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A NOTE CONCERNING HOLDER'S INEQUALITY

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Abstract. The aim of this paper is to give some variants of the well-known Holder inequality for one variable functions for two variable functions and on $L^p(\mu)$ spaces, using the refined Young inequalities presented by F. Kittaneh and Y. Manasrah. A variant of Minkowski's inequality will be given and also as applications we will use a variant of Holder's inequality for generalizing some Hermite-Hadamard inequalities.

1. Introduction

It is well-known the famous Young inequality:

$$(1-\nu)a + \nu b \geq a^{1-\nu}b^{\nu}$$

where $a,\ b$ are nonnegative real numbers and $\nu \in [0,1]$.

This inequality was recently refined by F. Kittaneh and Y. Manasrah, see [6].

Theorem 1. *([6]) For $a,\ b \geq 0$ and $\nu \in [0,1]$, we have*

$$(1-\nu)a + \nu b \geq a^{1-\nu}b^{\nu} + r_0(\sqrt{a}-\sqrt{b})^2,$$

where $r_0 = \min\{\nu, 1-\nu\}$.

As applications, in [6] and [7] are given the following results and also the reverse scalar Young inequality with consequences:

Corollary 1. *For $a,\ b \geq 0$, and $0 \leq \nu \leq 1$ we have*

$$(a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu})^2 + 2r_0(a-b)^2 \leq (a+b)^2,$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Remark 1. *([6]) For $a,\ b \geq 0$, and $0 \leq \nu \leq 1$ we have*

$$(a^{\nu}b^{1-\nu})^2 + r_0^2(a-b)^2 \leq \nu a^2 + (1-\nu)b^2$$

where $r_0 = \min\{\nu, 1-\nu\}$.

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Theorem 2. *If* $a,\ b \geq 0$ *and* $0 \leq \nu \leq 1$ *then*

$$\nu a + (1-\nu)b \leq R_0(\sqrt{a} - \sqrt{b})^2 + a^\nu b^{1-\nu}$$

and

$$(\nu a + (1-\nu)b)^2 \leq R_0^2(a-b)^2 + (a^\nu b^{1-\nu})^2,$$

where $R_0 = \max\{\nu, 1-\nu\}$.

Corollary 2. *If* $a,\ b \geq 0$ *and* $0 \leq \nu \leq 1$ *then*

$$(a+b)^2 \leq 2R_0(a-b)^2 + (a^\nu b^{1-\nu} + a^{1-\nu} b^\nu)^2$$

where $R_0 = \max\{\nu, 1-\nu\}$.

We recall the well-known Holder′s integral inequality which can be stated as follows, see [9], [3] and then Theorem 2.1, see [3].

Theorem 3. *If* $f(x) \geq 0,\ g(x) \geq 0$ *and* $f(x) \in L^p[a,b],\ g(x) \in L^q[a,b]$ *and* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{1}$$

Theorem 4. *If the conditions of Theorem 3 are satisfied and* $t > 0$ *then*

$$\int_a^b f(x)g(x)dx \leq C(p,t) \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{2}$$

where $C(p,t) = \frac{1}{p} t^{\frac{1}{p}-1} + (1-\frac{1}{p}) t^{\frac{1}{p}}$.

Now if we consider $(\Omega, \mathcal{F}, \mu)$ a measure space and p a real number with $p \geq 1$ then the space $L^p = L^p(\Omega, \mathcal{F}, \mu)$ is the collection of all complex-valued Borel measurable functions f such that $\int_\Omega |f|^p d\mu < \infty$ and $||f||_p = (\int_\Omega |f|^p d\mu)^{1/p},\ f \in L^p$.

Holder Inequality([2]) Let $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $q \in L^q$ then $fg \in L^1$ and $||fg||_1 \leq ||f||_p ||g||_q$.

We recall that the function $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}},\ h \neq 1$ where h is positive real numbers is called Specht's ratio , see [4].

2. The results

Theorem 5. *Let* $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p$ *and* $g \in L^q$ *with* $||f||_p > 0,\ ||g||_q > 0$ *then*

$$\frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{||fg||_1}{||f||_p ||g||_q} + \frac{1}{\max\{p,q\}} [\frac{1}{p} \left(\frac{s}{t}\right)^{1/q} + \frac{1}{q} \left(\frac{t}{s}\right)^{1/p} -$$

$$- \frac{2}{\sqrt{pq}} \left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \frac{\int_\Omega |f|^{p/2} |g|^{q/2} d\mu}{||f||_p^{p/2} ||g||_q^{q/2}}] \le \frac{1}{p^2} \left(\frac{s}{t}\right)^{1/q} + \frac{1}{q^2} \left(\frac{t}{s}\right)^{1/p},$$

where $s,\ t > 0$.

Proof. Using that $|f|^{p/2}|g|^{q/2} \le \frac{1}{2}(|f|^p + |g|^q)$ and $f \in L^p,\ g \in L^q$ we have that $\int_\Omega |f|^{p/2}|g|^{q/2} d\mu \le \frac{1}{2}(\int_\Omega |f|^p d\mu + \int_\Omega |g|^q d\mu) < \infty$.

