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APPLICATIONS OF KATO'S INEQUALITY FOR n-TUPLES OF OPERATORS IN HILBERT SPACES, (I)

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Abstract. In this paper, by the use of the famous Kato's inequality for bounded linear operators, we establish some inequalities for n-tuples of operators and apply them for functions of normal operators defined by power series as well as for some norms and numerical radii that arise in multivariate Operator Theory.

1. Introduction

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [16, p. 240]:

If the operator $A \in \mathcal{B}(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$. If A is invertible, then so is B.

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A| := \sqrt{A^*A}$.

In 1952, Kato [17] proved the following generalization of Schwarz inequality:

$$|\langle Tx, y\rangle|^2 \le \langle (T^*T)^\alpha x, x\rangle \left\langle (TT^*)^{1-\alpha} y, y\right\rangle, \tag{1.1}$$

for any $x, y \in H$, $\alpha \in [0,1]$ and T is a bounded linear operator on H.

Utilizing the modulus notation introduced before, we can write (1.1) as follows

$$|\langle Tx, y\rangle|^2 \le \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle. \tag{1.2}$$

For related results to the Kato's inequality, see [6]-[15], [17]-[23] and [31].

In the recent paper [3], by employing Kato's inequality (1.2), Dragomir established the following results for sequences of bonded linear operators on complex Hilbert spaces:

Theorem 1. *Let $(T_1, ..., T_n) \in \mathcal{B}(H) \times ... \times \mathcal{B}(H) := \mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ an n-tuple of nonnegative weights not all of them equal to zero. Then we have*

$$\sum_{j=1}^n p_j |\langle T_j x, y\rangle|^2 \le \left\langle \sum_{j=1}^n p_j |T_j|^2 x, x\right\rangle^\alpha \left\langle \sum_{j=1}^n p_j \left|T_j^*\right|^2 y, y\right\rangle^{1-\alpha} \tag{1.3}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

He also obtained the result:

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Theorem 2. *With the assumptions in Theorem 1 we have*

$$\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right| \le \left\langle \sum_{j=1}^{n} p_j \left|T_j\right|^{2\alpha} x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right|^{2(1-\alpha)} y, y\right\rangle^{1/2} \tag{1.4}$$

for any $x, y \in H$.

For various related results, see the papers [1], [24]-[28] and [4]-[5].

Motivated by the above results, we establish in this paper other similar inequalities for n-tuples of bounded linear operators that can be obtained from the Kato's result (1.2) and apply them for functions of normal operators defined by power series as well as for some norms and numerical radii that can be associated with this n-tuples of bonded linear operators on Hilbert spaces.

2. Some Inequalities for n-Tuple of Linear Operators

Employing Kato's inequality (1.2) we can state the following new result:

Theorem 3. *Let* $(T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$ *be an* n*-tuple of bounded linear operators on the Hilbert space* $(H; \langle\cdot,\cdot\rangle)$ *and* $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ *an* n*-tuple of nonnegative weights not all of them equal to zero. Then we have*

$$\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right| \le \left\langle \sum_{j=1}^{n} p_j \left(\frac{\left|T_j\right|^{2\alpha} + \left|T_j\right|^{2(1-\alpha)}}{2}\right) x, x\right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n} p_j \left(\frac{\left|T_j^*\right|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2}\right) y, y\right\rangle^{1/2} \tag{2.1}$$

for any $x, y \in H, \alpha \in [0,1]$ *and, in particular, for* $\alpha = \frac{1}{2}$

$$\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right| \le \left\langle \sum_{j=1}^{n} p_j \left|T_j\right| x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right| y, y\right\rangle^{1/2} \tag{2.2}$$

for any $x, y \in H$.

Proof. Utilising Kato's inequality we have

$$\left|\langle T_j x, y\rangle\right| \le \left\langle \left|T_j\right|^{2\alpha} x, x\right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y\right\rangle^{1/2}$$

and, by replacing α with $1-\alpha$,

$$\left|\langle T_j x, y\rangle\right| \le \left\langle \left|T_j\right|^{2(1-\alpha)} x, x\right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2\alpha} y, y\right\rangle^{1/2},$$

which, by summation gives

$$\left|\langle T_j x, y\rangle\right| \le \frac{1}{2}\left[\left\langle \left|T_j\right|^{2\alpha} x, x\right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y\right\rangle^{1/2} + \left\langle \left|T_j\right|^{2(1-\alpha)} x, x\right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2\alpha} y, y\right\rangle^{1/2}\right] \tag{2.3}$$

for any $j \in \{1, ..., n\}$ and $x, y \in H$.

