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HERMITE-HADAMARD TYPE INEQUALITIES

LOREDANA CIURDARIU

ABSTRACT. Two inequalities will be presented using Barnes-Gudunova-Levin inequality. Then a similar lemma will be presented for partial differentiable mapping and an related to the right side of Hermite-Hadamard type inequality for co-ordinated convex functions in two variables are obtained.

1. INTRODUCTION

We recall the well-known Holder′s integral inequality which can be stated as follows, see [18], [9] and then Theorem 2.1, see [9].

Theorem 1. *If* $f(x) \geq 0$, $g(x) \geq 0$ *and* $f(x) \in L^p[a,b]$, $g(x) \in L^q[a,b]$ *and* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{1}$$

Theorem 2. *If the conditions of Theorem 1 are satisfied and* $t > 0$ *then*

$$\int_a^b f(x)g(x)dx \leq C(p,t) \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{2}$$

where $C(p,t) = \frac{1}{p}t^{\frac{1}{p}-1} + (1-\frac{1}{p})t^{\frac{1}{p}}$.

We also need to recall the definition of the (α, m) -convex functions, see for example [22].

Definition 1. *The function* $f : [0,b] \to \mathbb{R}$, $b > 0$ *is said to be* (α, m) *-convex, where* $(\alpha, m) \in [0,1]^2$ *if we have*

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0,b]$ *and* $t \in [0,1]$.

We will use below in the proof of some theorem the following inequality, called Barnes-Gudunova-Levin inequality, see [29] and [24], [25], [26].

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Let f, g be nonnegative concave functions on $[a,b]$. Then, for $p, q > 1$ we have

$$\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)^q dx\right)^{\frac{1}{q}} \leq B(p,q) \int_a^b f(x)g(x)dx,$$

where

$$B(p,q) = \frac{6(b-a)^{(\frac{1}{p})+(\frac{1}{q})-1}}{(p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}.$$

In the special case $q = p$ we have

$$\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)^p dx\right)^{\frac{1}{p}} \leq B(p,p) \int_a^b f(x)g(x)dx,$$

with

$$B(p,p) = \frac{6(b-a)^{(\frac{2}{p})-1}}{(p+1)^{\frac{2}{p}}}.$$

We also need the following two results from [29] and [19]:

Remark 1. *(Remark 1.1 [29]) Observe that whenever, f^p is concave on $[a,b]$ the nonnegative function f is also concave on $[a,b]$, where $p > 1$.*

For $q > 1$ similarly if g^q is concave on $[a,b]$, the nonnegative function g is concave on $[a,b]$.

Lemma 1. *(Lemma 1 [19]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a,\ b \in I$ with $a < b$ and $f'' \in L[a,b]$. Then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)dt.$$

Lemma 2. *(Lemma 2 [21]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a,\ b \in I$ with $a < b$ and $r \in \mathbb{R}^+$. Then the following equality holds:*

$$\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx =$$
$$= (b-a)^2 \int_0^1 k(t)f''(tb+(1-t)a)dt$$

where

$$k(t) = \begin{cases} \frac{t}{r}(\frac{1}{r+1} - t),\ t \in [0, \frac{1}{2}) \\ (1-t)(\frac{t}{r} - \frac{1}{r+1}),\ t \in [\frac{1}{2}, 1]. \end{cases} \tag{1.1}$$

Lemma 3. *(Lemma 2.1 [5]) Suppose that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on I°, the interior of I. Assume that $a,\ b \in I^\circ$ with $a < b$ and f'' is integrable on $[a,b]$. Then the following equality holds,*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx =$$
$$= \frac{(b-a)^2}{16}\int_0^1 (1-t^2)(f''(\frac{1+t}{2}a + \frac{1-t}{2}b)) + (f''(\frac{1-t}{2}a + \frac{1+t}{2}b))dt$$

Let $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ be a bidimensional interval. We recall, see [13] that a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w),$$

holds for all $(x, y),\ (z, w) \in \Delta$ and $\lambda \in [0, 1]$. Now we recal below also the definition of co-ordinated convex functions, see [13].

Definition 2. *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$f(tx+(1-t)y, su+(1-s)w) \leq tsf(x, u)+t(1-s)f(x, w)+s(1-t)f(y, u)+(1-t)(1-s)f(y, w),$$

holds for all $t, s \in [0, 1]$ and $(x, u), (y, w) \in \Delta$.

Also we need to recall the following the definition of quasi-convex functions on the co-ordinates, see [14] and we will use in our proof the second, which is the formal definition.

Definition 3. *A function $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is said to be quasi-convex on Δ if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \max\{f(x, y), f(z, w)\},$$

holds for all $\lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

Definition 4. *A function $f : \Delta \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if*

$$f(tx + (1-t)z, sy + (1-s)w) \leq \max\{f(x, y), f(x, w), f(z, y), f(z, w)\}$$

for all $(x, y), (z, w) \in \Delta$ $t, s \in [0, 1]$.

A formal definition for co-ordinates s- convex functions in the second sense, which can be found in [15], is given below:

Definition 5. *A mapping $f : \Delta \to \mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ if the inequality*

$$f(tx + (1-t)y, ru + (1-r)w) \leq$$

$$\leq t^s r^s f(x, u) + t^s(1-r)^s f(x, w) + r^s(1-t)^s f(y, u) + (1-t)^s(1-r)^s f(y, w),$$

holds for all $t, r \in [0, 1],\ (x, u), (y, w) \in \Delta$ and for some fixed $s \in (0, 1]$.

2. Some Hermite-Hadamard's type inequalities for (α, m)- convex functions

We give an analog of Lemma 1 from [30] for the second derivative of f. Then using this lemma and Theorem 1 we give some variant of right hand left Hermite-Hadamard inequality for functions whose powers of second derivative in absolute value are (α, m)- convex.