Taking now in the refined Young inequality, see Theorem 1([6]), $\nu = \frac{1}{p},\ 1-\nu = \frac{1}{q}$, $a = \frac{1}{p^{1/p}} \left(\frac{s}{t}\right)^{\frac{1}{pq}} \frac{|f|}{||f||_p}$ and $b = \frac{1}{q^{1/q}} \left(\frac{t}{s}\right)^{\frac{1}{pq}} \frac{|g|}{||g||_q}$ we will obtain:

$$\frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{|f||g|}{||f||_p ||g||_q} + \frac{1}{\max\{p,q\}} [\frac{1}{\sqrt{p}} \left(\frac{s}{t}\right)^{\frac{1}{2q}} \frac{|f|^{p/2}}{||f||_p^{p/2}} - \frac{1}{\sqrt{q}} \left(\frac{t}{s}\right)^{\frac{1}{2p}} \frac{|g|^{q/2}}{||g||_q^{q/2}}] \le$$

$$\le \frac{1}{p^2} \left(\frac{s}{t}\right)^{\frac{1}{q}} \frac{|f|^p}{||f||_p^p} + \frac{1}{q^2} \left(\frac{t}{s}\right)^{\frac{1}{p}} \frac{|g|^q}{||g||_q^q}.$$

Integration of both sizes yields:

$$\frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{\int_\Omega |fg| d\mu}{||f||_p ||g||_q} + \frac{1}{\max\{p,q\}} \int_\Omega [\frac{1}{p} \left(\frac{s}{t}\right)^{\frac{1}{q}} \frac{|f|^p}{||f||_p^p} + \frac{1}{q} \left(\frac{t}{s}\right)^{\frac{1}{p}} \frac{|g|^q}{||g||_q^q} -$$

$$- \frac{2}{\sqrt{pq}} \left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \frac{|f|^{p/2}|g|^{q/2}}{||f||_p^{p/2} ||g||_q^{q/2}}] d\mu \le \frac{1}{p^2} \left(\frac{s}{t}\right)^{\frac{1}{q}} \frac{\int_\Omega |f|^p d\mu}{||f||_p^p} + \frac{1}{q^2} \left(\frac{t}{s}\right)^{\frac{1}{p}} \frac{\int_\Omega |g|^q d\mu}{||g||_q^q} =$$

$$= \frac{1}{p^2} \left(\frac{s}{t}\right)^{\frac{1}{q}} + \frac{1}{q^2} \left(\frac{t}{s}\right)^{\frac{1}{p}}$$

and by calculus we obtain the desired inequality.

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Theorem 6. *Let* $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p$ *and* $g \in L^q$ *with* $||f||_p > 0,\ ||g||_q > 0$ *then*

$$\frac{1}{p^2} \left(\frac{s}{t}\right)^{1/q} + \frac{1}{q^2} \left(\frac{t}{s}\right)^{1/p} \le \frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{\int_\Omega |f||g| d\mu}{||f||_p ||g||_q} + \frac{1}{\min\{p,q\}} [\frac{1}{p} \left(\frac{s}{t}\right)^{1/q} + \frac{1}{q} \left(\frac{t}{s}\right)^{1/p} -$$

$$- \frac{2}{\sqrt{pq}} \left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \frac{\int_\Omega |f|^{p/2}|g|^{q/2} d\mu}{||f||_p^{p/2} ||g||_q^{q/2}}],$$

where $s,\ t > 0$.

Proof. If we take in Theorem 2, which is the reverses scalar Young inequality, see [6] and [7], $\nu = \frac{1}{p},\ 1-\nu = \frac{1}{q}$, $a = \frac{1}{p^{1/p}} \left(\frac{s}{t}\right)^{\frac{1}{pq}} \frac{|f|}{||f||_p}$ and $b = \frac{1}{q^{1/q}} \left(\frac{t}{s}\right)^{\frac{1}{pq}} \frac{|g|}{||g||_q}$ we will obtain as in the previous theorem the desired inequality.

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Remark 2. *(i) Under the above conditions if we put $s=t$ in Theorem 5 and then in Theorem 6 respectively, we will have:*

$$\frac{1}{p^{1/p}}\frac{1}{q^{1/q}}\frac{||fg||_1}{||f||_p||g||_q}+\frac{1}{\max\{p,q\}}[1-\frac{2}{\sqrt{pq}}\frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}]\leq 1-\frac{2}{pq}$$

and

$$1-\frac{2}{pq}\leq \frac{1}{p^{1/p}}\frac{1}{q^{1/q}}\frac{\int_\Omega |f||g|d\mu}{||f||_p||g||_q}+\frac{1}{\min\{p,q\}}[1-\frac{2}{\sqrt{pq}}\frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}]$$

respectively.

