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INTEGRAL INEQUALITIES FOR SOME CONVEX FUNCTIONS

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ABSTRACT. In this paper, we established some new integral inequalities for different kinds of convex functions by using some classical inequalities.

1. Introduction

We recall following definitions.

The function $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. We can define starshaped functions on $[0,b]$ which satisfy the condition

$$f(tx)\leq tf(x)$$

for $t\in[0,1]$. In [8], Toader defined the concept of $m-$convexity as the following:

Definition 1. *The function $f:[0,b]\to\mathbb{R}$ is said to be $m-$convex, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have:*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0,b]$ for which $f(0)\leq 0$.

Some interesting and important inequalities for $m-$convex functions can be found in [4], [3], [2], [6], [7], [8] and [9].

In [5], Hudzik and Maligranda considered among others the class of functions which are $s-$convex in the second sense.

Definition 2. *A function $f:\mathbb{R}^+\to\mathbb{R}$, where $\mathbb{R}^+=[0,\infty)$, is said to be $s-$convex in the second sense if*

$$f(\alpha x+\beta y)\leq\alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$. This class of $s-$convex functions in the second sense is usually denoted by K_s^2.

In [10], $s-$convexity introduced by Breckner as a generalization of convex functions. Also, Breckner proved the fact that the set valued map is $s-$convex only if the associated support function is $s-$convex function in [11].

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Theorem 1. *[See [12]] Suppose that $f:[0,\infty)\to[0,\infty)$ is an $s-$convex function in the second sense, where $s\in(0,1]$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L_1[0,1]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (2.3). The above inequalities are sharp.

Several properties of $s-$convexity in the first sense are discussed in the paper [5]. Obviously, $s-$convexity means just convexity when $s=1$. Some new Hermite-Hadamard type inequalities based on concavity and $s-$convexity established by Kırmacı *et al.* in [13]. For related results see the papers [12] and [13].

In [1], Dragomir proved the following theorem.

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be an $m-$convex function with $m\in(0,1]$ and $0\leq a<b$. If $f\in L_1[a,b]$, then one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}dx \\ &\leq \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}\right]. \end{aligned} \tag{1.2}$$

In [16], Miheşan gave definition of $(\alpha,m)-$convexity as following;

Definition 3. *The function $f:[0,b]\to\mathbb{R}$, $b>0$ is said to be $(\alpha,m)-$convex, where $(\alpha,m)\in[0,1]^2$, if we have*

$$f(tx+m(1-t)y)\leq t^{\alpha}f(x)+m(1-t^{\alpha})f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha,m)-$convex functions on $[0,b]$ for which $f(0)\leq 0$. If we choose $(\alpha,m)=(1,m)$, it can be easily seen that $(\alpha,m)-$convexity reduces to $m-$convexity and for $(\alpha,m)=(1,1)$, we have ordinary convex functions on $[0,b]$. For the recent results based on the above definition see the papers [2], [4], [15], [17], [18], and [19].

Definition 4. *[See [20]] A function $f:I\to[0,\infty)$ is said to be log-convex or multiplicatively convex if log f is convex, or, equivalently, if for all $x,y\in I$ and $t\in[0,1]$ one has the inequality:*

$$f(tx+(1-t)y)\leq[f(x)]^t[f(y)]^{1-t} \tag{1.3}$$

We note that a $\log-$convex function is convex, but the converse may not necessarily be true.

Theorem 3. *Let $f,g:[a,b]\to\mathbb{R}$ be real valued non-negative convex functions and $F(x,y)(t),G(x,y)(t):[0,1]\to\mathbb{R}^+$ are defined as the following ([14]);*

$$\begin{aligned} F(x,y)(t) &= \frac{1}{2}\left[f(tx+(1-t)y)+f((1-t)x+ty)\right] \\ G(x,y)(t) &= \frac{1}{2}\left[g(tx+(1-t)y)+g((1-t)x+ty)\right]. \end{aligned}$$

