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ON HERMITE-HADAMARD INEQUALITY FOR TWICE DIFFERENTIABLE FUNCTIONS BOUNDED BY EXPONENTIALS

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Abstract. Some Hermite-Hadamard type inequalities for twice differentiable functions whose second derivatives are bounded below and above by exponentials are given. Applications for special means are provided as well.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the papers [1] – [58] and the references therein.

In this paper we establish some Hermite-Hadamard type inequalities for twice differentiable functions whose second derivatives are bounded below and above by exponential functions. Applications for special means are provided as well.

2. The Results

The following result holds:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function with the property that there exists the constants $\alpha,m,M\in\mathbb{R}$ with $\alpha\neq 0, m<M$ and such that*

$$me^{\alpha t}\leq f''(t)\leq Me^{\alpha t} \tag{2.1}$$

for any $t\in(a,b)$.

Then we have the inequalities

$$\begin{aligned}\frac{m}{\alpha^2}&\left(\frac{e^{\alpha a}+e^{\alpha b}}{2}-\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}\right)\\&\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\\&\leq\frac{M}{\alpha^2}\left(\frac{e^{\alpha a}+e^{\alpha b}}{2}-\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}\right)\end{aligned} \tag{2.2}$$

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and

$$\frac{m}{\alpha^2}\left(\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}-e^{\alpha\left(\frac{a+b}{2}\right)}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{M}{\alpha^2}\left(\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}-e^{\alpha\left(\frac{a+b}{2}\right)}\right). \tag{2.3}$$

Proof. Consider the auxiliary function $g_{m,\alpha}:[a,b]\to\mathbb{R}$ given by $g_{m,\alpha}(t):=f(t)-\frac{m}{\alpha^2}e^{\alpha t}$. This function is twice differentiable and since $g''_{m,\alpha}(t):=f''(t)-me^{\alpha t}\geq 0$ we have that $g_{m,\alpha}$ is convex.

By the definition of convexity we have that

$$\begin{aligned} 0 &\leq \lambda g_{m,\alpha}(a)+(1-\lambda)g_{m,\alpha}(b)-g_{m,\alpha}(\lambda a+(1-\lambda)b) \\ &= \lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b) \\ &\quad -\frac{m}{\alpha^2}\left(\lambda e^{\alpha a}+(1-\lambda)e^{\alpha b}-e^{\alpha(\lambda a+(1-\lambda)b)}\right) \end{aligned} \tag{2.4}$$

for any $\lambda\in[0,1]$.

This is equivalent with

$$\frac{m}{\alpha^2}\left(\lambda e^{\alpha a}+(1-\lambda)e^{\alpha b}-e^{\alpha(\lambda a+(1-\lambda)b)}\right) \leq \lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b) \tag{2.5}$$

for any $\lambda\in[0,1]$.

Utilising the auxiliary function $g_{M,\alpha}:[a,b]\to\mathbb{R}$ given by $g_{M,\alpha}(t):=\frac{M}{\alpha^2}e^{\alpha t}-f(t)$ we also get

$$\lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b) \leq \frac{M}{\alpha^2}\left(\lambda e^{\alpha a}+(1-\lambda)e^{\alpha b}-e^{\alpha(\lambda a+(1-\lambda)b)}\right) \tag{2.6}$$

for any $\lambda\in[0,1]$.

Integrating the inequality (2.5) over $\lambda\in[0,1]$ and taking into account that

$$\int_0^1 e^{\alpha(\lambda a+(1-\lambda)b)}d\lambda=\frac{1}{b-a}\int_a^b e^{\alpha s}ds=\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}$$

and

$$\int_0^1 f(\lambda a+(1-\lambda)b)\,d\lambda=\frac{1}{b-a}\int_a^b f(t)\,dt$$

we obtain the first inequality in (2.2).

The second part of (2.2) follows by (2.6) in the same way.

Now if we use (2.5) and (2.6) for $\lambda=\frac{1}{2}$ we get

$$\frac{m}{\alpha^2}\left(\frac{e^{\alpha u}+e^{\alpha v}}{2}-e^{\alpha\left(\frac{u+v}{2}\right)}\right)\leq\frac{f(u)+f(v)}{2}-f\left(\frac{u+v}{2}\right) \leq\frac{M}{\alpha^2}\left(\frac{e^{\alpha u}+e^{\alpha v}}{2}-e^{\alpha\left(\frac{u+v}{2}\right)}\right) \tag{2.7}$$

for any $u,v\in[a,b]$.