Lemma 4. *Let $f : I^\circ \to \mathbb{R}$, $I^\circ \subset [0,\infty)$ be a twice differentiable function on I° where a, $b \in I^\circ$, $a < b$. If $f^{''} \in L[a,b]$. Then we have:*

$$\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2}) = \frac{(a-b)^2}{2}[\int_0^{\frac{1}{2}} t^2 f^{''}(ta+(1-t)b)dt+$$

$$+\int_{\frac{1}{2}}^1 (t-1)^2 f^{''}(ta+(1-t)b)dt].$$

Proof. As in the proof of Lemma 1, [30] we note that

$$\int_0^{\frac{1}{2}} t^2 f^{''}(ta+(1-t)b)dt = \frac{f^{'}(\frac{a+b}{2})}{4(a-b)} - \frac{1}{(a-b)^2} f(\frac{a+b}{2}) + \frac{2}{(a-b)^2}\int_0^{\frac{1}{2}} f(ta+(1-t)b)dt$$

and

$$\int_{\frac{1}{2}}^1 (t-1)^2 f^{''}(ta+(1-t)b)dt = -\frac{f^{'}(\frac{a+b}{2})}{4(a-b)} - \frac{1}{(a-b)^2} f(\frac{a+b}{2}) + \frac{2}{(a-b)^2}\int_{\frac{1}{2}}^1 f(ta+(1-t)b)dt$$

using the integration by parts.

Using the last two equalities and the substitution $x = ta + (1-t)b$ we obtain:

$$\frac{2}{(a-b)^3}\int_a^b f(x)dx - \frac{2}{(a-b)^2} f(\frac{a+b}{2}) = \int_0^{\frac{1}{2}} t^2 f^{''}(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)^2 f^{''}(ta+(1-t)b)dt$$

i.e. the desired inequality.

∎

Theorem 3. *Let $f : I \subset [0,b^*] \to \mathbb{R}$ be a twice differentiable function on I° such that $f^{''} \in L[a,b]$ where $a, b \in I^\circ$ with $a < b$, $b^* > 0$. If $|f^{''}|$ is (α,m) - convex on $[a,b]$ for $(\alpha,m) \in (0,1)\times(0,1)$ then the following inequality holds:*

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(a-b)^2}{2}[m|f^{''}(\frac{a}{m})|(\frac{1}{\alpha+3} - \frac{2}{\alpha+2} + \frac{1}{\alpha+1} -$$

$$-\frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}}) + |f^{''}(b)|(\frac{1}{12} - \frac{1}{\alpha+3} + \frac{2}{\alpha+2} - \frac{1}{\alpha+1} + \frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}})].$$

Proof. Using now Lemma 4 and then Definition 1 we will obtain:

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(a-b)^2}{2}[\int_0^{\frac{1}{2}} t^2|f^{''}(ta+(1-t)b)|dt+$$

$$+\int_{\frac{1}{2}}^1 (t-1)^2|f^{''}(ta+(1-t)b)|dt] \leq \frac{(a-b)^2}{2}[\int_0^{\frac{1}{2}} t^2(mt^\alpha|f^{''}(\frac{a}{m})| + (1-t^\alpha)|f^{''}(b)|)dt+$$

$$+\int_{\frac{1}{2}}^1 (t-1)^2(mt^\alpha|f^{''}(\frac{a}{m})| + (1-t^\alpha)|f^{''}(b)|)dt].$$

By calculus we obtain

$$|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})| \leq \frac{(a-b)^2}{2}[m|f^{''}(\frac{a}{m})|(\frac{1}{\alpha+3} - \frac{2}{\alpha+2} + \frac{1}{\alpha+1} -$$

$$-\frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}}) + |f^{''}(b)|(\frac{1}{12} - \frac{1}{\alpha+3} + \frac{2}{\alpha+2} - \frac{1}{\alpha+1} + \frac{1}{(\alpha+1)(\alpha+2)2^{\alpha+1}})].$$

∎

Next we will find a lower bound for the expression from the Hermite-Hadamard inequality for functions which satisfy the conditions from the Barnes-Gudunova-Levin inequality.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a,\ b \in I^\circ$ with $a < b$ and $f^{''} \in L[a,b]$. If $f^{''}$ is nonnegative on $[a,b]$ and $(f^{''})^q$ is concave on $[a,b]$ for some fixed $m \in (0,1]$, $\alpha \in (0,1)$ and $q > 1$ then the following inequality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{12}(q+1)^{\frac{1}{q}}\frac{1}{2^{\frac{1}{q}}}\frac{(\Gamma(p+1))^{\frac{1}{p}}}{((2p+1)...(p+2))^{\frac{1}{p}}}((f^{''}(a))^q+(f^{''}(b))^q)^{\frac{1}{q}}.$$

where $\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}dx,\ t > 0$ is the function Γ of Euler, $\frac{1}{p}+\frac{1}{q} = 1$, and $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ is the function β of Euler.

Proof. Using Lemma 1, see [19] and the Barnes-Gudunova-Levin inequality for $f^{''}$ and the function $t(1-t),\ t \in [0,1]$ we obtain

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f^{''}(ta+(1-t)b)dt \geq$$

$$\geq \frac{1}{B(p,q)}\left(\int_0^1 (t(1-t))^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (f^{''}(ta+(1-t)b))^q dt\right)^{\frac{1}{q}} \geq$$

$$\geq \frac{1}{B(p,q)}(\beta(p+1,p+1))^{\frac{1}{p}}\left(\int_0^1 (t(f^{''}(a))^q+(1-t)(f^{''}(b))^q)dt\right)^{\frac{1}{q}} =$$

$$= \frac{1}{B(p,q)}(\beta(p+1,p+1))^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left((f^{''}(a))^q+(f^{''}(b))^q\right)^{\frac{1}{q}} =$$

$$= \left(\frac{1}{2}\right)^{\frac{1}{q}}\frac{1}{B(p,q)}\left(\frac{(\Gamma(p+1))^2}{\Gamma(2p+2)}\right)^{\frac{1}{p}}\left((f^{''}(a))^q+(f^{''}(b))^q\right)^{\frac{1}{q}}.$$