(ii) Under the conditions of Theorem 5 and using its proof we also obtain:

$$||fg||_1\leq C(p,s,t)||f||_p||g||_q,$$

where

$$C(p,s,t)=\frac{q^{\frac{1}{q}}}{p^{1+\frac{1}{q}}}\left(\frac{s}{t}\right)^{1/q}+\frac{p^{\frac{1}{p}}}{q^{1+\frac{1}{p}}}\left(\frac{t}{s}\right)^{1/p}.$$

(iii) Taking now in (ii), $\frac{s}{t}=\frac{p}{q}$ we will obtain the inequality from Theorem 2 [1] *which is a generalization of Holder's inequality, see* [2] *i.e.*

$$1-\frac{1}{\min\{p,q\}}||\frac{|f|^{p/2}}{||f||_p^{p/2}}-\frac{|g|^{q/2}}{||g||_q^{q/2}}||^2\leq\frac{||fg||_1}{||f||_p||g||_q}\leq$$

$$\leq 1-\frac{2}{\max\{p,q\}}\left(1-\frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}\right)=1-\frac{1}{\max\{p,q\}}||\frac{|f|^{p/2}}{||f||_p^{p/2}}-\frac{|g|^{q/2}}{||g||_q^{q/2}}||^2.$$

(iv) If we consider in (ii) the one variable functions f and g defined on $[a,b]$ and satisfying the conditions of Theorem 2.1, [3] *(Theorem 4) then we will obtain the inequality from Theorem 4.*

(v) If we consider in Theorem 5 and Theorem 6 respectively, the two variable functions $f(x,y)$ and $g(x,y)$ defined on $[a,b]\times[c,d]$ then the inequality become:

$$\frac{1}{p^{1/p}}\frac{1}{q^{1/q}}\frac{\int_a^b\int_c^d f(x,y)g(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}+\frac{1}{\max\{p,q\}}\cdot$$

$$\cdot[\frac{1}{p}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q}\left(\frac{t}{s}\right)^{1/p}-\frac{2}{\sqrt{pq}}\left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\frac{\int_a^b\int_c^d f^{p/2}(x,y)g^{q/2}(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}]\leq$$

$$\leq\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p},$$

and the reverses

$$\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p}\leq$$

$$\leq\frac{1}{p^{1/p}}\frac{1}{q^{1/q}}\frac{\int_a^b\int_c^d f(x,y)g(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}+\frac{1}{\min\{p,q\}}\cdot$$

$$\cdot[\frac{1}{p}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q}\left(\frac{t}{s}\right)^{1/p}-\frac{2}{\sqrt{pq}}\left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\frac{\int_a^b\int_c^d f^{p/2}(x,y)g^{q/2}(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}]$$

respectively.

(vi) Theorem 5 for example (and also Theorem 6) can be rewritten as below:

$$\frac{1}{p^{1/p}q^{1/q}}\frac{||fg||_1}{||f||_p||g||_q} \le \frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q} + \frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p} -$$

$$-\frac{1}{\max\{p,q\}}||\frac{1}{\sqrt{p}}\left(\frac{s}{t}\right)^{1/(2q)}\frac{|f|^{p/2}}{||f||_p^{p/2}} - \frac{1}{\sqrt{q}}\left(\frac{t}{s}\right)^{1/(2p)}\frac{|g|^{q/2}}{||g||_q^{q/2}}||_2^2.$$

The following result will present a variant of Minkowski's inequality using an additional positive parameter:

Theorem 7. *Let* $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p$ *and* $g \in L^q$ *with* $||f||_p > 0,\ ||g||_q > 0$ *then*

$$||f+g||_p \le p^{1/p}q^{1/q}[||f||_p(\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q} + \frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p} - \frac{1}{\max\{p,q\}}\cdot$$

$$\cdot||\frac{1}{\sqrt{p}}\left(\frac{s}{t}\right)^{1/(2q)}\frac{|f+g|^{p/2}}{||f+g||_p^{p/2}} - \frac{1}{\sqrt{q}}\left(\frac{t}{s}\right)^{1/(2p)}\frac{|f|^{p/2}}{||f||_p^{p/2}}||_2^2)+$$

$$+||g||_p(\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q} + \frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p} - \frac{1}{\max\{p,q\}}\cdot$$

$$\cdot||\frac{1}{\sqrt{p}}\left(\frac{s}{t}\right)^{1/(2q)}\frac{|f+g|^{p/2}}{||f+g||_p^{p/2}} - \frac{1}{\sqrt{q}}\left(\frac{t}{s}\right)^{1/(2p)}\frac{|g|^{p/2}}{||g||_p^{p/2}}||_2^2)].$$

Proof. We will use as in [1] that

$$||f+g||_p = \left(\int_\Omega |f+g|^p d\mu\right)^{\frac{1}{p}} = \int_\Omega \frac{|f+g|^{p-1}|f+g|}{||\ |f+g|^{p-1}||_q}d\mu \le$$

$$\le \int_\Omega \frac{|f+g|^{p-1}|f|}{||\ |f+g|^{p-1}||_q}d\mu + \int_\Omega \frac{|f+g|^{p-1}|g|}{||\ |f+g|^{p-1}||_q}d\mu$$

and Theorem 5.