If we write this inequality for $u = \lambda a + (1-\lambda) b$ and $v = (1-\lambda) a + \lambda b$ then we get

$$\begin{aligned}
(2.8)\qquad & \frac{m}{\alpha^2}\left(\frac{e^{\alpha(\lambda a+(1-\lambda)b)}+e^{\alpha((1-\lambda)a+\lambda b)}}{2}-e^{\alpha\left(\frac{a+b}{2}\right)}\right) \\
& \le \frac{f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)}{2}-f\left(\frac{a+b}{2}\right) \\
& \le \frac{M}{\alpha^2}\left(\frac{e^{\alpha(\lambda a+(1-\lambda)b)}+e^{\alpha((1-\lambda)a+\lambda b)}}{2}-e^{\alpha\left(\frac{a+b}{2}\right)}\right)
\end{aligned}$$

for any $\lambda \in [0,1]$.

Integrating the inequality (2.8) over λ on the interval $[0,1]$ and taking into account that

$$\int_0^1 e^{\alpha(\lambda a+(1-\lambda)b)}d\lambda = \int_0^1 e^{\alpha((1-\lambda)a+\lambda b)}d\lambda = \frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}$$

and

$$\begin{aligned}
\int_0^1 f(\lambda a+(1-\lambda)b)\,d\lambda &= \int_0^1 f((1-\lambda)a+\lambda b)\,d\lambda \\
&= \frac{1}{b-a}\int_a^b f(t)\,dt
\end{aligned}$$

then we get the desired result (2.3). $\square$

Remark 1. *If $0 < x < y$ and the function $f : [\ln x, \ln y] \to \mathbb{R}$ satisfies the condition (2.1) on the interval $[\ln x, \ln y]$, then we have the inequalities*

$$\begin{aligned}
(2.9)\qquad & \frac{m}{\alpha^2}\left(A(x^\alpha,y^\alpha)-L(x^\alpha,y^\alpha)\right) \\
& \le A(f(\ln x), f(\ln y)) - \frac{1}{\ln y-\ln x}\int_{\ln x}^{\ln y} f(t)\,dt \\
& \le \frac{M}{\alpha^2}\left(A(x^\alpha,y^\alpha)-L(x^\alpha,y^\alpha)\right)
\end{aligned}$$

and

$$\begin{aligned}
(2.10)\qquad & \frac{m}{\alpha^2}\left(L(x^\alpha,y^\alpha)-G(x^\alpha,y^\alpha)\right) \\
& \le \frac{1}{\ln y-\ln x}\int_{\ln x}^{\ln y} f(t)\,dt - f(\ln G(x,y)) \\
& \le \frac{M}{\alpha^2}\left(L(x^\alpha,y^\alpha)-G(x^\alpha,y^\alpha)\right),
\end{aligned}$$

where $A(p,q) := \frac{p+q}{2}$ is the arithmetic mean, $G(p,q) := \sqrt{pq}$ is the geometric mean and $L(p,q) := \frac{p-q}{\ln p-\ln q}$ is the logarithmic mean.

We need the following result that is of interest itself. It provides lower and upper bounds for the Jensen's difference

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right)$$

in the case of twice differentiable functions whose second derivatives are bounded by exponentials as in (2.1).

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function with the property that there exists the constants $\alpha,m,M\in\mathbb{R}$ with $\alpha\neq 0, m<M$ and such that (2.1) is valid.*

Then for any $x_i\in[a,b]$ and $p_i\geq 0$ with $i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$ we have the inequalities

$$\begin{aligned}(2.11)\quad \frac{m}{\alpha^2}\left(\sum_{i=1}^n p_i e^{\alpha x_i}-e^{\alpha\left(\sum_{i=1}^n p_i x_i\right)}\right)&\\ \leq \sum_{i=1}^n p_i f(x_i)-f\left(\sum_{i=1}^n p_i x_i\right)&\\ \leq \frac{M}{\alpha^2}\left(\sum_{i=1}^n p_i e^{\alpha x_i}-e^{\alpha\left(\sum_{i=1}^n p_i x_i\right)}\right)&.\end{aligned}$$