∎

Remark 2. *(i) Under the above conditions, using the power-mean inequality, the inequality can be also written as:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{12}(q+1)^{\frac{1}{q}}\frac{1}{2^{\frac{1}{q}}}\frac{(\Gamma(p+1))^{\frac{1}{p}}}{((2p+1)...(p+2))^{\frac{1}{p}}}(f^{''}(a)+f^{''}(b)).$$

(ii) If we consider in previous theorem $p = q > 1$ not that $\frac{1}{p}+\frac{1}{q} = 1$ then by Barnes-Gudunova-Levin inequality we have:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^{3-(\frac{2}{p})}}{12}\frac{1}{2^{\frac{1}{q}}}\frac{(\Gamma(p+1))^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}{(2p+1)^{\frac{1}{p}}...(p+2)^{\frac{1}{p}}}\left((f^{''}(a))^p+(f^{''}(b))^p\right)^{\frac{1}{p}}.$$

In the following theorem we will find a lower bound for the left member of the equality from Lemma 2, see [21] when $r \in (0,1)$ and f has some properties.

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f^{''} \in L_1[a,b]$, where $a,\ b \in I$ with $a < b$ and $r \in (0,1)$. If $f^{''}$ is nonnegative on $[a,b]$ and $(f^{''})^q$ -is concave on $[a,b]$ then the following inequality holds:*

$$\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{2r(r+1)}[\frac{1}{B(p_1,q)}(\frac{\Gamma(p_1+1)}{2(2p_1+1)...(p_1+1)})^{\frac{1}{p_1}}(\frac{1}{8}(f^{''}(b))^q + \frac{3}{8}(f^{''}(a))^q)^{\frac{1}{q}} +$$

$$+\frac{1}{B(p_2,q_2)}(\frac{\Gamma(p_2+1)}{2(2p_2+1)...(p_2+1)})^{\frac{1}{p_2}}(\frac{3}{8}(f^{''}(b))^q + \frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}].$$

where $p_1,\ q,\ p_2 > 1$ and $B(p,q)$ is given in Barnes-Gudunova-Levin inequality.

Proof. From Lemma 2 we can notice that

$$\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx =$$

$$= (b-a)^2[\int_0^{\frac{1}{2}} \frac{t}{r}(\frac{1}{r+1} - t)dt + \int_{\frac{1}{2}}^1 (1-t)(\frac{t}{r} - \frac{1}{r+1})dt].$$

Taking into account that for $r \in (0,1)$, $k(t) > 0$ and that $\frac{t}{r}(\frac{1}{r+1} - t)$ and $(1-t)(\frac{t}{r} - \frac{1}{r+1})$ are concave because its second derivative is negative we apply the Barnes-Gudunova-Levin inequality obtaining:

$$\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx \geq$$

$$\geq (b-a)^2[\frac{1}{B(p_1,q)}\left(\int_0^{\frac{1}{2}}(\frac{t}{r}(\frac{1}{r+1} - t))^{p_1}dt\right)^{\frac{1}{p_1}}\left(\int_0^{\frac{1}{2}}(f^{''}(tb+(1-t)a))^q dt\right)^{\frac{1}{q}} +$$

$$+\frac{1}{B(p_2,q)}\left(\int_{\frac{1}{2}}^1((1-t)(\frac{t}{r} - \frac{1}{r+1}))^{p_2}dt\right)^{\frac{1}{p_2}}\left(\int_{\frac{1}{2}}^1(f^{''}(tb+(1-t)a))^q dt\right)^{\frac{1}{q}}] \geq$$

$$\geq \frac{(b-a)^2}{r(r+1)}[\frac{1}{B(p_1,q)}(\int_0^{\frac{1}{2}} t^{p_1}(1-t(r+1))^{p_1}dt)^{\frac{1}{p_1}}(\int_0^{\frac{1}{2}}(t(f^{''}(b))^q + (1-t)(f^{''}(a))^q)dt)^{\frac{1}{q}} +$$

$$+\frac{1}{B(p_2,q)}(\int_{\frac{1}{2}}^1(1-t)^{p_2}(t(r+1)-r)^{p_2}dt)^{\frac{1}{p_2}}(\int_{\frac{1}{2}}^1(t(f^{''}(b))^q + (1-t)(f^{''}(a))^q)dt)^{\frac{1}{q}}] =$$

$$= \frac{(b-a)^2}{r(r+1)}[\frac{1}{B(p_1,q)}(\frac{1}{8}(f^{''}(b))^q + \frac{3}{8}(f^{''}(a))^q)^{\frac{1}{q}}(\int_0^{\frac{1}{2}} t^{p_1}(1-t(r+1))^{p_1}dt)^{\frac{1}{p_1}} +$$

$$+\frac{1}{B(p_2,q)}(\frac{3}{8}(f^{''}(b))^q + \frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}(\int_{\frac{1}{2}}^1(1-t)^{p_2}(t(r+1)-r)^{p_2}dt)^{\frac{1}{p_2}}].$$

If we denote $\int_0^{\frac{1}{2}} t^p(1-t(r+1))^p dt$ by $I_1(p)$ and $\int_{\frac{1}{2}}^1(1-t)^p(t(r+1)-r)^p dt$ by $I_2(p)$ then using the substitution $u = 1-t$ we have $I_2(p) = \int_0^{\frac{1}{2}} u^p((1-u)(r+1)-r)^p du = I_1(p)$.