By the elementary inequality

$$ab + cd \le \left(a^2 + c^2\right)^{1/2} \left(b^2 + d^2\right)^{1/2}, a, b, c, d \ge 0 \tag{2.4}$$

we have

$$\left[\left\langle |T_j|^{2\alpha} x, x\right\rangle^{1/2} \left\langle |T_j^*|^{2(1-\alpha)} y, y\right\rangle^{1/2} + \left\langle |T_j|^{2(1-\alpha)} x, x\right\rangle^{1/2} \left\langle |T_j^*|^{2\alpha} y, y\right\rangle^{1/2}\right]$$
$$\leq \left[\left\langle \left(|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}\right) x, x\right\rangle\right]^{1/2} \left[\left\langle \left(|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}\right) y, y\right\rangle\right]^{1/2},$$

which, by (2.3), produces

$$|\langle T_j x, y\rangle| \leq \left\langle \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2}\right) x, x\right\rangle^{1/2} \times \left\langle \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right) y, y\right\rangle^{1/2} \tag{2.5}$$

for any $j \in \{1, ..., n\}$ and $x, y \in H$.

Multiplying the inequalities (2.5) with the positive weights p_j, summing over j from 1 to n and utilizing the weighted Cauchy-Buniakowski-Schwarz inequality

$$\sum_{j=1}^{n} p_j a_j b_j \leq \left(\sum_{j=1}^{n} p_j a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j b_j^2\right)^{1/2}$$

where $(a_1, ..., a_n), (b_1, ..., b_n) \in \mathbb{R}_+^n$, we have

$$\begin{aligned} \sum_{j=1}^{n} p_j |\langle T_j x, y\rangle| &\leq \sum_{j=1}^{n} p_j \left(\left\langle \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2}\right) x, x\right\rangle^{1/2} \right. \\ &\quad \left. \times \left\langle \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right) y, y\right\rangle^{1/2}\right) \\ &\leq \left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2}\right) x, x\right\rangle^{1/2} \\ &\quad \times \left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right) y, y\right\rangle^{1/2} \end{aligned} \tag{2.6}$$

for any $j \in \{1, ..., n\}$ and $x, y \in H$, and the inequality in (2.1) is proved. □

For vectors of norm one, the second inequality from (2.1) and (2.2) can be refined as follows:

Remark 1. *With the assumptions in Theorem 3 we have*

$$\begin{aligned}
(2.7)\qquad \sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right| &\le \left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2}\right) x, x\right\rangle^{1/2} \\
&\quad \times \left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right) y, y\right\rangle^{1/2} \\
&\le \left\langle \left[\sum_{j=1}^{n} p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2}\right)\right]^{1/2} x, x\right\rangle \\
&\quad \times \left\langle \left[\sum_{j=1}^{n} p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right)\right]^{1/2} y, y\right\rangle \\
&\le \frac{1}{2}\left[\left\langle \left[\sum_{j=1}^{n} p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2}\right)\right]^{1/2} x, x\right\rangle^2\right. \\
&\quad \left. + \left\langle \left[\sum_{j=1}^{n} p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right)\right]^{1/2} y, y\right\rangle^2\right] \\
&\le \frac{1}{2}\left[\left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2}\right) x, x\right\rangle\right. \\
&\quad \left. + \left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right) y, y\right\rangle\right]
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\begin{aligned}
(2.8)\qquad & \sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right| \\
&\le \left\langle \sum_{j=1}^{n} p_j |T_j| x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |T_j^*| y, y\right\rangle^{1/2} \\
&\le \left\langle \left(\sum_{j=1}^{n} p_j |T_j|\right)^{1/2} x, x\right\rangle \left\langle \left(\sum_{j=1}^{n} p_j |T_j^*|\right)^{1/2} y, y\right\rangle
\end{aligned}$$

$$\leq \frac{1}{2}\left[\left\langle \left(\sum_{j=1}^{n} p_j \left|T_j\right|\right)^{1/2} x, x\right\rangle^2 + \left\langle \left(\sum_{j=1}^{n} p_j \left|T_j^*\right|\right)^{1/2} y, y\right\rangle^2\right]$$
$$\leq \frac{1}{2}\left[\left\langle \sum_{j=1}^{n} p_j \left|T_j\right| x, x\right\rangle + \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right| y, y\right\rangle\right]$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

The proof follow by utilizing the Hölder-McCarthy inequalities (see for instance [29]) $\langle P^r x, x\rangle \leq \langle Px, x\rangle^r$ and $\langle Px, x\rangle^s \leq \langle P^s x, x\rangle$ that hold for the positive operator P, for $r \in (0,1)$, $s \in [1, \infty)$ and $x \in H$ with $\|x\| = 1$. The details are omitted.