For all $t \in [0,1]$ *we have*

$$\frac{1}{b-a}\int_a^b F\left(x,\frac{a+b}{2}\right)(t)G\left(x,\frac{a+b}{2}\right)(t)dx \tag{1.4}$$
$$\leq \frac{1}{4(b-a)}\int_a^b f(x)g(x)dx+\frac{3}{16}\left[M(a,b)+N(a,b)\right]$$

and

$$\frac{2}{(b-a)^2}\int_a^b\int_a^b F(x,y)(t)G(x,y)(t)dxdy \tag{1.5}$$
$$\leq \frac{1}{(b-a)}\int_a^b f(x)g(x)dx+\frac{1}{4}\left[M(a,b)+N(a,b)\right]$$

where

$$\begin{aligned} M(a,b) &= f(a)g(a)+f(b)g(b)\\ N(a,b) &= f(a)g(b)+f(b)g(a). \end{aligned}$$

The main purpose of this paper is to prove some new inequalities as above, but now for $m-$convex and $s-$convex functions by modified the mappings $F(x,y)(t)$ and $G(x,y)(t)$.

2. Main Results

Theorem 4. *Let* $f,g:[0,\infty)\to\mathbb{R}^+$ *be* $m-$*convex functions with* $m\in(0,1]$, $0\leq a<b$ *and* $f,\ g,\ fg\in L_1[a,b]$. $F(x,y)(t)$, $G(x,y)(t):[0,1]\to\mathbb{R}^+$ *are defined as the following:*

$$\begin{aligned} F(x,y)(t) &= \frac{1}{2}\left[f(tx+m(1-t)y)+f((1-t)x+mty)\right]\\ G(x,y)(t) &= \frac{1}{2}\left[g(tx+m(1-t)y)+g((1-t)x+mty)\right]. \end{aligned}$$

For all $t\in[0,1]$, *we have*

(2.1)
$$\int_a^b F\left(x,\frac{a+b}{2}\right)(t)G\left(x,\frac{a+b}{2}\right)(t)dx$$
$$\leq \frac{1}{4}\int_a^b f(x)g(x)dx+\frac{m}{4}\left[\mu_1\int_a^b f(x)dx+\mu_2\int_a^b g(x)dx\right]+\frac{m^2}{4}(b-a)\mu_1\mu_2$$

where

$$\begin{aligned} \mu_1 &= \frac{m+1}{4}\left(\frac{g(a)+g(b)}{2}+m\frac{g(\frac{a}{m})+g(\frac{b}{m})}{2}\right)\\ \mu_2 &= \frac{m+1}{4}\left(\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}\right) \end{aligned}$$

and

(2.2)

$$\begin{aligned}&\frac{2}{(b-a)^2}\int_a^b\int_a^b F(x,y)(t)G(x,y)(t)dxdy\\ \leq\ &\frac{m^2+1}{2}\int_a^b f(x)g(x)dx+\frac{m}{b-a}\int_a^b f(x)dx\int_a^b g(y)dy.\end{aligned}$$

Proof. Since f and g are $m-$convex functions, we can write

$$\begin{aligned}F(x,y)(t) &\leq \frac{1}{2}\left[tf(x)+m(1-t)f(y)+(1-t)f(x)+mtf(y)\right]\\ &= \frac{1}{2}\left[f(x)+mf(y)\right],\end{aligned}$$

$$F\left(x,\frac{a+b}{2}\right)(t)\leq\frac{1}{2}\left[f(x)+mf\left(\frac{a+b}{2}\right)\right] \tag{2.3}$$

and analogously, if we set $x=x$ and $y=\frac{a+b}{2}$, we can write

$$\begin{aligned}G(x,y)(t) &\leq \frac{1}{2}\left[tg(x)+m(1-t)g(y)+(1-t)g(x)+mtg(y)\right]\\ &= \frac{1}{2}\left[g(x)+mg(y)\right],\end{aligned}$$

$$G\left(x,\frac{a+b}{2}\right)(t)\leq\frac{1}{2}\left[g(x)+mg\left(\frac{a+b}{2}\right)\right]. \tag{2.4}$$

By multiplying the inequalities (2.3) and (2.4), we get

(2.5)

$$\begin{aligned}&F\left(x,\frac{a+b}{2}\right)(t)G\left(x,\frac{a+b}{2}\right)(t)\\ \leq\ &\frac{1}{4}\left[f(x)+mf\left(\frac{a+b}{2}\right)\right]\left[g(x)+mg\left(\frac{a+b}{2}\right)\right]\\ =\ &\frac{1}{4}\left[f(x)g(x)+mf\left(\frac{a+b}{2}\right)g(x)+mg\left(\frac{a+b}{2}\right)f(x)+m^2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

Integrating the above inequality with respect to x on $[a,b]$, we obtain the following inequality:

(2.6)

$$\begin{aligned}&\int_a^b F\left(x,\frac{a+b}{2}\right)(t)G\left(x,\frac{a+b}{2}\right)(t)dx\\ \leq\ &\frac{1}{4}\left\{\int_a^b f(x)g(x)dx+\int_a^b mf\left(\frac{a+b}{2}\right)g(x)dx\right.\\ &\left.+\int_a^b mg\left(\frac{a+b}{2}\right)f(x)dx+m^2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)\right\}\end{aligned}$$

Using the inequalities in (1.2), we rewrite the (2.6) as the following:

$$
\begin{aligned}
& \int_a^b F\left(x,\frac{a+b}{2}\right)(t)G\left(x,\frac{a+b}{2}\right)(t)dx \\
\leq\; & \frac{1}{4}\left[\int_a^b f(x)g(x)dx + m\frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}dx\int_a^b g(x)dx\right. \\
& + m\frac{1}{b-a}\int_a^b \frac{g(x)+mg(\frac{x}{m})}{2}dx\int_a^b f(x)dx \\
& \left. +m^2\frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}dx\frac{1}{b-a}\int_a^b \frac{g(x)+mg(\frac{x}{m})}{2}dx\,(b-a)\right] \\
\leq\; & \frac{1}{4}\left[\int_a^b f(x)g(x)dx + m\frac{m+1}{4}\left(\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}\right)\int_a^b g(x)dx\right. \\
& +m\frac{m+1}{4}\left(\frac{g(a)+g(b)}{2}+m\frac{g(\frac{a}{m})+g(\frac{b}{m})}{2}\right)\int_a^b f(x)dx \\
& \left. + m^2\left(\frac{m+1}{4}\right)^2\left(\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}\right)\left(\frac{g(a)+g(b)}{2}+m\frac{g(\frac{a}{m})+g(\frac{b}{m})}{2}\right)(b-a)\right] \\
=\; & \frac{1}{4}\int_a^b f(x)g(x)dx + \frac{m}{4}\left[\mu_1\int_a^b f(x)dx + \mu_2\int_a^b g(x)dx\right] + \frac{m^2}{4}(b-a)\,\mu_1\mu_2.
\end{aligned}
$$

The proof is completed. □

Remark 1. *If we choose $m=1$, inequalities (2.1) and (2.2) reduces to (1.4) and (1.5) respectively.*

Theorem 5. *Let $f,g:[0,\infty)\to\mathbb{R}$ be $s-$convex functions in the second sense and $F(x,y)(t)$, $G(x,y)(t):[0,1]\to\mathbb{R}^+$ are defined as the following:*

$$
\begin{aligned}
F(x,y)(t) &= \frac{1}{2}\left[f(t^s x+(1-t)^s y)+f((1-t)^s x+t^s y)\right] \\
G(x,y)(t) &= \frac{1}{2}\left[g(t^s x+(1-t)^s y)+f((1-t)^s x+t^s y)\right].
\end{aligned}
$$

If $f,\ g,\ fg\in L_1[a,b]$, for all $t\in[0,1]$ we have

$$
\int_0^1 \left[F\left(a,b\right)(t)+G\left(a,b\right)(t)\right]dt \leq \frac{f(a)+f(b)+g(a)+g(b)}{s+1}
$$

and

$$
\begin{aligned}
& \int_0^1 F\left(a,b\right)(t)G\left(a,b\right)(t)dt \\
\leq\; & [M(a,b)+N(a,b)]\left\{\frac{1}{2(2s+1)}+\frac{1}{2}\beta(s+1,s+1)\right\} \\
=\; & [M(a,b)+N(a,b)]\left\{\frac{1}{2(2s+1)}+\frac{1}{2}p^{s+1}(1+p)^{2(s+1)}\int_0^1 \frac{t^s(1-t)^s}{(t+p)^{2(s+1)}}dt\right\}
\end{aligned}
$$

where

$$\begin{aligned} M(a,b) &= f(a)g(a)+f(b)g(b) \\ N(a,b) &= f(a)g(b)+f(b)g(a) \end{aligned}$$

and the Euler Beta function is defined by

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y>0.$$

Proof. Since f and g are $s-$convex functions in the second sense, we can write

$$F(x,y)(t) \leq \frac{1}{2}\left[t^s f(x)+(1-t)^s f(y)+(1-t)^s f(x)+t^s f(y)\right], \tag{2.7}$$

$$G(x,y)(t) \leq \frac{1}{2}\left[t^s g(x)+(1-t)^s g(y)+(1-t)^s g(x)+t^s g(y)\right]. \tag{2.8}$$