Proof. Since the auxiliary function $g_{m,\alpha}:[a,b]\to\mathbb{R}$ given by $g_{m,\alpha}(t):=f(t)-\frac{m}{\alpha^2}e^{\alpha t}$ is convex, then by Jensen's inequality we have

$$\begin{aligned}0\leq \sum_{i=1}^n p_i g_{m,\alpha}(x_i)-g_{m,\alpha}\left(\sum_{i=1}^n p_i x_i\right)&\\ =\sum_{i=1}^n p_i f(x_i)-f\left(\sum_{i=1}^n p_i x_i\right)&\\ -\frac{m}{\alpha^2}\left(\sum_{i=1}^n p_i e^{\alpha x_i}-e^{\alpha\left(\sum_{i=1}^n p_i x_i\right)}\right)&\end{aligned}$$

which produces the first inequality.

The second inequality follows in a similar way by employing the auxiliary function $g_{M,\alpha}:[a,b]\to\mathbb{R}$ given by $g_{M,\alpha}(t):=\frac{M}{\alpha^2}e^{\alpha t}-f(t)$. □

Remark 2. *If $0<x<y$ and the function $f:[\ln x,\ln y]\to\mathbb{R}$ satisfies the condition (2.1) on the interval $[\ln x,\ln y]$, then we have the inequalities*

$$\begin{aligned}(2.12)\quad \frac{m}{\alpha^2}\left(\sum_{i=1}^n p_i y_i^{\alpha}-\prod_{i=1}^n y_i^{\alpha p_i}\right)\leq \sum_{i=1}^n p_i f(\ln y_i)-f\left(\ln\prod_{i=1}^n y_i^{p_i}\right)&\\ \leq \frac{M}{\alpha^2}\left(\sum_{i=1}^n p_i y_i^{\alpha}-\prod_{i=1}^n y_i^{\alpha p_i}\right)&,\end{aligned}$$

where $0<x\leq y_i\leq y$ for $i\in\{1,...,n\}$.

Utilising the Jensen's type inequality (2.11) we are able to provide some upper and lower bounds for the difference of the integral means

$$\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{(b-a)^n}\int_a^b ...\int_a^b f\left(\sum_{i=1}^n p_i x_i\right)dx_1...dx_n$$

where $p_i>0$ with $i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function with the property that there exists the constants $\alpha,m,M\in\mathbb{R}$ with $\alpha\neq 0, m<M$ and such that (2.1) is valid.*

Then for any $p_i > 0$ *with* $i \in \{1, ..., n\}$ *and* $\sum_{i=1}^{n} p_i = 1$ *we have the inequalities*

$$\frac{m}{\alpha^2}\left(\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}-\frac{1}{\alpha^n\prod_{i=1}^{n}p_i}\prod_{i=1}^{n}\frac{e^{\alpha p_i b}-e^{\alpha p_i a}}{b-a}\right) \tag{2.13}$$
$$\leq \frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{(b-a)^n}\int_a^b \dots \int_a^b f\left(\sum_{i=1}^{n}p_i x_i\right)dx_1\dots dx_n$$
$$\leq \frac{M}{\alpha^2}\left(\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)}-\frac{1}{\alpha^n\prod_{i=1}^{n}p_i}\prod_{i=1}^{n}\frac{e^{\alpha p_i b}-e^{\alpha p_i a}}{b-a}\right).$$

Proof. We integrate the inequality (2.11) on $[a,b]^n$ to get

$$\frac{m}{\alpha^2}\left(\sum_{i=1}^{n}p_i\int_a^b\dots\int_a^b e^{\alpha x_i}dx_1\dots dx_n-\int_a^b\dots\int_a^b e^{\alpha\left(\sum_{i=1}^{n}p_i x_i\right)}dx_1\dots dx_n\right) \tag{2.14}$$
$$\leq \sum_{i=1}^{n}p_i\int_a^b\dots\int_a^b f(x_i)\,dx_1\dots dx_n-\int_a^b\dots\int_a^b f\left(\sum_{i=1}^{n}p_i x_i\right)dx_1\dots dx_n$$
$$\leq \frac{M}{\alpha^2}\left(\sum_{i=1}^{n}p_i\int_a^b\dots\int_a^b e^{\alpha x_i}dx_1\dots dx_n-\int_a^b\dots\int_a^b e^{\alpha\left(\sum_{i=1}^{n}p_i x_i\right)}dx_1\dots dx_n\right).$$

Observe that

$$\int_a^b\dots\int_a^b e^{\alpha x_i}dx_1\dots dx_n=(b-a)^{n-1}\int_a^b e^{\alpha x_i}dx_i$$
$$=(b-a)^n\frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)},$$

$$\int_a^b\dots\int_a^b e^{\alpha\left(\sum_{i=1}^{n}p_i x_i\right)}dx_1\dots dx_n=\int_a^b\dots\int_a^b\prod_{i=1}^{n}e^{\alpha p_i x_i}dx_1\dots dx_n$$
$$=\prod_{i=1}^{n}\int_a^b e^{\alpha p_i x_i}dx_i=\prod_{i=1}^{n}\frac{e^{\alpha p_i b}-e^{\alpha p_i a}}{\alpha p_i}$$
$$=\frac{(b-a)^n}{\alpha^n\prod_{i=1}^{n}p_i}\prod_{i=1}^{n}\frac{e^{\alpha p_i b}-e^{\alpha p_i a}}{b-a}$$

and

$$\int_a^b\dots\int_a^b f(x_i)\,dx_1\dots dx_n=(b-a)^{n-1}\int_a^b f(x)\,dx.$$

From (2.14) we then get

$$\frac{m}{\alpha^2}\left((b-a)^n \frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)} - \frac{(b-a)^n}{\alpha^n \prod_{i=1}^n p_i}\prod_{i=1}^n \frac{e^{\alpha p_i b}-e^{\alpha p_i a}}{b-a}\right)$$
$$\leq (b-a)^{n-1}\int_a^b f(x)\,dx - \int_a^b \dots \int_a^b f\left(\sum_{i=1}^n p_i x_i\right) dx_1 \dots dx_n$$
$$\leq \frac{M}{\alpha^2}\left((b-a)^n \frac{e^{\alpha b}-e^{\alpha a}}{\alpha(b-a)} - \frac{(b-a)^n}{\alpha^n \prod_{i=1}^n p_i}\prod_{i=1}^n \frac{e^{\alpha p_i b}-e^{\alpha p_i a}}{b-a}\right),$$

which by division with $(b-a)^n$ produces the desired result (2.13). □

3. Some Applications

The above inequalities may be applied for various functions in Analysis for which simple upper and lower bounds for the function $\frac{f''(\cdot)}{e^{\alpha \cdot}}$ can be found.

Consider, for instance, the function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ given by $f(t)=\frac{1}{t}$. We have $f''(t)=\frac{2}{t^3}$ for $t\in[a,b]$ and

$$\frac{2}{b^3 e^b} \leq \frac{f''(t)}{e^t} \leq \frac{2}{a^3 e^a} \tag{3.1}$$

for any $t\in[a,b]$.

Utilising the inequality (2.2) we obtain

$$\frac{2}{b^3 e^b}\left(A\left(e^a,e^b\right)-L\left(e^a,e^b\right)\right) \leq \frac{L(a,b)-H(a,b)}{L(a,b)\,H(a,b)} \leq \frac{2}{a^3 e^a}\left(A\left(e^a,e^b\right)-L\left(e^a,e^b\right)\right) \tag{3.2}$$

and from (2.3)

$$\frac{2}{b^3 e^b}\left(L\left(e^a,e^b\right)-e^{A(a,b)}\right) \leq \frac{A(a,b)-L(a,b)}{A(a,b)\,L(a,b)} \leq \frac{2}{a^3 e^a}\left(L\left(e^a,e^b\right)-e^{A(a,b)}\right), \tag{3.3}$$

where $H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}}$ is the harmonic mean.

Now, consider the function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ given by $f(t)=e^{\beta t}$ with $\beta>\alpha$. Then we have

$$\beta^2 e^{(\beta-\alpha)a} \leq \frac{f''(t)}{e^{\alpha t}} \leq \beta^2 e^{(\beta-\alpha)b} \tag{3.4}$$

for any $t\in[a,b]$.

If we apply the inequality (2.11) for the function $f(t) = e^{\beta t}$, $t \in [a,b]$, then we have the inequalities

$$
\begin{aligned}
(3.5)\quad & \frac{\beta^2}{\alpha^2} e^{(\beta-\alpha)a}\left(\sum_{i=1}^{n} p_i e^{\alpha x_i} - e^{\alpha\left(\sum_{i=1}^{n} p_i x_i\right)}\right) \\
& \le \sum_{i=1}^{n} p_i e^{\beta x_i} - e^{\beta\left(\sum_{i=1}^{n} p_i x_i\right)} \\
& \le \frac{\beta^2}{\alpha^2} e^{(\beta-\alpha)b}\left(\sum_{i=1}^{n} p_i e^{\alpha x_i} - e^{\alpha\left(\sum_{i=1}^{n} p_i x_i\right)}\right)
\end{aligned}
$$

for any $x_i \in [a,b]$ and $p_i \ge 0$ with $i \in \{1,...,n\}$ and $\sum_{i=1}^{n} p_i = 1$.

Now, assume that $0 < s \le y_i \le S < \infty$ for any $i \in \{1,...,n\}$. On choosing $x_i = \ln y_i$ for any $i \in \{1,...,n\}$, then we have $\ln s \le x_i \le \ln S$.

If we write the inequality (3.5) for these $x_i = \ln y_i$ for any $i \in \{1,...,n\}$ we get for $\beta > \alpha$

$$
\begin{aligned}
(3.6)\quad & \frac{\beta^2}{\alpha^2} s^{(\beta-\alpha)}\left(\sum_{i=1}^{n} p_i y_i^{\alpha} - \prod_{i=1}^{n} y_i^{\alpha p_i}\right) \le \sum_{i=1}^{n} p_i y_i^{\beta} - \prod_{i=1}^{n} y_i^{\beta p_i} \\
& \le \frac{\beta^2}{\alpha^2} S^{(\beta-\alpha)}\left(\sum_{i=1}^{n} p_i y_i^{\alpha} - \prod_{i=1}^{n} y_i^{\alpha p_i}\right)
\end{aligned}
$$

provided that $0 < s \le y_i \le S < \infty, p_i \ge 0$ for any $i \in \{1,...,n\}$ and $\sum_{i=1}^{n} p_i = 1$.

If in this inequality we take $\beta = 1$ and $\alpha = -1$ then we get

$$
(3.7)\quad s^2\left(\sum_{i=1}^{n} \frac{p_i}{y_i} - \frac{1}{\prod_{i=1}^{n} y_i^{p_i}}\right) \le \sum_{i=1}^{n} p_i y_i - \prod_{i=1}^{n} y_i^{p_i} \le S^2\left(\sum_{i=1}^{n} \frac{p_i}{y_i} - \frac{1}{\prod_{i=1}^{n} y_i^{p_i}}\right).
$$

Finally, on applying the inequality (2.13) for the exponential function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ given by $f(t) = e^{\beta t}$ with $\beta > \alpha$, we obtain

$$
\begin{aligned}
(3.8)\quad & \frac{\beta^2}{\alpha^2} e^{(\beta-\alpha)a}\left(\frac{e^{\alpha b} - e^{\alpha a}}{\alpha(b-a)} - \frac{1}{\alpha^n \prod_{i=1}^{n} p_i} \prod_{i=1}^{n} \frac{e^{\alpha p_i b} - e^{\alpha p_i a}}{b-a}\right) \\
& \le \frac{e^{\beta b} - e^{\alpha a}}{\beta(b-a)} - \frac{1}{\beta^n \prod_{i=1}^{n} p_i} \prod_{i=1}^{n} \frac{e^{\beta p_i b} - e^{\beta p_i a}}{b-a} \\
& \le \frac{\beta^2}{\alpha^2} e^{(\beta-\alpha)b}\left(\frac{e^{\alpha b} - e^{\alpha a}}{\alpha(b-a)} - \frac{1}{\alpha^n \prod_{i=1}^{n} p_i} \prod_{i=1}^{n} \frac{e^{\alpha p_i b} - e^{\alpha p_i a}}{b-a}\right)
\end{aligned}
$$

for any $p_i > 0$ with $i \in \{1,...,n\}$ and $\sum_{i=1}^{n} p_i = 1$.

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