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As an application, using now Remark 2 (v) in the proof of Theorem 3, see [8], we will obtain:

Proposition 1. *Let* $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta = [a,b] \times [c,d]$ *with* $a < b,\ c < d$. *If* $|\frac{\partial^2 f}{\partial s \partial t}|^q$ *is convex on the co-ordinates on* Δ *and* $p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality holds:*

$$|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f(\frac{a_b}{2}, \frac{c+d}{2}) - A| \le$$

$$\le \frac{(b-a)(d-c)}{4(p+1)^{2/p}}C(p,m,n)[\frac{|\frac{\partial^2}{\partial s\partial t}(a,c)|^q + |\frac{\partial^2}{\partial s\partial t}(a,d)|^q + |\frac{\partial^2}{\partial s\partial t}(b,c)|^q + |\frac{\partial^2}{\partial s\partial t}(b,d)|^q}{4}]^{1/q},$$

where

$$A = \frac{1}{b-a}\int_a^b f(x, \frac{c+d}{2})dx + \frac{1}{d-c}\int_c^d f(\frac{a+b}{2}, y)dy$$

and

$$C(p,m,n)=\frac{q^{\frac{1}{q}}}{p^{1+\frac{1}{q}}}\left(\frac{m}{n}\right)^{1/q}+\frac{p^{\frac{1}{p}}}{q^{1+\frac{1}{p}}}\left(\frac{n}{m}\right)^{1/p},$$

with $m,n>0$.

Also as applications we can think to use Remark 2 (v) (or (ii)) instead of Holder's inequality, in the proof of Theorem 4 and then Theorem 5, see [10].

Then we will have the following variant of Theorem 4 (see [10]):

Consequence 1. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$, p, P, $\hat{P}$, $\tilde{P}$, $\bar{P}$ *be as in Theorem 3([10]), then ,*

$$|f(x,y)-\frac{1}{P(a,c)}\int_a^b\int_c^d p(s,t)f(s,t)dsdt|\le M_1(x)+M_2(y)+M_3(x,y)$$

where

$$M_1(x)=\frac{p_1^{1/p_1}q_1^{1/q_1}}{|P(a,c)|}(\int_a^b|\hat{P}(x,s)|^{q_1}ds)^{1/q_1}||\frac{\partial f}{\partial s}||_{p_1}[\frac{1}{p_1^2}\left(\frac{m_1}{n_1}\right)^{\frac{1}{q_1}}+\frac{1}{q_1^2}\left(\frac{n_1}{m_1}\right)^{\frac{1}{p_1}}-$$

$$-\frac{1}{\max\{p_1,q_1\}}\left(\frac{1}{p_1}(\frac{m_1}{n_1})^{\frac{1}{q_1}}+\frac{1}{q_1}(\frac{n_1}{m_1})^{\frac{1}{p_1}}-\frac{2}{\sqrt{p_1q_1}}(\frac{m_1}{n_1})^{\frac{1}{2}(\frac{1}{q_1}-\frac{1}{p_1})}\frac{\int_a^b|\hat{P}(x,s)|^{\frac{q_1}{2}}|\frac{\partial f}{\partial s}|^{\frac{p_1}{2}}ds}{||\frac{\partial f}{\partial s}||_{p_1}^{\frac{p_1}{2}}(\int_a^b|\hat{P}(x,s)|^{q_1}ds)^{\frac{1}{2}}}\right)],$$

if $\frac{\partial f}{\partial s}(s,t)\in L_{p_1}([a,b]\times[c,d])$, $\frac{1}{p_1}+\frac{1}{q_1}=1$

$$M_2(y)=\frac{p_2^{1/p_2}q_2^{1/q_2}}{|P(a,c)|}(\int_c^d|\tilde{P}(y,t)|^{q_2}dt)^{1/q_2}||\frac{\partial f}{\partial t}||_{p_2}[\frac{1}{p_2^2}\left(\frac{m_2}{n_2}\right)^{\frac{1}{q_2}}+\frac{1}{q_2^2}\left(\frac{n_2}{m_2}\right)^{\frac{1}{p_2}}-$$

$$-\frac{1}{\max\{p_2,q_2\}}\left(\frac{1}{p_2}(\frac{m_2}{n_2})^{\frac{1}{q_2}}+\frac{1}{q_2}(\frac{n_2}{m_2})^{\frac{1}{p_2}}-\frac{2}{\sqrt{p_2q_2}}(\frac{m_2}{n_2})^{\frac{1}{2}(\frac{1}{q_2}-\frac{1}{p_2})}\frac{\int_c^d|\tilde{P}(y,t)|^{\frac{q_2}{2}}|\frac{\partial f}{\partial t}|^{\frac{p_2}{2}}dt}{||\frac{\partial f}{\partial t}||_{p_2}^{\frac{p_2}{2}}(\int_c^d|\tilde{P}(y,t)|^{q_2}dt)^{\frac{1}{2}}}\right)],$$

if $\frac{\partial f}{\partial t}(s,t)\in L_{p_2}([a,b]\times[c,d])$, $\frac{1}{p_2}+\frac{1}{q_2}=1$

$$M_3(x,y)=\frac{p_3^{1/p_3}q_3^{1/q_3}}{|P(a,c)|}(\int_a^b\int_c^d|\bar{P}(x,s,y,t)|^{q_3}dsdt)^{1/q_3}||\frac{\partial^2 f}{\partial s\partial t}||_{p_3}[\frac{1}{p_3^2}\left(\frac{m_3}{n_3}\right)^{\frac{1}{q_3}}+$$

$$+\frac{1}{q_3^2}\left(\frac{n_3}{m_3}\right)^{\frac{1}{p_3}}-\frac{1}{\max\{p_3,q_3\}}(\frac{1}{p_3}(\frac{m_3}{n_3})^{\frac{1}{q_3}}+\frac{1}{q_3}(\frac{n_3}{m_3})^{\frac{1}{p_3}}-$$

$$-\frac{2}{\sqrt{p_3q_3}}(\frac{m_3}{n_3})^{\frac{1}{2}(\frac{1}{q_3}-\frac{1}{p_3})}\frac{\int_a^b\int_c^d|\bar{P}(x,s,y,t)|^{\frac{q_3}{2}}|\frac{\partial^2 f}{\partial s\partial t}|^{\frac{p_3}{2}}dsdt}{||\frac{\partial^2 f}{\partial s\partial t}||_{p_3}^{\frac{p_3}{2}}(\int_a^b\int_c^d|\bar{P}(x,s,y,t)|^{q_3}dsdt)^{\frac{1}{2}}})],$$

if $\frac{\partial^2 f}{\partial s\partial t}(s,t)\in L_{p_3}([a,b]\times[c,d])$, $\frac{1}{p_3}+\frac{1}{q_3}=1$ *for all* $(x,y)\in[a,b]\times[c,d]$ *and* $m_1,\ n_1,\ m_2,\ n_2,\ m_3,\ n_3>0$.

Consequence 2. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be continuous on* $[a,b]\times[c,d]$, *and* $\frac{\partial^2 f}{\partial s\partial t}(s,t)$ *exist on* $(a,b)\times(c,d)$ *and* $|\frac{\partial^2 f}{\partial s\partial t}(s,t)|^p$ *an integrable function such that*

$$||\frac{\partial^2 f}{\partial s\partial t}(s,t)||_p=(\int_a^b\int_c^d|\frac{\partial^2 f}{\partial s\partial t}(s,t)|^p dsdt)^{\frac{1}{p}}<\infty$$

, i. e. the conditions from Theorem 5 ([10]). Then it follows that:

$$|\int_a^b\int_c^d p(s,t)f(s,t)dsdt-[\int_a^b\int_c^d p(s,t)f(x,t)dt+\int_a^b\int_c^d p(s,t)f(s,y)ds-f(x,y)P(a,c)]|\leq$$

$$\leq p^{\frac{1}{p}}q^{\frac{1}{q}}\left(\int_a^b\int_c^d|\bar{P}(x,s,y,t)|^q dsdt\right)^{\frac{1}{q}}||\frac{\partial^2 f}{\partial s\partial t}(s,t)||_p[\frac{1}{p^2}(\frac{m}{n})^{\frac{1}{q}}+\frac{1}{q^2}(\frac{n}{m})^{\frac{1}{p}}-$$

$$-\frac{1}{\sqrt{pq}}(\frac{1}{p}(\frac{m}{n})^{\frac{1}{q}}+\frac{1}{q}(\frac{n}{m})^{\frac{1}{p}}-\frac{2}{\sqrt{pq}}\left(\frac{m}{n}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\frac{\int_a^b\int_c^d|\bar{P}(x,s,y,t)|^{\frac{q}{2}}|\frac{\partial^2 f}{\partial s\partial t}(s,t)|^{\frac{q}{2}}dsdt}{(\int_a^b\int_c^d|\bar{P}(x,s,y,t)|^q dsdt)^{\frac{1}{2}}||\frac{\partial^2 f}{\partial s\partial t}(s,t)||_p^{\frac{p}{2}}})]$$

for all $(x,y)\in[a,b]\times[c,d]$ where $\frac{1}{p}+\frac{1}{q}=1$ and $m,\ n>0$.

Now we will improve Remark 2 (iii) using Specht's ratio.

Theorem 8. *We suppose that the two functions $f:[a,b]\to\mathbb{R}_+$ and $g:[a,b]\to\mathbb{R}_+$ satisfy $f\in L^p[a,b]$, $g\in L^q[a,b]$ with $p>1$, $\int_a^b f^p(x)dx>0$, $\int_a^b g^q(x)dx>0$ and $0<m\leq\frac{f^p(x)}{\int_a^b f^p(x)dx}\leq M$, $0<m\leq\frac{g^q(x)}{\int_a^b g^q(x)dx}\leq M$ where $m,M\in\mathbb{R}$. Then we have*

$$S(\sqrt{h})\frac{\int_a^b f(x)g(x)dx}{(\int_a^b f^p(x)dx)^{\frac{1}{p}}(\int_a^b g^q(x)dx)^{\frac{1}{q}}}\geq 1-2r(1-\frac{\int_a^b f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)dx}{(\int_a^b f^p(x)dx\int_a^b g^q(x)dx)^{\frac{1}{2}}})$$

where $h=\frac{M}{m}>1$ and $r=\frac{1}{\max\{p,q\}}$.

Proof. We will use the proof of Theorem 3.3 see [4]. In Lemma 3.1 see [4], we take $a=1$ and then for all $b>0$ we have

$$S(\sqrt{b})b^{\nu}\geq\nu b+(1-\nu)-r(\sqrt{b}-1)^2.$$

That means

$$\max_{m\leq t\leq M}S(\sqrt{t})b^{\nu}\geq\nu b+(1-\nu)-r(\sqrt{b}-1)^2,$$

for all $b\in[m,M]$. Taking now $b=\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}$ and using that

$$\frac{1}{h}=\frac{m}{M}\leq\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}\leq\frac{M}{m}=h>1$$

and the fact that $S(x)$ is monotone decreasing for $0<x<1$ and monotone increasing for $x>1$ and $S(h)=S(\frac{1}{h}),\ h\neq 1$ we obtain

$$S(\sqrt{h})\frac{(f^p(x))^{\nu}}{(\int_a^b f^p(x)dx)^{\nu}}\frac{(\int_a^b g^q(x)dx)^{\nu}}{(g^q(x))^{\nu}}\geq\nu\frac{f^p(x)}{g^q(x)}\frac{\int_a^b g^q(x)dx}{\int_a^b f^p(x)dx}+(1-\nu)-$$

$$-r\left(\sqrt{\frac{f^p(x)}{\int_a^b f^p(x)dx}}-\sqrt{\frac{g^q(x)}{\int_a^b g^q(x)dx}}\right)^2\frac{\int_a^b g^q(x)dx}{g^q(x)}.$$

If we put above $\nu=\frac{1}{p}$ previous inequality becomes

$$S(\sqrt{h})\frac{f(x)g(x)}{(\int_a^b f^p(x)dx)^{\frac{1}{p}}(\int_a^b g^q(x)dx)^{\frac{1}{q}}}\geq\frac{1}{p}\frac{f^p(x)}{\int_a^b f^p(x)dx}+\frac{1}{q}\frac{g^q(x)}{\int_a^b g^q(x)dx}-$$

$$-r\left(\sqrt{\frac{f^p(x)}{\int_a^b f^p(x)dx}}-\sqrt{\frac{g^q(x)}{\int_a^b g^q(x)dx}}\right)^2$$

and by calculus and integrating both sides of the inequality over a to b, we obtain the desired inequality.

∎

If we consider now two variable function the above result becomes:

Proposition 2. *(i) Previous result will be true also if the functions f and g are two variable functions defined on* $[a,b]\times[c,d]$ *the conditions of the classical Holder's inequality for two variable functions take place and* $0<m\le \frac{f^p(x,y)}{\int_a^b\int_c^d f^p(x,y)dydx}\le M$, $0<m\le \frac{g^q(x,y)}{\int_a^b\int_c^d g^q(x,y)dydx}\le M$ *where* $m,M\in\mathbb{R},\ p>1\ h=\frac{M}{m}>1,\ r=\frac{1}{\max\{p,q\}}$ *are satisfied:*

$$S(\sqrt{h})\frac{\int_a^b\int_c^d f(x,y)g(x,y)dydx}{(\int_a^b\int_c^d f^p(x,y)dydx)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dydx)^{\frac{1}{q}}}\ge$$

$$\ge 1-2r(1-\frac{\int_a^b\int_c^d f^{\frac{p}{2}}(x,y)g^{\frac{q}{2}}(x,y)dydx}{(\int_a^b\int_c^d f^p(x,y)dydx\int_a^b\int_c^d g^q(x,y)dydx)^{\frac{1}{2}}})$$

(ii) Moreover if we put in previous proof $b=\frac{q}{p}\left(\frac{s}{t}\right)\frac{f^p(x,y)}{\int_a^b\int_c^d f^p(x,y)dydx}\frac{\int_a^b\int_c^d g^q(x,y)dydx}{g^q(x,y)}$ *and* $0<mp\left(\frac{t}{s}\right)^{\frac{1}{q}}\le\frac{f^p(x,y)}{\int_a^b\int_c^d f^p(x,y)dydx}\le Mp\left(\frac{t}{s}\right)^{\frac{1}{q}}$, $0<mq\left(\frac{s}{t}\right)^{\frac{1}{p}}\le\frac{g^q(x,y)}{\int_a^b\int_c^d g^q(x,y)dydx}\le Mq\left(\frac{s}{t}\right)^{\frac{1}{p}}$ *where* $m,M\in\mathbb{R},\ s,t>0$ *then under the above conditions we have:*

$$S(\sqrt{h})\frac{s^{\frac{1}{p}}t^{\frac{1}{q}}}{p^{\frac{1}{p}}q^{\frac{1}{q}}}\frac{\int_a^b\int_c^d f(x,y)g(x,y)dydx}{(\int_a^b\int_c^d f^p(x,y)dydx)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dydx)^{\frac{1}{q}}}\ge$$

$$\ge\frac{1}{p^2}s+\frac{1}{q^2}t-r(\frac{1}{p}s+\frac{1}{q}t-2\sqrt{\frac{st}{pq}}\frac{\int_a^b\int_c^d f^{\frac{p}{2}}(x,y)g^{\frac{q}{2}}(x,y)dydx}{(\int_a^b\int_c^d f^p(x,y)dydx\int_a^b\int_c^d g^q(x,y)dydx)^{\frac{1}{2}}}).$$

(iii) Let $1<p<\infty,\ 1<q<\infty,\ \frac{1}{p}+\frac{1}{q}=1$. *If* $f\in L^p$ *and* $g\in L^q$ *with* $||f||_p>0,\ ||g||_q>0$ *with* $mp\left(\frac{t}{s}\right)^{\frac{1}{q}}||f||_p^p\le|f|^p\le Mp\left(\frac{t}{s}\right)^{\frac{1}{q}}q||f||_p^p$ *and* $mq\left(\frac{s}{t}\right)^{\frac{1}{p}}||g||_q^q\le|g|^q\le Mq\left(\frac{s}{t}\right)^{\frac{1}{p}}||g||_q^q$ *then*

$$S(\sqrt{h})\frac{s^{\frac{1}{p}}t^{\frac{1}{q}}}{p^{\frac{1}{p}}q^{\frac{1}{q}}}\frac{||\,|f||g|\,||_1}{||f||_p||g||_q}\ge$$

$$\ge\frac{s}{p^2}+\frac{t}{q^2}-r(\frac{1}{p}s+\frac{1}{q}t-2\sqrt{\frac{st}{pq}}\frac{\int_\Omega|f|^{\frac{p}{2}}|g|^{\frac{q}{2}}d\mu}{||f||_p^{\frac{p}{2}}||g||_q^{\frac{q}{2}}})$$

where $s,\ t>0$.

Using a similar method see, [4], and [11] we can prove the following result:

Proposition 3. *Let* $1<p<\infty,\ 1<q<\infty,\ \frac{1}{p}+\frac{1}{q}=1$. *If* $f\in L^p$ *and* $g\in L^q$ *with* $||f||_p>0,\ ||g||_q>0$ *with* $mp\left(\frac{t}{s}\right)^{\frac{1}{q}}||f||_p^p\le|f|^p\le Mp\left(\frac{t}{s}\right)^{\frac{1}{q}}||f||_p^p$ *and* $mq\left(\frac{s}{t}\right)^{\frac{1}{p}}||g||_q^q\le|g|^q\le Mq\left(\frac{s}{t}\right)^{\frac{1}{p}}||g||_q^q$ *then*

$$h\sqrt{M}\mathcal{L}(\sqrt{M},\sqrt{m})logS(\sqrt{h})\ge\frac{s}{p^2}+\frac{t}{q^2}-\frac{s^{\frac{1}{p}}t^{\frac{1}{q}}}{p^{\frac{1}{p}}q^{\frac{1}{q}}}\frac{||\,|f||g|\,||_1}{||f||_p||g||_q}-r(\frac{s}{p}+\frac{t}{q}-2\sqrt{\frac{st}{pq}}\frac{\int_\Omega|f|^{\frac{p}{2}}|g|^{\frac{q}{2}}d\mu}{||f||_p^{\frac{p}{2}}||g||^{\frac{q}{2}}})$$

where $\mathcal{L}(x,y) = \frac{y-x}{\log y - \log x}$, $x \neq y$ *is the logarithmic mean* (x, y *being two real positive numbers)* $h = \frac{M}{m} > 1$, s, $t > 0$ *and* $r = \frac{1}{\max\{p,q\}}$.

Now we will give several forms of Holder's inequality using the method from [12], Theorem 7.

Theorem 9. *Let* $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p$ *and* $g \in L^q$ *with* $||f||_p > 0$, $||g||_q > 0$ *and positive real numbers* m, $m^{'}$, M, $M^{'}$ *satisfy either of the following conditions: (i)* $0 < m^{'}p\left(\frac{t}{s}\right)^{\frac{1}{q}} \leq \frac{|f|^p}{||f||_p^p} \leq mp\left(\frac{t}{s}\right)^{\frac{1}{q}}$, $tmp \leq Mqs$, $Mq\left(\frac{s}{t}\right)^{\frac{1}{p}} \leq \frac{|g|^q}{||g||_q^q} \leq M^{'}q\left(\frac{s}{t}\right)^{\frac{1}{p}}$

(ii) $0 < m^{'}q\left(\frac{s}{t}\right)^{\frac{1}{p}} \leq \frac{|g|^q}{||g||_q^q} \leq mq\left(\frac{s}{t}\right)^{\frac{1}{p}}$, $mqs \leq Mpt$, $Mp\left(\frac{t}{s}\right)^{\frac{1}{q}} \leq \frac{|f|^p}{||f||_p^p} \leq M^{'}p\left(\frac{t}{s}\right)^{\frac{1}{q}}$

then

$$\frac{1}{p^2}s + \frac{1}{q^2}t \geq K(h,2)^r \frac{s^{\frac{1}{p}}t^{\frac{1}{q}}}{p^{\frac{1}{p}}q^{\frac{1}{q}}} \frac{||\,|f||g|\,||_1}{||g||_q||f||_p}$$

$h = \frac{M}{m}$, $h^{'} = \frac{M^{'}}{m^{'}}$, $K(h,2) = \frac{(t+2)^2}{4t}$, $t > 0$ *is the Kantorovici constant(see* [12]*) and* s, $t > 0$.

