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SOME NEW INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES OF ABSOLUTE VALUES ARE CONVEX AND CONCAVE

M. EMIN ÖZDEMIR■, ALPER EKINCI♠, AND AHMET OCAK AKDEMIR♠

Abstract. In this paper, we prove some new inequalities for the functions whose derivatives absolute values are convex and concave by dividing the interval $[a,b]$ to $n+1$ equal even sub-intervals. We obtain some new results involving intermediate values of $|f'|$ in $[a,b]$ by using some classical inequalities like Hermite-Hadamard, Hölder, Power-Mean and Jensen.

1. INTRODUCTION

The function $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. Geometrically, this means that if P,Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR. A huge amount of the researchers interested in this definition and there are several papers based on convexity. See the papers [1]-[7].

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and let $a,b\in I$, with $a<b$. The following double inequality;

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality [2]. Both inequalities hold in the reversed direction if f is concave.

In a recent paper [1], Latif and Dragomir proved following Theorems:

Theorem 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$ then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le\left(\frac{b-a}{96}\right)\left[|f'(a)|+4\left|f'\left(\frac{3a+b}{4}\right)\right|+2\left|f'\left(\frac{a+b}{2}\right)\right|+4\left|f'\left(\frac{a+3b}{2}\right)\right|+|f'(b)|\right]. \tag{1.1}$$

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Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q > 1$, then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{b-a}{16}\right) \tag{1.2}$$
$$\times \left\{ \left(\left| f\left(\frac{3a+b}{4}\right)\right|^q + |f'(a)|^q \right)^{\frac{1}{q}} + \left(\left| f\left(\frac{a+b}{2}\right)\right|^q + \left| f'\left(\frac{3a+b}{4}\right)\right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. \left(\left| f\left(\frac{a+3b}{4}\right)\right|^q + \left| f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} + \left(\left| f\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \left(\frac{1}{2}\right) \left(\frac{1}{3}\right)^{\frac{1}{q}} \left(\frac{b-a}{16}\right) \tag{1.3}$$
$$\times \left\{ \left(|f'(a)|^q + 2\left| f'\left(\frac{3a+b}{4}\right)\right|^q \right)^{\frac{1}{q}} + \left(\left| f\left(\frac{a+b}{2}\right)\right|^q + 2\left| f'\left(\frac{3a+b}{4}\right)\right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. \left(\left| f'\left(\frac{a+b}{2}\right)\right|^q + 2\left| f'\left(\frac{a+3b}{4}\right)\right|^q \right)^{\frac{1}{q}} + \left(2\left| f\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\},$$

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is concave on $[a,b]$ for some fixed $q > 1$, then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \tag{1.4}$$
$$\leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{b-a}{16}\right) \left[\left| f'\left(\frac{7a+b}{8}\right)\right| + \left| f'\left(\frac{5a+3b}{8}\right)\right| \right.$$
$$\left. + \left| f'\left(\frac{3a+5b}{8}\right)\right| + \left| f'\left(\frac{a+7b}{8}\right)\right| \right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed*

$q>1$, *then the following inequality holds:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{1.5}$$
$$\leq \left(\frac{b-a}{32}\right)\left[\left|f'\left(\frac{13a+3b}{12}\right)\right|+\left|f'\left(\frac{11a+5b}{12}\right)\right|\right.$$
$$\left.+\left|f'\left(\frac{5a+13b}{12}\right)\right|+\left|f'\left(\frac{3a+13b}{12}\right)\right|\right].$$

The main aim of this paper is to establish some new inequalities involving values of $|f'|$ at intermediate points of $[a,b]$ interval for functions whose absolute values of derivatives are convex and concave. In order to prove our results we divide the interval $[a,b]$ to $n+1$ equal even sub-intervals.

2. MAIN RESULTS

We need following lemma to prove our main Theorems:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$ and n is an odd number then the following equality holds:*

$$\left(\tfrac{b-a}{n+1}\right)\sum_{k=0}^{(n-1)/2}\left[\int_0^1 tf'\left(t\tfrac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\tfrac{a(n-2k+1)+b.2k}{n+1}\right)dt\right.$$
$$\left.+\int_0^1(1-t).f'\left(t\tfrac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\tfrac{a(n-2k)+b.(2k+1)}{n+1}\right)dt\right] \tag{2.1}$$
$$=\sum_{k=0}^{(n-1)\;/2}2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx.$$

Proof. Firstly, we take

$$I_{1k} = \int_0^1\left[tf'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b.2k}{n+1}\right)\right]dt$$
$$I_{2k} = \int_0^1\left[(1-t)\,f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b.(2k+1)}{n+1}\right)\right]dt.$$

Hence, it is obvious that for $k=0$, we have

$$I_{10}+I_{20}=\int_0^1 tf'\left(t\frac{an+b}{n+1}+(1-t)a\right)dt+\int_0^1 tf'\left(t\frac{a(n-1)+2b}{n+1}+(1-t)\frac{an+b}{n+1}\right)dt.$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1 tf'\left(t\frac{an+b}{n+1}+(1-t)a\right)dt+\int_0^1 (t-1)f'\left(t\frac{a(n-1)+2b}{n+1}+(1-t)\frac{an+b}{n+1}\right)dt \\
= &\left[\frac{(n+1)t}{b-a}f\left(t\frac{an+b}{n+1}+(1-t)a\right)\Big|_0^1-\frac{n+1}{b-a}\int_0^1 f\left(t\frac{an+b}{n+1}+(1-t)a\right)dt\right] \\
&+\left[\frac{(n+1)(t-1)}{b-a}f\left(t\frac{a(n-1)+2b}{n+1}+(1-t)\frac{an+b}{n+1}\right)\Big|_0^1\right. \\
&\left.-\frac{n+1}{b-a}\int_0^1 f\left(t\frac{a(n-1)+2b}{n+1}+(1-t)\frac{an+b}{n+1}\right)dt\right].
\end{aligned}$$

By making use of the substitutions $x=t.\frac{an+b}{n+1}+(1-t)a$ and $y=t\frac{a(n-1)+2b}{n+1}+(1-t)\frac{an+b}{n+1}$, we get

$$I_{10}+I_{20}=\frac{2(n+1)}{b-a}f\left(\frac{an+b}{n+1}\right)-\left(\frac{n+1}{b-a}\right)^2\left[\int_a^{\frac{an+b}{n+1}}f(x)\,dx+\int_{\frac{an+b}{n+1}}^{\frac{a(n-1)+2b}{n+1}}f(x)dx\right].$$

By a similar argument, for $k=1$,we have

$$I_{11}+I_{21}=\frac{2(n+1)}{b-a}f\left(\frac{a(n-2)+3b}{n+1}\right)-\left(\frac{n+1}{b-a}\right)^2\left[\int_{\frac{a(n-1)+2b}{n+1}}^{\frac{a(n-2)+3b}{n+1}}f(x)\,dx+\int_{\frac{a(n-2)+3b}{n+1}}^{\frac{a(n-3)+4b}{n+1}}f(x)dx\right].$$

If we apply same calculations from $k=2$ to $(n-1)/2$, we have

$$\frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}(I_{1k}+I_{2k})=\sum_{k=0}^{(n-1)/2}2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx.$$

Which is the desired result. □

Theorem 6. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$ and n is an odd number then the following inequality holds:*

$$\begin{aligned}
&\left|\sum_{k=0}^{(n-1)/2}2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le\tfrac{b-a}{6(n+1)}\sum_{k=0}^{(n-1)/2}\left(4\left|f'\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\right. \\
&\left.+\left|f'\left(\tfrac{a(n-2k+1)+b(2k)}{n+1}\right)\right|+\left|f'\left(\tfrac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right.
\end{aligned}\tag{2.2}$$

Proof. By using Lemma 1 and properties of modulus, we have

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left(\int_0^1 \left|tf'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|dt\right.$$
$$\left.+\int_0^1\left|(1-t)\,f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|dt\right).$$

By using the convexity of $|f'|$, we obtain

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \tfrac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left|f'\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\int_0^1 t^2dt+\left|f'\left(\tfrac{a(n-2k+1)+b.2k}{n+1}\right)\right|\int_0^1 t(1-t)dt$$
$$\left.+\left|f'\left(\tfrac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\int_0^1 t\,(1-t)\,dt+\left|f'\left(\tfrac{a(n-2k)+b.(2k+1)}{n+1}\right)\right|\int_0^1(1-t)^2dt\right)$$
$$=\tfrac{b-a}{6(n+1)}\sum_{k=0}^{(n-1)/2}\left(4\left|f'\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|+\left|f'\left(\tfrac{a(n-2k+1)+b.2k}{n+1}\right)\right|+\left|f'\left(\tfrac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right).$$

Which completes the proof. □

Corollary 1. *If we choose $n=1$ in (2.2) we obtain the following result:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{6}\left(4f'\left(\frac{a+b}{2}\right)+f'(a)+f'(b)\right).$$

Corollary 2. *Under the conditions of Theorem 6, the following inequality holds:*

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\frac{n-2k}{n+1}\right)|f'(a)|+\left(\frac{2k+1}{n+1}\right)|f'(b)|\right].$$

Besides if $|f'(x)|\leq M$, for all $x\in[a,b]$, then we have also following inequality:

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\left(\frac{b-a}{n+1}\right)\left(\frac{n-1}{2}\right)M.$$

Proof. It follows directly from Theorem 6 and using the convexity of $|f'|$. □

Remark 1. *If we choose $n=3$ in (2.2) , this inequality reduces to (1.1).*

Theorem 7. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q\geq 1$ and n is an odd number, then the following inequality holds:*

$$\begin{aligned}
&\left|\sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq\left(\tfrac{b-a}{n+1}\right)\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\sum_{k=0}^{(n-1)/2}\left\{\left[\tfrac{1}{2}\left|f'\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right.\right. \\
&\qquad\left.+\tfrac{1}{2}\left|f'\left(\tfrac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\right]^{\frac{1}{q}} \\
&\left.+\left[\tfrac{1}{2}\left|f'\left(\tfrac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+\tfrac{1}{2}\left|f'\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}\tag{2.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and by using the Hölder inequality, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq\tfrac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left\{\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\right. \\
&\times\left(\int_0^1\left|f'\left(t\tfrac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\tfrac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1(1-t)^p dt\right)^{\frac{1}{p}} \\
&\left.\times\left(\int_0^1\left|f'\left(t\tfrac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}
\end{aligned}\tag{2.4}$$

Since $|f|^q$ is convex on $[a,b]$, we have

$$\begin{aligned}
&\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\,(2k)}{n+1}\right)\right|^q dt \\
\leq\;&\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\int_0^1 tdt+\left|f'\left(\frac{a(n-2k+1)+b\,(2k)}{n+1}\right)\right|^q\int_0^1(1-t)dt \\
=\;&\frac{1}{2}\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q+\left|f'\left(\frac{a(n-2k+1)+b\,(2k)}{n+1}\right)\right|^q\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|\int_0^1\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right| \\
\leq\ & \frac{1}{2}\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right].
\end{aligned}$$

By using the last two inequalities in (2.4), we obtain the desired result. □

Corollary 3. *If we choose $n=1$ in (2.3), we obtain the following result:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{b-a}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^q \\
&\times\left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right]^{\frac{1}{q}}+\left[\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 4. *Under the conditions of Theorem 7, the following inequality holds:*

$$\begin{aligned}
&\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \left(\frac{b-a}{n+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\sum_{k=0}^{(n-1)/2}\left\{\left[\left(\frac{2n-4k+1}{n+1}\right)\left|f'(a)\right|+\left(\frac{4k+1}{n+1}\right)\left|f'(b)\right|\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\frac{2n-4k-1}{n+1}\right)\left|f'(a)\right|+\left(\frac{4k+3}{n+1}\right)\left|f'(b)\right|\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

Besides if $|f'(x)|^q \leq M$, for all $x\in[a,b]$, then we have also following inequality:

$$\begin{aligned}
&\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \left(\frac{b-a}{n+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(n-1)^2}{2}\right)(M)^{\frac{1}{q}}.
\end{aligned}$$

Proof. It follows from Theorem 7 using the convexity of $|f'|^q$ and the fact

$$\sum_{k=1}^{n}(u_k+v_k)^s \leq \sum_{k=1}^{n}(u_k)^s+\sum_{k=1}^{n}(v_k)^s,\ u_k,v_k\geq 0,\ 1\leq k\leq n,\ 0\leq s<1.$$ □

Remark 2. *If we choose $n=3$ in (2.3), this inequality reduces to (1.2).*

Theorem 8. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed*

$q\geq 1$ and n is an odd number, then the following inequality holds:

$$\begin{aligned}
&\left|\sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \left(\tfrac{b-a}{n+1}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)^{\frac{1}{q}}\sum_{k=0}^{(n-1)/2}\left\{\left[2\left|f'\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right.\right. \\
&\left.+\left|f'\left(\tfrac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\right]^{\frac{1}{q}} \\
&\left.+\left[2\left|f'\left(\tfrac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+\left|f'\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.5}$$

Proof. From Lemma 1 and by using the well-known Power mean inequality, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \tfrac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left\{\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1 t\left|f'\left(t\tfrac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\tfrac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 (1-t)dt\right)^{\left(1-\frac{1}{q}\right)} \\
&\left.\times\left(\int_0^1 (1-t)\left|f'\left(t\tfrac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.6}$$

Since $|f|^q$ is convex on $[a,b]$, we have

$$\begin{aligned}
&\int_0^1 t\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt \\
\leq\ & \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\int_0^1 t^2dt+\left|f'\left(\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q\int_0^1 t(1-t)dt \\
=\ & \frac{1}{3}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q+\frac{1}{6}\left|f'\left(\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|\int_0^1 (1-t)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right| \\
\leq\ & \frac{1}{6}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+\frac{1}{3}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q
\end{aligned}$$

Using the last two inequalities in (2.6), we get the result. □

Corollary 5. *If we choose $n = 1$ in (2.5), we obtain the following result:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \left(\frac{b-a}{8}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}$$
$$\times \left[2\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right]^{\frac{1}{q}} + \left[2|f'(b)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 6. *Under the conditions of Theorem 8, using the same arguments as in Corollary 4, the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$
$$\le \left(\frac{b-a}{n+1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}} \sum_{k=0}^{(n-1)/2} \left\{ \left[\left(\frac{3n-6k+1}{n+1}\right)|f'(a)| + \left(\frac{6k+2}{n+1}\right)|f'(b)|\right]^{\frac{1}{q}} \right.$$
$$\left. + \left[\left(\frac{3n-6k-2}{n+1}\right)|f'(a)| + \left(\frac{6k+5}{n+1}\right)|f'(b)|\right]^{\frac{1}{q}} \right\}.$$

Besides if $|f'(x)|^q \le M$, for all $x \in [a,b]$, then we have also following inequality:

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$
$$\le \left(\frac{b-a}{n+1}\right)\left(\frac{1}{2}\right)^{\frac{1}{q}} (n-1)(M)^{\frac{1}{q}}.$$

Remark 3. *If we choose $n = 3$ in (2.3), this inequality reduces to (1.3).*

Theorem 9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathring{I}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$, and n is an odd number, then the following inequality holds:*

(2.7)

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right) - \tfrac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$
$$\le \left(\tfrac{b-a}{n+1}\right)\left(\tfrac{q-1}{2q-1}\right) \sum_{k=0}^{(n-1)/2} \left(\left|f'\left(\tfrac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q + \left|f'\left(\tfrac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|^q \right)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and using the well-known Hölder inequality for $q>1$ and $p=\frac{q}{q-1}$, we have
(2.8)

$$\left|\sum_{k=0}^{(n-1)/2}2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \tfrac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left(\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\tfrac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\tfrac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\tfrac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

Since $|f|^q$ is concave on $[a,b]$ and by using the Hadamard inequality for concave functions, we have

$$\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt$$
$$\leq\left|f'\left(\frac{\frac{a(n-2k)+b(2k+1)}{n+1}+\frac{a(n-2k+1)+b2k}{n+1}}{2}\right)\right|^q=\left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q$$

and similarly,

$$\int_0^1\left|f'\left(t.\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b.(2k+1)}{n+1}\right)\right|^q dt$$
$$\leq\left|f'\left(\frac{\frac{a(n-2k-1)+b(2k+2)}{n+1}+\frac{a(n-2k)+b.(2k+1)}{n+1}}{2}\right)\right|^q=\left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|^q.$$

Using these two inequalities in (2.8), we get the desired result. □

Corollary 7. *If we choose $n=1$ in (2.7), we obtain the following result:*

$$\left|2f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\left(\frac{b-a}{2}\right)\left(\frac{q-1}{2q-1}\right)\left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{3b+a}{4}\right)\right|^q\right].$$

Corollary 8. *Under the conditions of Theorem 9 and assume that $|f'|$ is a linear function, the following inequality holds:*

$$\left|\sum_{k=0}^{(n-1)/2}2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\left(\tfrac{b-a}{n+1}\right)\left(\tfrac{q-1}{2q-1}\right)2\sum_{k=0}^{(n-1)/2}\left[\left(\tfrac{n-k}{n+1}\right)|f'(a)|+\left(\tfrac{k+1}{n+1}\right)|f'(b)|\right].$$

Proof. It follows directly from Theorem 9 and linearity of $|f'|$. □

Remark 4. *If we choose* $n=3$ *in* (2.7), *this inequality reduces to* (1.4).

Theorem 10. *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* $\overset{\circ}{I}$ *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^q$ *is concave on* $[a,b]$ *for some fixed* $q>1$ *and* n *is an odd number, then the following inequality holds:*

$$\begin{aligned}(2.9)\quad &\left|\sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le \tfrac{b-a}{(n+1)}\sum_{k=0}^{(n-1)/2}\left[\left|f'\left(\tfrac{2}{3}\tfrac{a(n-2k)+b(2k+1)}{n+1}+\tfrac{1}{3}\tfrac{a(n-2k+1)+b(2k)}{n+1}\right)\right|\right.\\ &\left.+\left|f'\left(\tfrac{1}{3}\tfrac{a(n-2k-1)+b(2k+2)}{n+1}+\tfrac{2}{3}\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\right].\end{aligned}$$

Proof. We know that if $|f'|^q$ is concave, then $|f'|$ is also concave (See: [1]). From Lemma 1 and property of modulus, we have

$$\begin{aligned}&\left|\sum_{k=0}^{(n-1)/2} 2f\left(\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)-\tfrac{n+1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le \tfrac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\int_0^1 t\left|f'\left(t.\tfrac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\tfrac{a(n-2k+1)+b(2k)}{n+1}\right)\right|dt\right.\quad .\\ &\left.+\int_0^1(1-t).\left|f'\left(t\tfrac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\tfrac{a(n-2k)+b(2k+1)}{n+1}\right)\right|dt\right]\end{aligned}$$

Since $|f'|$ is concave, by using Jensen Inequality, we obtain

$$\begin{aligned}&\le \tfrac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b(2k)}{n+1}\right)dt}{\int_0^1 tdt}\right)\right|\right.\\ &\left.+\left(\int_0^1(1-t)\,dt\right)\left|f'\left(\frac{\int_0^1(1-t)\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)dt}{\int_0^1(1-t)dt}\right)\right|\right].\end{aligned}$$

which is equivalent to (2.9). This completes proof. □

Corollary 9. *If we choose* $n=1$ *in* (2.9), *we obtain the following result:*

$$\begin{aligned}&\left|2f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right|\\ \le\ &\frac{b-a}{4}\left[\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right].\end{aligned}$$

Corollary 10. *Under the conditions of Theorem 10 and assume that $|f'|$ is a linear function, the following inequality holds:*

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{2(b-a)}{n+1}\sum_{k=0}^{(n-1)/2}\left[(n-2k)\left|f'(a)\right|+(2k+n)\left|f'(b)\right|\right].$$

Proof. It follows directly from Theorem 10 and linearity of $|f'|$. □

Remark 5. *If we choose $n=3$ in (2.9), this inequality reduces to (1.5).*

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■Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, ERZURUM TURKEY

E-mail address: emos@atauni.edu.tr

♠Ağri İbrahim Çeçen University, Faculty of Science and Letters, Department of Mathematics, 04100 AĞRI TURKEY

E-mail address: alperekinci@hotmail.com

♠Ağri İbrahim Çeçen University, Faculty of Science and Letters, Department of Mathematics, 04100 AĞRI TURKEY

E-mail address: ahmetakdemir@agri.edu.tr