.* **2011**, Art. ID 358981, 14 pp.

[16] J. Mićić, J. Pečarić and Y. Seo, Converses of Jensen's operator inequality. *Oper. Matrices* **4** (2010), no. 3, 385–403.

[17] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19**(1993), 405-420.

[18] F. Riesz and B. Sz-Nagy, *Functional Analysis*, New York, Dover Publications, 1990.

[19] R. Sharma, On Jensen's inequality for positive linear functionals. *Int. J. Math. Sci. Eng. Appl.* **5** (2011), no. 5, 263–271.

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa