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FURTHER BOUNDS FOR TWO MAPPINGS RELATED TO THE HERMITE-HADAMARD INEQUALITY

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Abstract. Some new results concerning two mappings associated to the celebrated Hermite-Hadamard integral inequality for twice differentiable functions with applications for special means are given.

1. Introduction

The Hermite-Hadamard integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for special means.

In order to provide various refinements of this result, the first author introduced in 1991, see [2], the following associated mapping $H:[0,1]\to\mathbb{R}$ defined by

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

for a given convex function $f:[a,b]\to\mathbb{R}$.

Some of the main properties of H are explored in [2], [3], [4] and [9].

The corresponding double integral mapping in connection with the Hermite-Hadamard inequalities was considered first in [3] and is defined as

$$F:[0,1]\to\mathbb{R}, F(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy.$$

Some of the main results concerning this mapping can be seen in [3] (see also [4]).

For other related results, see for instance the research papers [1], [11], [12], [13], [15], [14], [16], [17], [18], the monograph online [10] and the references therein.

In the recent paper [7] we proved the following result where upper and lower bounds for the associated functions

$$\frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)f\left(\frac{a+b}{2}\right)-H(t)$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx-F(t)$$

with $t\in[0,1]$, have been given.

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$
\begin{aligned}
(1.1)\qquad & 0\leq 2\min\{t,1-t\} \\
&\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right] \\
&\leq \frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)\,f\left(\frac{a+b}{2}\right)-H(t) \\
&\leq 2\max\{t,1-t\} \\
&\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(1.2)\qquad & 0\leq 2\min\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx-F\left(\frac{1}{2}\right)\right] \\
&\leq \frac{1}{b-a}\int_a^b f(x)\,dx-F(t) \\
&\leq 2\max\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx-F\left(\frac{1}{2}\right)\right],
\end{aligned}
$$

for any $t\in[0,1]$.

Employing a different technique, in [8] we obtained the following result as well:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$
\begin{aligned}
(1.3)\qquad & \frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)\,f\left(\frac{a+b}{2}\right)-H(t) \\
&\leq t(1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(1.4)\qquad & \frac{1}{b-a}\int_a^b f(x)\,dx-F(t) \\
&\leq 2t(1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]
\end{aligned}
$$

for any $t\in[0,1]$.

Motivated by the above results we establish in this paper some new bounds involving these two mappings. Applications for special means are also provided.

2. The Results

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on the interval (a,b) and assume that there exists the constants $k<K$ such that*

$$
(2.1)\qquad k\leq f''(s)\leq K \text{ for any } s\in(a,b).
$$

Then we have

$$\begin{aligned}(2.2)\qquad & \frac{1}{24}k(1-t)t(b-a)^2 \\ & \le \frac{t}{b-a}\int_a^b f(x)\,dx + (1-t)f\left(\frac{a+b}{2}\right) - H(t) \\ & \le \frac{1}{24}K(1-t)t(b-a)^2\end{aligned}$$

and

$$(2.3)\qquad \frac{1}{12}k(1-t)t(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \le \frac{1}{12}K(1-t)t(b-a)^2$$

for any $t \in [0,1]$.

Proof. Consider the auxiliary function $g_k : [a,b] \to \mathbb{R}$, $g_k(s) := f(s) - \frac{1}{2}ks^2$. This function is twice differentiable and $g_k''(s) = f''(s) - k \ge 0$ by (2.1), which shows that g_k is convex on $[a,b]$.

By the definition of convexity we have

$$\begin{aligned}0 &\le tg_k(x) + (1-t)g_k(y) - g_k(tx+(1-t)y) \\ &= tf(x) + (1-t)f(y) - f(tx+(1-t)y) \\ &- \frac{1}{2}k\left[tx^2 + (1-t)y^2 - (tx+(1-t)y)^2\right] \\ &= tf(x) + (1-t)f(y) - f(tx+(1-t)y) \\ &- \frac{1}{2}k(1-t)t(x-y)^2\end{aligned}$$

for any $x,y \in [a,b]$ and for any $t \in [0,1]$.

Therefore we have

$$(2.4)\qquad \frac{1}{2}k(1-t)t(x-y)^2 \le tf(x) + (1-t)f(y) - f(tx+(1-t)y)$$

for any $x,y \in [a,b]$ and for any $t \in [0,1]$.

By utilising the auxiliary function $g_K : [a,b] \to \mathbb{R}$, $g_K(s) := \frac{1}{2}Ks^2 - f(s)$ we also get

$$(2.5)\qquad tf(x) + (1-t)f(y) - f(tx+(1-t)y) \le \frac{1}{2}K(1-t)t(x-y)^2$$

for any $x,y \in [a,b]$ and for any $t \in [0,1]$.

Now, from (2.4) we get

$$\begin{aligned}(2.6)\qquad & \frac{1}{2}k(1-t)t\left(x - \frac{a+b}{2}\right)^2 \\ & \le tf(x) + (1-t)f\left(\frac{a+b}{2}\right) - f\left(tx + (1-t)\frac{a+b}{2}\right)\end{aligned}$$

for any $x \in [a,b]$ and for any $t \in [0,1]$.

Integrating the inequality (2.4) over $x \in [a,b]$ we have

$$\begin{aligned}(2.7)\qquad & \frac{1}{2}k(1-t)t\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\ & \le t\int_a^b f(x)\,dx + (1-t)f\left(\frac{a+b}{2}\right) - \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx\end{aligned}$$

and since

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{1}{12}(b-a)^3$$

then we get from (2.7) the first inequality in (2.2).

The second inequality in (2.2) follows from (2.5) by a similar argument.

Integrating the inequality (2.4) over x and y on $[a,b]$ we have

$$\begin{aligned}
&\frac{1}{2}k(1-t)t\int_a^b\int_a^b (x-y)^2\,dxdy \\
&\le t(b-a)\int_a^b f(x)\,dx + (1-t)(b-a)\int_a^b f(y)\,dy \\
&-\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy \\
&= (b-a)\int_a^b f(x)\,dx - \int_a^b\int_a^b f(tx+(1-t)y)\,dxdy.
\end{aligned} \tag{2.8}$$

Since

$$\int_a^b\int_a^b (x-y)^2\,dxdy = \frac{1}{6}(b-a)^4$$

then from (2.8) we get the first inequality in (2.3).

The second inequality in (2.3) follows from (2.5) by a similar argument. □

The following result also holds:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$\frac{1}{12}\left(t-\frac{1}{2}\right)^2 k(b-a)^2 \le F(t) - F\left(\frac{1}{2}\right) \le \frac{1}{12}\left(t-\frac{1}{2}\right)^2 K(b-a)^2 \tag{2.9}$$

for any $t \in [0,1]$.

Proof. By taking $t = \frac{1}{2}$, $x = u$ and $y = v$ in the inequalities (2.4) and (2.5) we get

$$\frac{1}{8}k(u-v)^2 \le \frac{f(u)+f(v)}{2} - f\left(\frac{u+v}{2}\right) \le \frac{1}{8}K(u-v)^2 \tag{2.10}$$

for any $u.v \in [a,b]$.

Now, if we write the inequality (2.10) for $u = tx+(1-t)y$ and $v = ty+(1-t)x$ the we get

$$\begin{aligned}
&\frac{1}{2}k\left(t-\frac{1}{2}\right)^2(x-y)^2 \\
&\le \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2} - f\left(\frac{x+y}{2}\right) \\
&\le \frac{1}{2}K\left(t-\frac{1}{2}\right)^2(x-y)^2
\end{aligned} \tag{2.11}$$

for any $x, y \in [a,b]$ and for any $t \in [0,1]$.

Integrating the inequality (2.11) over x and y on $[a,b]$ we have

$$\begin{aligned}
(2.12)\qquad & \frac{1}{2}k\left(t-\frac{1}{2}\right)^2\int_a^b\int_a^b(x-y)^2\,dxdy \\
& \leq \int_a^b\int_a^b\frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}dxdy \\
& -\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\
& \leq \frac{1}{2}K\left(t-\frac{1}{2}\right)^2\int_a^b\int_a^b(x-y)^2\,dxdy
\end{aligned}$$

and since

$$\begin{aligned}
& \int_a^b\int_a^b\frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}dxdy \\
& =\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy=F(t)
\end{aligned}$$

we deduce from (2.12) the desired inequality (2.9). □

3. Applications for L_p-means

Let us consider the convex mapping $f:(0,\infty)\to\mathbb{R},\ f(x)=x^p,\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a<b$. Define the mapping

$$H_p(t):=\frac{1}{b-a}\int_a^b(tx+(1-t)A(a,b))^p\,dx,\ t\in[0,1].$$

It is obvious that $H_p(0)=A^p(a,b),\ H_p(1)=L_p^p(a,b)$ where, we recall that $A(a,b)=\frac{a+b}{2}$,

$$L_p^p(a,b):=\frac{1}{p+1}\frac{b^{p+1}-a^{p+1}}{b-a},\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$$

and for $t\in(0,1)$ we have

$$\begin{aligned}
(3.1)\qquad H_p(t) & =\frac{1}{[tb+(1-t)A(a,b)]-[ta+(1-t)A(a,b)]}\int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)}y^p dy \\
& =L_p^p(ta+(1-t)A(a,b),tb+(1-t)A(a,b)).
\end{aligned}$$

Now, consider the function

$$F_p(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b(tx+(1-t)y)^p\,dxdy.$$

We observe that $F_p(1)=F_p(0)=L_p^p(a,b)$ and for $t\in(0,1)$ we have

$$\begin{aligned}
(3.2)\qquad F_p(t) & =\frac{1}{b-a}\int_a^b\left(\frac{1}{b-a}\int_a^b(tx+(1-t)y)^p\,dx\right)dy \\
& =\frac{1}{b-a}\int_a^b\left(\frac{1}{[tb+(1-t)y]-[ta+(1-t)y]}\int_{ta+(1-t)y}^{tb+(1-t)y}s^p ds\right)dy \\
& =\frac{1}{b-a}\int_a^b L_p^p(ta+(1-t)y,tb+(1-t)y)\,dy.
\end{aligned}$$

We can calculate the double integral

$$F_p\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{x+y}{2}\right)^p dxdy$$
$$= \begin{cases} \frac{4}{(b-a)^2(p+1)(p+2)}\left[b^{p+2}-2\left(\frac{b+a}{2}\right)^{p+2}+a^{p+2}\right] & p\neq -2, \\ \frac{8}{(b-a)^2}\ln\left(\frac{A(a,b)}{G(a,b)}\right) & p=-2 \end{cases}$$

for $p\neq -1$, where $G(a,b)$ denotes the geometric mean of a,b (see [7]).

Let us consider the convex mapping $f_p:(0,\infty)\to\mathbb{R}$, $f_p(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a<b$. Define the quantities

$$K_p := p(p-1)\times\begin{cases} b^{p-2}, & \text{if } p\geq 2 \\ a^{p-2}, & \text{if } p\in(-\infty,0)\cup[1,2)\setminus\{-1\} \end{cases}$$

and

$$k_p := p(p-1)\times\begin{cases} a^{p-2}, & \text{if } p\geq 2 \\ b^{p-2}, & \text{if } p\in(-\infty,0)\cup[1,2)\setminus\{-1\}. \end{cases}$$

We observe that with these notations we have that

$$k_p\leq f_p''(x)\leq K_p$$

for any $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a\leq x\leq b$.

We can state the following result:

Proposition 1. *We have the following inequalities:*

$$\frac{1}{24}k_p(1-t)t(b-a)^2\leq tL_p^p(a,b)+(1-t)A^p(a,b)-H_p(t)\leq\frac{1}{24}K_p(1-t)t(b-a)^2, \tag{3.3}$$

$$\frac{1}{12}k_p(1-t)t(b-a)^2\leq L_p^p(a,b)-F_p(t)\leq\frac{1}{12}K_p(1-t)t(b-a)^2 \tag{3.4}$$

and

$$\frac{1}{12}\left(t-\frac{1}{2}\right)^2k_p(b-a)^2\leq F_p(t)-F_p\left(\frac{1}{2}\right)\leq\frac{1}{12}\left(t-\frac{1}{2}\right)^2K_p(b-a)^2$$

for any $t\in[0,1]$.

The proof follows by Theorem 3 and 4 and the details are omitted.

References

[1] A.G. AZPEITIA, Convex functions and the Hadamard inequality. *Rev. Colombiana Mat.* **28** (1994), no. 1, 7–12.

[2] S.S. DRAGOMIR, A mapping in connection to Hadamard's inequalities, *An. Öster. Akad. Wiss. Math.-Natur.,* (Wien), **128**(1991), 17-20. MR 934:26032. ZBL No. 747:26015.

[3] S.S. DRAGOMIR, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.,* **167**(1992), 49-56. MR:934:26038, ZBL No. 758:26014.

[4] S.S. DRAGOMIR, On Hadamard's inequalities for convex functions, *Mat. Balkanica,* **6**(1992), 215-222. MR: 934: 26033.

[5] S.S. DRAGOMIR, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure & Appl. Math.*, 3(2002), No. 3, Art. 35. [Online: http://www.emis.de/journals/JIPAM/article187.html?sid=187].

[6] S.S. DRAGOMIR, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 471-476.

[7] S.S. DRAGOMIR and I. GOMM, Bounds for two mappings associated to the Hermite-Hadamard inequality, *Aust. J. Math. Anal. Appl.*, **8**(2011), Art. 5, 9 pages.

[8] S.S. DRAGOMIR and I. GOMM, Some new bounds for two mappings related to the Hermite-Hadamard inequality for convex functions, *Num. Alg. Cont. & Opt.* **2**(2012), No. 2, pp. 271-278.

[9] S.S. DRAGOMIR, D.S. MILOŚEVIĆ and J. SÁNDOR, On some refinements of Hadamard's inequalities and applications, *Univ. Belgrad, Publ. Elek. Fak. Sci. Math.,* **4**(1993), 21-24.

[10] S.S. DRAGOMIR and C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[11] A. GUESSAB and G. SCHMEISSER, Sharp integral inequalities of the Hermite-Hadamard type. *J. Approx. Theory* **115** (2002), no. 2, 260–288.

[12] E. KILIANTY and S.S. DRAGOMIR, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* **13** (2010), no. 1, 1–32.

[13] M. MERKLE, Remarks on Ostrowski's and Hadamard's inequality, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **10** (1999), 113–117.

[14] C. E. M. PEARCE and A. M. RUBINOV, P-functions, quasi-convex functions, and Hadamard type inequalities, *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[15] J. PEČARIĆ and A. VUKELIĆ, Hadamard and Dragomir-Agarwal inequalities, the Euler formulae and convex functions. *Functional Equations, Inequalities and Applications*, 105–137, Kluwer Acad. Publ., Dordrecht, 2003.

[16] G. TOADER, Superadditivity and Hermite-Hadamard's inequalities, *Studia Univ. Babeş-Bolyai Math.* **39** (1994), no. 2, 27–32.

[17] G.-S. YANG and M.-C. HONG, A note on Hadamard's inequality, *Tamkang J. Math.* **28** (1997), no. 1, 33–37.

[18] G.-S. YANG and K.-L. TSENG, On certain integral inequalities related to Hermite-Hadamard inequalities, *J. Math. Anal. Appl.* **239** (1999), no. 1, 180–187.

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