Proof. Using Corollary 3, see [12], putting $x = \frac{|g|^q}{||g||_q^q}\frac{||f||_p^p}{|f|^p}\frac{pt}{qs}$ in

$$(1-\nu) + \nu x \geq \min_{h \leq x \leq h^{'}} K(x,2)^r x^{\nu}, \ (\forall) x > 0$$

and taking into account the hypothesis and that $K(x,2)$ is an increasing function for $x > 1$ and decreasing for $0 < x < 1$ and $K(h,2)^r = K(\frac{1}{h},2)^r$ we have for $\nu = \frac{1}{q}$:

$$\frac{1}{p^2}s\frac{|f|^p}{||f||_p^p} + \frac{1}{q^2}t\frac{|g|^q}{||g||_q^q} \geq K(h,2)^r \frac{s^{\frac{1}{p}}t^{\frac{1}{q}}}{p^{\frac{1}{p}}q^{\frac{1}{q}}} \frac{|g||f|}{||g||_q||f||_p}.$$

By integration we obtain the desired result. ∎

As a particular case we obtain:

Consequence 3. *Let* $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p$ *and* $g \in L^q$ *with* $||f||_p > 0$, $||g||_q > 0$ *and positive real numbers* m, $m^{'}$, M, $M^{'}$ *satisfy either of the following conditions: (i)* $0 < m^{'} \leq \frac{|f|^p}{||f||_p^p} \leq m \leq M \leq \frac{|g|^q}{||g||_q^q} \leq M^{'}$

(ii) $0 < m^{'} \leq \frac{|g|^q}{||g||_q^q} \leq m \leq M \leq \frac{|f|^p}{||f||_p^p} \leq M^{'}$

then

$$1 \geq K(h,2)^r \frac{||\,|f||g|\,||_1}{||g||_q||f||_p}$$

$h = \frac{M}{m}$, $h^{'} = \frac{M^{'}}{m^{'}}$ *and* $K(t,2)$ *is the Kantorovici constant as in* [12] *i.e* $K(t,2) = \frac{(t+2)^2}{4t}$, $t > 0$.

Using now the demonstration of Theorem 2.3 see [5] we find the following result:

Theorem 10. *Let* $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p}+\frac{1}{q}=1$. *If* $f \in L^p$ *and* $g \in L^q$ *with* $||f||_p > 0,\ ||g||_q > 0$ *and positive real numbers* $m,\ M$ *satisfying* $mp\left(\frac{t}{s}\right)^{\frac{1}{q}} \leq \frac{|f|^p}{||f||_p^p} \leq \frac{|g|^q}{||g||_q^q} \leq Mq\left(\frac{s}{t}\right)^{\frac{1}{p}} \leq 1$ *with* $h=\frac{M}{m}$ *we have the following inequalities:*

(i) (Ratio-type reverse inequality)

$$\frac{s}{p^2}+\frac{t}{q^2} \leq \frac{t^{\frac{1}{q}}s^{\frac{1}{p}}}{q^{\frac{1}{q}}p^{\frac{1}{p}}}\frac{||\,|f||g|\,||_1}{||f||_p||g||_q}\exp\frac{1}{qp}(1-\frac{1}{h})^2$$

(ii) (Difference-type reverse inequality)

$$\frac{s}{p^2}+\frac{t}{q^2}-\frac{t^{\frac{1}{q}}s^{\frac{1}{p}}}{q^{\frac{1}{q}}p^{\frac{1}{p}}}\frac{||\,|f||g|\,||_1}{||f||_p||g||_q} \leq \frac{1}{pq}(\log h)^2.$$

Proof. (i) From (10), Corollary 2.2, see [5], with $a \leq b$ we obtain $(1-\lambda)t_1+\lambda \leq t_1^{1-\lambda}\exp\lambda(1-\lambda)(1-\frac{1}{t_1})^2$ for $0 < t_1 \leq 1$. So for t_1 which satisfies $0 < m \leq t_1 \leq M \leq 1$ results

$$(1-\lambda)t_1+\lambda \leq t_1^{1-\lambda}\max_{m\leq t_1\leq M} e^{\lambda(1-\lambda)(1-\frac{1}{t_1})^2}.$$

If we put $t_1=\frac{q}{p}\frac{s}{t}\frac{|f|^p||g||_q^q}{|g|^q||f||_p^p}$ and $\lambda=\frac{1}{q}$ then $\frac{1}{h}=\frac{m}{M}\leq t_1 \leq \frac{M}{n}=h$ and after calculus inequality becomes:

$$\frac{s}{p^2}\frac{|f|^p}{||f||_p^p}+\frac{t}{q^2}\frac{|g|^q}{||g||_q^q} \leq \frac{t^{\frac{1}{q}}s^{\frac{1}{p}}}{q^{\frac{1}{q}}p^{\frac{1}{p}}}\frac{|f||g|}{||f||_p||g||_q}e^{\frac{1}{pq}}(1-\frac{1}{h})^2$$

which by integration leads to the desired inequality. For (ii) we use (11) Corollary 2.2, see [5]. ∎

Consequence 4. *Let* $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p}+\frac{1}{q}=1$. *If* $f \in L^p$ *and* $g \in L^q$ *with* $||f||_p > 0,\ ||g||_q > 0$ *and positive real numbers* $m,\ M$ *satisfying* $m \leq \frac{|f|^p}{||f||_p^p} \leq \frac{|g|^q}{||g||_q^q} \leq M \leq 1$ *with* $h=\frac{M}{m}$ *we have the following inequalities:*

(i) (Ratio-type reverse inequality)

$$1 \leq \frac{||\,|f||g|\,||_1}{||f||_p||g||_q}\exp\frac{1}{qp}(1-\frac{1}{h})^2$$

(ii) (Difference-type reverse inequality)

$$1-\frac{||\,|f||g|\,||_1}{||f||_p||g||_q} \leq \frac{1}{pq}(\log h)^2.$$

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