Therefore we obtain,

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{r(r+1)}[\frac{1}{B(p_1,q)}I_1(p_1)^{\frac{1}{p_1}}(\frac{1}{8}(f^{''}(b))^q+\frac{3}{8}(f^{''}(a))^q)^{\frac{1}{q}}+$$

$$+\frac{1}{B(p_2,q)}I_1(p_2)^{\frac{1}{p_2}}(\frac{3}{8}(f^{''}(b))^q+\frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}].$$

Now we consider

$$I_1(p)=\int_0^{\frac{1}{2}} t^p(1-t(r+1))^p dt,\ r\in(0,1),\ p>1.$$

Then it is easy to see that $t^p(1-t(r+1))^p > t^p(1-2t)^p \geq 0$ and

$$\int_0^{\frac{1}{2}} t^p(1-t(r+1))^p dt > \int_0^{\frac{1}{2}} t^p(1-2t)^p dt.$$

If we use substitution $2t=u$ in last integral, we obtain

$$I=\int_0^{\frac{1}{2}} t^p(1-2t)^p dt=\frac{1}{2^{p+1}}\int_0^1 u^p(1-u)^p du=\frac{1}{2^{p+1}}\beta(p+1,p+1).$$

Thus

$$I_1(p)>\frac{1}{2^{p+1}}\beta(p+1,p+1)=\frac{\Gamma(p+1)}{2^{p+1}(2p+1)...(p+1)}$$

and then the inequality becomes:

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{2r(r+1)}[\frac{1}{B(p_1,q)}(\frac{\Gamma(p_1+1)}{2(2p_1+1)...(p_1+1)})^{\frac{1}{p_1}}(\frac{1}{8}(f^{''}(b))^q+\frac{3}{8}(f^{''}(a))^q)^{\frac{1}{q}}+$$

$$+\frac{1}{B(p_2,q)}(\frac{\Gamma(p_2+1)}{2(2p_2+1)...(p_2+1)})^{\frac{1}{p_2}}(\frac{3}{8}(f^{''}(b))^q+\frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}].$$

∎

Remark 3. *(i) Under the above conditions if $p_1=p_2=p$ then the inequality becomes:*

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{r(r+1)B(p,q)}(\frac{\beta(p+1,p+1)}{2^{p+1}})^{\frac{1}{p}}[(\frac{1}{8}(f^{''}(b))^q+\frac{3}{8}(f^{''}(a))^q)^{\frac{1}{q}}+(\frac{3}{8}(f^{''}(b))^q+\frac{1}{8}(f^{''}(a))^q)^{\frac{1}{q}}]$$

(ii) Under the above conditions if $p_1=p_2=q_1=q_2=p$ then the inequality becomes:

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^{3-(2/p)}}{12r(r+1)2^{4/p}}(\frac{\Gamma(p+1)(p+1)}{(2p+1)...(p+2)})^{\frac{1}{p}}[(f^{''}(b))^p+3(f^{''}(a))^p)^{\frac{1}{p}}+(3(f^{''}(b))^p+8(f^{''}(a))^p)^{\frac{1}{p}}].$$

Next theorem will give a lower bound for the left member of the equality from Lemma 3, see[5].

Theorem 6. *Suppose that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on I°, the interior of I. Assume that $a,\ b \in I^{\circ}$ with $a < b$ and $f^{''}$ is integrable on $[a, b]$. If $f^{''}$ is nonnegative on $[a, b]$ and $(f^{''})^q$ is concave, $q > 1,\ p_1 > 1,\ p_2 > 1$ then the following inequality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}\Big[\frac{\big(\frac{\sqrt{\pi}\Gamma(p_1+1)}{\Gamma(p_1+\frac{3}{2})}\big)^{\frac{1}{p_1}}}{B(p_1,q)}\Big(\frac{3}{4}(f^{''}(a))^q + \frac{1}{4}(f^{''}(b))^q\Big)^{\frac{1}{q}} +$$

$$+\frac{\big(\frac{\sqrt{\pi}\Gamma(p_2+1)}{\Gamma(p_2+\frac{3}{2})}\big)^{\frac{1}{p_2}}}{B(p_2,q)}\Big(\frac{1}{4}(f^{''}(a))^q + \frac{3}{4}(f^{''}(b))^q\Big)^{\frac{1}{q}}\Big].$$

Proof. Using Lemma 3 and the Barnes-Gudunova-Levin inequality we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{16}\int_0^1 (1-t^2)\Big(f^{''}\big(\frac{1+t}{2}a + \frac{1-t}{2}b\big)\Big) + \Big(f^{''}\big(\frac{1-t}{2}a + \frac{1+t}{2}b\big)\Big)dt \geq$$

$$\geq \frac{(b-a)^2}{16}\Big[\frac{1}{B(p_1,q)}\Big(\int_0^1 (1-t^2)^{p_1}dt\Big)^{\frac{1}{p_1}}\Big(\int_0^1 \Big(f^{''}\big(\frac{1+t}{2}a + \frac{1-t}{2}b\big)\Big)^q dt\Big)^{\frac{1}{q}} +$$

$$+\frac{1}{B(p_2,q)}\Big(\int_0^1 (1-t^2)^{p_2}dt\Big)^{\frac{1}{p_2}}\Big(\int_0^1 \Big(f^{''}\big(\frac{1-t}{2}a + \frac{1+t}{2}b\big)\Big)^q dt\Big)^{\frac{1}{q}}\Big] \geq$$

$$\geq \frac{(b-a)^2}{16}\Big[\frac{1}{B(p_1,q)}I(p_1)^{\frac{1}{p_1}}\Big(\int_0^1 \Big(\frac{1+t}{2}(f^{''}(a))^q + \frac{1-t}{2}(f^{''}(b))^q\Big)dt\Big)^{\frac{1}{q}} +$$

$$+\frac{1}{B(p_2,q)}I(p_2)^{\frac{1}{p_2}}\Big(\int_0^1 \Big(\frac{1-t}{2}(f^{''}(a))^q + \frac{1+t}{2}(f^{''}(b))^q\Big)dt\Big)^{\frac{1}{q}}\Big],$$

where we denoted $\int_0^1 (1-t^2)^p dt$ by $I(p)$. By calculus we obtain,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}\Big[\frac{I(p_1)^{\frac{1}{p_1}}}{B(p_1,q)}\Big(\frac{3}{4}(f^{''}(a))^q + \frac{1}{4}(f^{''}(b))^q\Big)^{\frac{1}{q}} + \frac{I(p_2)^{\frac{1}{p_2}}}{B(p_2,q)}\Big(\frac{1}{4}(f^{''}(a))^q + \frac{3}{4}(f^{''}(b))^q\Big)^{\frac{1}{q}}\Big].$$

But by substitution $t = \sqrt{1-s}$ we obtain $I(p) = \frac{1}{2}\int_0^1 s^p \frac{1}{\sqrt{1-s}}ds = \frac{1}{2}\beta(p+1, \frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}$, and replacing in previous inequality,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}\Big[\frac{\big(\frac{\sqrt{\pi}\Gamma(p_1+1)}{\Gamma(p_1+\frac{3}{2})}\big)^{\frac{1}{p_1}}}{B(p_1,q)}\Big(\frac{3}{4}(f^{''}(a))^q + \frac{1}{4}(f^{''}(b))^q\Big)^{\frac{1}{q}} +$$

$$+\frac{\big(\frac{\sqrt{\pi}\Gamma(p_2+1)}{\Gamma(p_2+\frac{3}{2})}\big)^{\frac{1}{p_2}}}{B(p_2,q)}\Big(\frac{1}{4}(f^{''}(a))^q + \frac{3}{4}(f^{''}(b))^q\Big)^{\frac{1}{q}}\Big].$$

∎

Remark 4. *(i) Under the above conditions, if* $p_1 = p_2$ *we have:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}\frac{\left(\frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}}{B(p,q)}\left[\left(\frac{3}{4}(f''(a))^q + \frac{1}{4}(f''(b))^q\right)^{\frac{1}{q}} + \left(\frac{1}{4}(f''(a))^q + \frac{3}{4}(f''(b))^q\right)^{\frac{1}{q}}\right].$$

(ii) Under the above conditions, if $p_1 = p_2 = q = p$ *we have:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \geq$$

$$\geq \frac{(b-a)^2}{16}\frac{\left(\frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\right)^{\frac{1}{p}}}{B(p,p)}\left[\left(\frac{3}{4}(f''(a))^p + \frac{1}{4}(f''(b))^p\right)^{\frac{1}{p}} + \left(\frac{1}{4}(f''(a))^p + \frac{3}{4}(f''(b))^p\right)^{\frac{1}{p}}\right].$$

3. Some Hermite-Hadamard's type inequalities for differentiable co-ordinated convex, quasi-convex and s-convex functions

We will give an analog of Lemma 1 see [11] for functions on rectangle from the plane $\mathbb{R}^2$. This result will be used then in the proof of next theorems.

Lemma 5. *Let* $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b]\times[c,d]$ *with* $a < b,\ c < d$. *If* $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$, *then the following identity takes place:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4},\frac{3c+d}{4}) + f(\frac{3a+b}{4},\frac{c+3d}{4}) + f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+f(\frac{a+3b}{4},\frac{c+3d}{4}) - \frac{2}{d-c}\int_c^d (f(\frac{3a+b}{4},y)+f(\frac{a+3b}{4},y))dy - \frac{2}{b-a}\int_a^b (f(x,\frac{3c+d}{4})+$$

$$+f(x,\frac{c+3d}{4}))dx + \frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy] =$$

$$= \sum_{i=0,j=0}^{3}(-1)^{i+j}\int_0^1\int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i, ry_{j+1}+(1-r)y_j)dtdr$$

where

$$s_i : [0,1]\to\mathbb{R},\ s_i(t) = t,\ i \in \{0,2\},$$
$$s_i : [0,1]\to\mathbb{R},\ s_i(t) = 1-t,\ i \in \{1,3\}$$

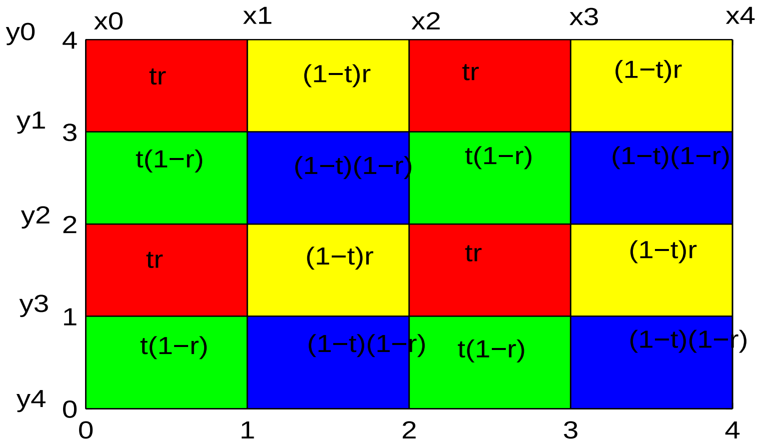
$x_i = \frac{(4-i)a+ib}{4},\ i = \overline{0,4}$ *and* $y_j = \frac{(4-j)c+jd}{4},\ j = \overline{0,4}$.

Proof. We denote

$$m_i(t) = tx_{i+1} + (1-t)x_i,\ i = \overline{0,3},$$
$$n_j(r) = ry_{j+1} + (1-r)y_j,\ j = \overline{0,3},$$

and

$$I_{ij} = \int_0^1\int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i, ry_{j+1}+(1-r)y_j)dtdr.$$



Now by integration by parts, we have

$$I_{ij} = \int_0^1 \int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r \partial t}(m_i(t), n_j(r))dtdr =$$

$$= \int_0^1 s_j(r)\frac{4}{b-a}[s_i(t)\frac{\partial f}{\partial r}(m_i(t), n_j(r))|_0^1 - \int_0^1 (-1)^i \frac{\partial f}{\partial r}(m_i(t), n_j(r))dt]dr =$$

$$= \frac{4}{b-a}\int_0^1 s_j(r)[\frac{1-(-1)^{i+1}}{2}\frac{\partial f}{\partial r}(m_i(1), n_j(r)) - \frac{1-(-1)^i}{2}\frac{\partial f}{\partial r}(m_i(0), n_j(r)) -$$

$$-(-1)^i \int_0^1 \frac{\partial f}{\partial r}(m_i(t), n_j(r))dt]dr = \frac{4}{(b-a)}[\frac{1-(-1)^{i+1}}{2}\int_0^1 s_j(r)\frac{\partial f}{\partial r}(m_i(1), n_j(r))dr -$$

$$-\frac{1-(-1)^i}{2}\int_0^1 s_j(r)\frac{\partial f}{\partial r}(m_i(0), n_j(r))dr - (-1)^i \int_0^1 \int_0^1 s_j(r)\frac{\partial f}{\partial r}(m_i(t), n_j(r))dtdr] =$$

$$= \frac{16}{(b-a)(d-c)}[\frac{1-(-1)^{i+1}}{2}(\frac{1-(-1)^{j+1}}{2}f(m_i(1), n_j(1)) - \frac{1-(-1)^j}{2}f(m_i(1), n_j(0)) -$$

$$-(-1)^j \int_0^1 f(m_i(1), n_j(r))dr) - \frac{1-(-1)^i}{2}(\frac{1-(-1)^{j+1}}{2}f(m_i(0), n_j(1)) -$$

$$-\frac{1-(-1)^j}{2}f(m_i(0), n_j(0)) - (-1)^j \int_0^1 f(m_i(0), n_j(r)dr) +$$

$$+(-1)^{i+1}\int_0^1 (\frac{1-(-1)^{j+1}}{2}f(m_i(t), n_j(1)) - \frac{1-(-1)^j}{2}f(m_i(t), n_j(0)) -$$

$$-\int_0^1 (-1)^j f(m_i(t), n_j(r))dr)dt] = \frac{16}{(b-a)(d-c)}[\frac{(1-(-1)^{i+1})(1-(-1)^{j+1})}{4}\cdot$$

$$\cdot f(m_i(1), n_j(1)) - \frac{(1-(-1)^{i+1})(1-(-1)^j)}{4}f(m_i(1), n_j(0)) - \frac{(1-(-1)^i)(1-(-1)^{j+1})}{4}\cdot$$

$$\cdot f(m_i(0), n_j(1)) + \frac{(1-(-1)^i)(1-(-1)^j)}{4}f(m_i(0), n_j(0)) + (-1)^{j+1}\frac{1-(-1)^{i+1}}{2}\cdot$$

$$\cdot \int_0^1 f(m_i(1), n_j(r))dr + (-1)^j \frac{1-(-1)^i}{2}\int_0^1 f(m_i(0), n_j(r))dr + (-1)^{i+1}\frac{1-(-1)^{j+1}}{2}\cdot$$

$$\cdot \int_0^1 f(m_i(t), n_j(1))dt + (-1)^i \frac{1-(-1)^j}{2}f(m_i(t), n_j(0)) + (-1)^{i+j}\int_0^1 \int_0^1 f(m_i(t), n_j(r))dtdr].$$

If we make use of the substitution $x = tx_{i+1} + (1-t)x_i,\ i = \overline{0,3},\ t \in [0,1]$ (that means $t \in [0,1] \Rightarrow x \in [x_i, x_{i+1}]$ and $dx = \frac{b-a}{4}dt$) $y = ry_{j+1} + (1-r)y_j,\ j = \overline{0,3}$, $r \in [0,1]$ (that means $r \in [0,1] \Rightarrow y \in [y_j, y_{j+1}]$ and $dy = \frac{d-c}{4}$) and take into account that $m_1(0) = m_0(1) = x_1,\ m_3(0) = m_2(1) = x_3$ and $n_0(1) = n_1(0) = y_1,\ n_3(0) = n_2(1) = y_3$ we obtain

$$\sum_{i=0,j=0}^{3}(-1)^{i+j}\int_0^1\int_0^1 s_i(t)s_j(r)\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i, ry_{j+1}+(1-r)y_j)dtdr =$$

$$= \frac{16}{(b-a)(d-c)}[4f(\frac{3a+b}{4},\frac{3c+d}{4}) + 4f(\frac{3a+b}{4},\frac{c+3d}{4}) + 4f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+4f(\frac{a+3b}{4},\frac{c+3d}{4}) + \frac{16}{(b-a)(d-c)}\sum_{i,j=0}^{3}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(x,y)dxdy+$$

$$+\frac{4}{d-c}(\sum_{i=0}^{3}\frac{-1+(-1)^{i+1}}{2}\sum_{j=0}^{3}\int_{y_j}^{y_{j+1}} f(x_{i+1},y)dy+\sum_{i=3}^{3}\frac{-1+(-1)^{i}}{2}\sum_{j=1}^{3}\int_{y_j}^{y_{j+1}} f(x_i,y)dy)+$$

$$+\frac{4}{b-a}(\sum_{j=0}^{3}\frac{-1+(-1)^{j+1}}{2}\sum_{i=0}^{3}\int_{x_i}^{x_{i+1}} f(x,y_{j+1})dx+\sum_{j=3}^{3}\frac{-1+(-1)^{j}}{2}\sum_{i=1}^{3}\int_{x_i}^{x_{i+1}} f(x,y_j)dx)) =$$

$$= \frac{16}{(b-a)(d-c)}[4f(\frac{3a+b}{4},\frac{3c+d}{4}) + 4f(\frac{3a+b}{4},\frac{c+3d}{4}) + 4f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+4f(\frac{a+3b}{4},\frac{c+3d}{4}) + \frac{16}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy+$$

$$+\frac{4}{d-c}(\sum_{i=0}^{3}\frac{-1+(-1)^{i+1}}{2}\int_c^d f(x_{i+1},y)dy + \sum_{i=3}^{3}\frac{-1+(-1)^{i}}{2}\int_c^3 f(x_i,y)dy)+$$

$$+\frac{4}{b-a}(\sum_{j=0}^{3}\frac{-1+(-1)^{j+1}}{2}\int_a^b f(x,y_{j+1})dx + \sum_{j=3}^{3}\frac{-1+(-1)^{j}}{2}\int_a^b f(x,y_j)dx)].$$