If we set $x=a$, $y=b$ in the above inequalities and by addition, then by integrating with respect to t over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 \left[F(a,b)(t)+G(a,b)(t)\right]dt \\ &\leq \frac{1}{2}\left[f(a)+f(b)+g(a)+g(b)\right]\left\{\int_0^1 t^s dt+\int_0^1 (1-t)^s dt\right\} \\ &= \frac{f(a)+f(b)+g(a)+g(b)}{s+1}. \end{aligned}$$

This completes the proof of the first inequality.

For the proof of the second inequality, if we multiply the inequalities (2.7) and (2.8) for $x=a$, $y=b$ and by integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 \left[F(a,b)(t)G(a,b)(t)\right]dt \\ &\leq \frac{1}{4}\left\{\left[f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)\right]\int_0^1 t^{2s}dt\right. \\ &\quad +\left[f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)\right]\int_0^1 (1-t)^{2s}dt \\ &\quad \left.+2\left[f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)\right]\int_0^1 t^s(1-t)^s dt\right\} \\ &= \left[M(a,b)+N(a,b)\right]\left\{\frac{1}{2(2s+1)}+\frac{1}{2}\beta(s+1,s+1)\right\}. \end{aligned}$$

By using following properties of the Beta function

$$\begin{aligned} \beta(x,y) &= \beta(y,x) \\ \beta(x,y) &= p^x(1+p)^{x+y}\int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+p)^{x+y}}dt, \ x,y,p>0 \end{aligned}$$

we obtain

$$\begin{aligned} &\int_0^1 \left[F(a,b)(t)+G(a,b)(t)\right]dt \\ &\leq \left[M(a,b)+N(a,b)\right]\left\{\frac{1}{2(2s+1)}+\frac{1}{2}p^{s+1}(1+p)^{2(s+1)}\int_0^1 \frac{t^s(1-t)^s}{(t+p)^{2(s+1)}}dt\right\}. \end{aligned}$$

The proof is completed. □

Theorem 6. *Let $f,g:[0,\infty)\to\mathbb{R}^{+}$ be $(\alpha,m)-$convex functions with $(\alpha,m)\in(0,1]^2$, $0\le a<b$ and $f,\ g,\ fg\in L_1[a,b]$. $F(x,y)(t),\ G(x,y)(t):[0,1]\to\mathbb{R}^{+}$ are defined as the following:*

$$\begin{aligned} F(x,y)(t) &= \frac{1}{2}\left[f(tx+m(1-t)y)+f(ty+m(1-t)x)\right] \\ G(x,y)(t) &= \frac{1}{2}\left[g(tx+m(1-t)y)+g(ty+m(1-t)x)\right]. \end{aligned}$$

For all $t\in[0,1]$, we have

(2.9)

$$\begin{aligned} &\int_0^1 F(a,b)(t)+G(a,b)(t)dt \\ \le\ &\frac{1}{2}\left[\frac{1+m\alpha}{1+\alpha}\right]\left[f(a)+f(b)+g(a)+g(b)\right] \end{aligned}$$

and

(2.10)

$$\begin{aligned} &\int_0^1 F(a,b)(t)G(a,b)(t)dt \\ \le\ &\frac{1}{4}\left[M(a,b)+N(a,b)\right]\left(\frac{\alpha\left(2m+m^2\right)}{\alpha+1}+\frac{m^2+1}{2\alpha+1}\right) \end{aligned}$$

where $M(a,b)$ and $N(a,b)$ as in Theorem 5.

Proof. Since f and g are $(\alpha,m)-$convex functions, we can write

$$F(x,y)(t)\le\frac{1}{2}\left[t^{\alpha}f(x)+m(1-t^{\alpha})f(y)+t^{\alpha}f(y)+m(1-t^{\alpha})f(x)\right]$$

If we set $x=a$ and $y=b$, we get

$$F(a,b)(t)\le\frac{1}{2}\left[\left(f(a)+f(b)\right)\left(t^{\alpha}+m(1-t^{\alpha})\right)\right] \tag{2.11}$$

and analogously, we have

$$G(a,b)(t)\le\frac{1}{2}\left[\left(g(a)+g(b)\right)\left(t^{\alpha}+m(1-t^{\alpha})\right)\right] \tag{2.12}$$

By adding the inequalities (2.11) and (2.12), we get

(2.13)

$$\begin{aligned} &F(a,b)(t)+G(a,b)(t) \\ \le\ &\frac{1}{2}\left[\left(f(a)+f(b)+g(a)+g(b)\right)\left(t^{\alpha}+m(1-t^{\alpha})\right)\right] \end{aligned}$$

Integrating the above inequality with respect to t on $[0,1]$, we obtain the inequality (2.9). For the proof of the inequality (2.10), by multiplying the inequalities (2.11) and (2.12), we have

$$\begin{aligned} &F(a,b)(t)G(a,b)(t) \\ \le\ &\frac{1}{4}\left[M(a,b)+N(a,b)\right]\left[t^{2\alpha}+2mt^{\alpha}\left(1-t^{\alpha}\right)+m^2\left(1-t^{\alpha}\right)^2\right]. \end{aligned}$$

By integrating the above inequality with respect to t on $[0,1]$, we get the inequality (2.10). □

Theorem 7. *Let $f,g:[a,b]\to\mathbb{R}^{+}$ be $\log-$convex functions and $f,\ g,\ fg\in L_{1}[a,b]$. $F(x,y)(t),\ G(x,y)(t):[0,1]\to\mathbb{R}^{+}$ are defined as in Theorem 3, then the following inequalities hold;*

(2.14)
$$\begin{aligned}&\int_{0}^{1}F\left(a,b\right)(t)G\left(a,b\right)(t)dt\\ \leq\ &\frac{1}{2}\left[L\left(f\left(a\right)g\left(a\right),f\left(b\right)g\left(b\right)\right)+L\left(f\left(a\right)g\left(b\right),f\left(b\right)g\left(a\right)\right)\right]\end{aligned}$$

for all $t\in[0,1]$, where

$$\begin{aligned}L\left(f\left(a\right)g\left(a\right),f\left(b\right)g\left(b\right)\right)&=\frac{f\left(a\right)g\left(a\right)-f\left(b\right)g\left(b\right)}{\ln f\left(a\right)g\left(a\right)-\ln f\left(b\right)g\left(b\right)}\\ L\left(f\left(a\right)g\left(b\right),f\left(b\right)g\left(a\right)\right)&=\frac{f\left(a\right)g\left(b\right)-f\left(b\right)g\left(a\right)}{\ln f\left(a\right)g\left(b\right)-\ln f\left(b\right)g\left(a\right)}.\end{aligned}$$

Proof. Since f,g are $\log-$convex functions on $[a,b]$, we can write

$$F(x,y)(t)\leq\frac{1}{2}\left[f^{t}(x)+f^{(1-t)}(y)+f^{(1-t)}(x)+f^{t}(y)\right]$$

and

$$G(x,y)(t)\leq\frac{1}{2}\left[g^{t}(x)+g^{(1-t)}(y)+g^{(1-t)}(x)+g^{t}(y)\right].$$

If we set $x=a$ and $y=b$, we have

$$F(a,b)(t)\leq\frac{1}{2}\left[f^{t}(a)+f^{(1-t)}(b)+f^{(1-t)}(a)+f^{t}(b)\right] \tag{2.15}$$

and

$$G(a,b)(t)\leq\frac{1}{2}\left[g^{t}(a)+g^{(1-t)}(b)+g^{(1-t)}(a)+g^{t}(b)\right]. \tag{2.16}$$

By multiplying the inequalities (2.15) and (2.16), we get

$$\begin{aligned}&F(a,b)(t)G(a,b)(t)\\ \leq\ &\frac{1}{4}\left[\left(f\left(a\right)g\left(a\right)^{t}\right)\left(f\left(b\right)g\left(b\right)^{1-t}+\left(f\left(a\right)g\left(b\right)^{t}\right)f\left(b\right)g\left(a\right)^{1-t}\right)\right.\\ &\left.+\left(f\left(a\right)g\left(a\right)^{t}\right)\left(f\left(b\right)g\left(b\right)^{1-t}+\left(f\left(a\right)g\left(b\right)^{t}\right)f\left(b\right)g\left(a\right)^{1-t}\right)\right].\end{aligned}$$

By integrating the above inequality with respect to t on $[0,1]$, we obtain the inequality (2.14). □

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