∎

Theorem 7. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b]\times[c,d]$ with $a < b,\ c < d$. If $|\frac{\partial^2 f}{\partial r\partial t}|$ is convex on the co-ordinates on Δ then the following inequality holds:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4},\frac{3c+d}{4}) + f(\frac{3a+b}{4},\frac{c+3d}{4}) + f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+f(\frac{a+3b}{4},\frac{c+3d}{4}) - \frac{2}{d-c}\int_c^d (f(\frac{3a+b}{4},y)+f(\frac{a+3b}{4},y))dy - \frac{2}{b-a}\int_a^b (f(x,\frac{3c+d}{4})+$$

$$+f(x,\frac{c+3d}{4}))dx + \frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy] \le$$

$$\le \frac{4}{9}[|\frac{\partial^2 f}{\partial r\partial t}(\frac{3a+b}{4},\frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r\partial t}(\frac{3a+b}{4},\frac{c+3d}{4})| + |\frac{\partial^2 f}{\partial r\partial t}(\frac{a+3b}{4},\frac{3c+d}{4})|+$$

$$+|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+3b}{4},\frac{c+3d}{4})|] + \frac{1}{36}[|\frac{\partial^2 f}{\partial r\partial t}(a,c)| + |\frac{\partial^2 f}{\partial r\partial t}(a,d)| + |\frac{\partial^2 f}{\partial r\partial t}(b,c)| + |\frac{\partial^2 f}{\partial r\partial t}(b,d)|+$$

$$+2|\frac{\partial^2 f}{\partial r\partial t}(a,\frac{c+d}{2})| + 2|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+b}{2},c)| + 2|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+b}{2},d)| + 2|\frac{\partial^2 f}{\partial r\partial t}(b,\frac{c+d}{2})|+$$

$$+4|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+b}{2},\frac{c+d}{2})|]+\frac{1}{9}[(|\frac{\partial^2 f}{\partial r\partial t}(\frac{3a+b}{4},c)|+|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+3b}{4},c)|+|\frac{\partial^2 f}{\partial r\partial t}(\frac{3a+b}{4},d)|+$$
$$+|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+3b}{4},d)|+2|\frac{\partial^2 f}{\partial r\partial t}(\frac{3a+b}{4},\frac{c+d}{2})|+2|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+3b}{4},\frac{c+d}{2})|)+(|\frac{\partial^2 f}{\partial r\partial t}(a,\frac{3c+d}{4})|+$$
$$+|\frac{\partial^2 f}{\partial r\partial t}(a,\frac{c+3d}{4})|+|\frac{\partial^2 f}{\partial r\partial t}(b,\frac{3c+d}{4})|+|\frac{\partial^2 f}{\partial r\partial t}(b,\frac{c+3d}{4})|+2|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+b}{2},\frac{3c+d}{4})|+$$
$$+2|\frac{\partial^2 f}{\partial r\partial t}(\frac{a+b}{2},\frac{c+3d}{4})|)].$$

Proof. Using previous lemma and Definition 2 we have

$$I=\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4},\frac{3c+d}{4})+f(\frac{3a+b}{4},\frac{c+3d}{4})+f(\frac{a+3b}{4},\frac{3c+d}{4})+$$
$$+f(\frac{a+3b}{4},\frac{c+3d}{4})-\frac{2}{d-c}\int_c^d(f(\frac{3a+b}{4},y)+f(\frac{a+3b}{4},y))dy-\frac{2}{b-a}\int_a^b(f(x,\frac{3c+d}{4})+$$
$$+f(x,\frac{c+3d}{4}))dx+\frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy]\leq$$
$$\leq\sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)|\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i,ry_{j+1}+(1-r)y_j)|dtdr\leq$$
$$\leq\sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)[tr|\frac{\partial^2 f}{\partial r\partial t}(x_{i+1},y_{j+1})|+t(1-r)|\frac{\partial^2 f}{\partial r\partial t}(x_{i+1},y_j)|+$$
$$+r(1-t)|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_{j+1})|+(1-t)(1-r)|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_j)|]=$$
$$=\sum_{i,j=0}^{3}[D_{i+1,j+1}\int_0^1\int_0^1 trs_i(t)s_j(r)dtdr+D_{i+1,j}\int_0^1\int_0^1 t(1-r)s_i(t)s_j(r)dtdr+$$
$$+D_{i,j+1}\int_0^1\int_0^1(1-t)rs_i(t)s_j(r)dtdr+D_{i,j}\int_0^1\int_0^1(1-t)(1-r)s_i(t)s_j(r)dtdr]=$$
$$=\sum_{i,j\in\{0,2\}}[D_{i+1,j+1}\int_0^1\int_0^1 t^2r^2dtdr+D_{i+1,j}\int_0^1\int_0^1 t^2(1-r)rdtdr+$$
$$+D_{i,j+1}\int_0^1\int_0^1(1-t)tr^2dtdr+D_{i,j}\int_0^1\int_0^1(1-t)t(1-r)rdtdr]+$$
$$+\sum_{i,j\in\{1,3\}}[D_{i+1,j+1}\int_0^1\int_0^1 tr(1-t)(1-r)dtdr+D_{i+1,j}\int_0^1\int_0^1 t(1-t)(1-r)^2dtdr+$$
$$+D_{i,j+1}\int_0^1\int_0^1(1-t)^2r(1-r)dtdr+D_{i,j}\int_0^1\int_0^1(1-t)^2(1-r)^2dtdr]+$$
$$+\sum_{i\in\{0,2\},j\in\{1,3\}}[D_{i+1,j+1}\int_0^1\int_0^1 t^2r(1-t)dtdr+D_{i+1,j}\int_0^1\int_0^1 t^2(1-r)^2dtdr+$$
$$+D_{i,j+1}\int_0^1\int_0^1 t(1-t)r(1-r)dtdr+D_{i,j}\int_0^1\int_0^1 t(1-t)(1-r)^2dtdr]+$$
$$+\sum_{i\in\{1,3\},j\in\{0,2\}}[D_{i+1,j+1}\int_0^1\int_0^1 t(1-t)r^2dtdr+D_{i+1,j}\int_0^1\int_0^1 t(1-t)r(1-r)dtdr+$$

$$+D_{i,j+1}\int_0^1\int_0^1(1-t)^2r^2dtdr+D_{i,j}\int_0^1\int_0^1(1-t)^2r(1-r)dtdr],$$

where $D_{i,j}=|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_j)|,\ i,j=\overline{0,4}$. By calculus we obtain:

$$I\le\sum_{i,j\in\{0,2\}}[D_{i+1,j+1}\frac{1}{9}+D_{i+1,j}\frac{1}{18}+D_{i,j+1}\frac{1}{18}+D_{i,j}\frac{1}{36}]+$$

$$+\sum_{i,j\in\{1,3\}}[D_{i+1,j+1}\frac{1}{36}+D_{i+1,j}\frac{1}{18}+D_{i,j+1}\frac{1}{18}+D_{i,j}\frac{1}{9}]+$$

$$+\sum_{i\in\{0,2\},j\in\{1,3\}}[D_{i+1,j+1}\frac{1}{18}+D_{i+1,j}\frac{1}{9}+D_{i,j+1}\frac{1}{36}+D_{i,j}\frac{1}{18}]+$$

$$+\sum_{i\in\{1,3\},j\in\{0,2\}}[D_{i+1,j+1}\frac{1}{18}+D_{i+1,j}\frac{1}{36}+D_{i,j+1}\frac{1}{9}+D_{i,j}\frac{1}{18}],$$

and then the inequality from theorem.

∎

We formulate below now the a similar result for quasi-convex functions on co-ordinates.

Theorem 8. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ with $a<b,\ c<d$. If $|\frac{\partial^2 f}{\partial r\partial t}|$ is quasi-convex on the co-ordinates on Δ then the following inequality holds:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4},\frac{3c+d}{4})+f(\frac{3a+b}{4},\frac{c+3d}{4})+f(\frac{a+3b}{4},\frac{3c+d}{4})+$$

$$+f(\frac{a+3b}{4},\frac{c+3d}{4})-\frac{2}{d-c}\int_c^d(f(\frac{3a+b}{4},y)+f(\frac{a+3b}{4},y))dy-\frac{2}{b-a}\int_a^b(f(x,\frac{3c+d}{4})+$$

$$+f(x,\frac{c+3d}{4}))dx+\frac{4}{(b-a)(c-d)}\int_a^b\int_c^d f(x,y)dxdy]\le$$

$$\le\frac{1}{4}\sum_{i,j=0}^{3}\max\{|\frac{\partial^2 f}{\partial r\partial t}(x_{i+i},y_{j+1})|,|\frac{\partial^2 f}{\partial r\partial t}(x_{i+1},y_j)|,|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_{j+1})|,|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_j)|\}.$$

Proof. Using now Lemma 5 and Definition 4 we find that the left member is less than the following expressions:

$$\sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)|\frac{\partial^2 f}{\partial r\partial t}(tx_{i+1}+(1-t)x_i,ry_{j+1}+(1-r)y_j)|dtdr\le$$

$$\le\sum_{i,j=0}^{3}\int_0^1\int_0^1 s_i(t)s_j(r)\max\{|\frac{\partial^2 f}{\partial r\partial t}(x_{i+i},y_{j+1})|,|\frac{\partial^2 f}{\partial r\partial t}(x_{i+1},y_j)|,|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_{j+1})|,|\frac{\partial^2 f}{\partial r\partial t}(x_i,y_j)|\}.$$

∎

For s- convex functions on the co-ordinates we can formulate also the following result:

Theorem 9. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partial differentiable mapping on Δ°, $\Delta = [a,b] \times [c,d]$ with $a < b,\ c < d,\ a, c \geq 0$ such that $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$. If $|\frac{\partial^2 f}{\partial r \partial t}|$ is s-convex on the co-ordinates on Δ then the following inequality holds:*

$$\frac{64}{(b-a)(d-c)}[f(\frac{3a+b}{4}, \frac{3c+d}{4}) + f(\frac{3a+b}{4}, \frac{c+3d}{4}) + f(\frac{a+3b}{4}, \frac{3c+d}{4})+$$

$$+f(\frac{a+3b}{4}, \frac{c+3d}{4}) - \frac{2}{d-c}\int_c^d (f(\frac{3a+b}{4}, y) + f(\frac{a+3b}{4}, y))dy - \frac{2}{b-a}\int_a^b (f(x, \frac{3c+d}{4})+$$

$$+f(x, \frac{c+3d}{4}))dx + \frac{4}{(b-a)(c-d)}\int_a^b \int_c^d f(x,y)dxdy] \leq$$

$$\leq \frac{1}{(s+2)^2}[|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{c+3d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{3c+d}{4})|+$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{c+3d}{4})|] + \frac{1}{(s+1)^2(s+2)^2}[|\frac{\partial^2 f}{\partial r \partial t}(a,c)| + |\frac{\partial^2 f}{\partial r \partial t}(a,d)| + |\frac{\partial^2 f}{\partial r \partial t}(b,c)|+$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(b,d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(a, \frac{c+d}{2})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, c)| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(b, \frac{c+d}{2})|+$$

$$+4|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{c+d}{2})|] + \frac{1}{(s+1)(s+2)^2}[(|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, c)| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, c)|+$$

$$+|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, d)| + |\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, d)| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{3a+b}{4}, \frac{c+d}{2})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+3b}{4}, \frac{c+d}{2})|)+$$

$$+(|\frac{\partial^2 f}{\partial r \partial t}(a, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(a, \frac{c+3d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(b, \frac{3c+d}{4})| + |\frac{\partial^2 f}{\partial r \partial t}(b, \frac{c+3d}{4})|+$$

$$+2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{3c+d}{4})| + 2|\frac{\partial^2 f}{\partial r \partial t}(\frac{a+b}{2}, \frac{c+3d}{4})|)].$$

Proof. The proof will be as in Theorem 7. ∎

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Department of Mathematics,"Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara

E-mail address, L. Ciurdariu: *ciu_ls@yahoo.com*