Remark 2. *We observe also that the choice $y = x$ in the inequality (2.8) produces the result*

$$\sum_{j=1}^{n} \left|\langle T_j x, x\rangle\right| \leq \left\langle \sum_{j=1}^{n} \left|T_j\right| x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} \left|T_j^*\right| x, x\right\rangle^{1/2} \tag{2.9}$$
$$\leq \left\langle \left(\sum_{j=1}^{n} \left|T_j\right|\right)^{1/2} x, x\right\rangle \left\langle \left(\sum_{j=1}^{n} \left|T_j^*\right|\right)^{1/2} x, x\right\rangle$$
$$\leq \frac{1}{2}\left[\left\langle \left(\sum_{j=1}^{n} \left|T_j\right|\right)^{1/2} x, x\right\rangle^2 + \left\langle \left(\sum_{j=1}^{n} \left|T_j^*\right|\right)^{1/2} x, x\right\rangle^2\right]$$
$$\leq \left\langle \sum_{j=1}^{n} \left[\frac{\left|T_j\right| + \left|T_j^*\right|}{2}\right] x, x\right\rangle$$

for any $\in H$ with $\|x\| = 1$.

Remark 3. *In order to provide some applications for functions of normal operators defined by power series, we need to state the inequality (2.1) for normal operators $N_j, j \in \{1, ..., n\}$, namely*

$$\sum_{j=1}^{n} p_j \left|\langle N_j x, y\rangle\right| \leq \left\langle \sum_{j=1}^{n} p_j \left(\frac{\left|N_j\right|^{2\alpha} + \left|N_j\right|^{2(1-\alpha)}}{2}\right) x, x\right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n} p_j \left(\frac{\left|N_j\right|^{2\alpha} + \left|N_j\right|^{2(1-\alpha)}}{2}\right) y, y\right\rangle^{1/2} \tag{2.10}$$

for any $\alpha \in [0,1]$ and for any $x, y \in H$.

From a different perspective that involves quadratics, we can state the following result as well:

Theorem 4. *Let $(T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ an n-tuple of nonnegative* *weights not all of them equal to zero. Then we have*

$$\begin{aligned}\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right|^2 & \\ \leq \frac{1}{2}\sum_{j=1}^{n} p_j \left(\|T_j x\|^{2\alpha}\left\|T_j^* y\right\|^{2(1-\alpha)} + \left\|T_j^* y\right\|^{2\alpha}\|T_j x\|^{2(1-\alpha)}\right) & \\ \leq \frac{1}{2}\left[\left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2\right)^{1-\alpha}\right. & \\ \left. + \left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{1-\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2\right)^{\alpha}\right] & \\ \leq \frac{1}{2}\sum_{j=1}^{n} p_j \left(\|T_j x\|^2 + \left\|T_j^* y\right\|^2\right) & \end{aligned} \tag{2.11}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Proof. We must prove the inequalities only in the case $\alpha \in (0,1)$, since the case $\alpha = 0$ or $\alpha = 1$ follows directly from the corresponding case of Kato's inequality.

Utilizing Kato's inequality for the operator $T_j, j \in \{1, ..., n\}$ we have

$$|\langle T_j x, y\rangle|^2 \leq \left\langle |T_j|^{2\alpha} x, x\right\rangle \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y\right\rangle \tag{2.12}$$

and, by replacing α with $1-\alpha$,

$$|\langle T_j x, y\rangle|^2 \leq \left\langle |T_j|^{2(1-\alpha)} x, x\right\rangle \left\langle \left|T_j^*\right|^{2\alpha} y, y\right\rangle, \tag{2.13}$$

for any $x, y \in H$.

By Hölder-McCarthy inequalities $\langle P^r x, x\rangle \leq \langle Px, x\rangle^r$, that holds for the positive operator P, for $r \in (0,1)$ and $x \in H$ with $\|x\| = 1$ we also have

$$\left\langle |T_j|^{2\alpha} x, x\right\rangle \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y\right\rangle \leq \left\langle |T_j|^2 x, x\right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y\right\rangle^{1-\alpha} \tag{2.14}$$

and

$$\left\langle |T_j|^{2(1-\alpha)} x, x\right\rangle \left\langle \left|T_j^*\right|^{2\alpha} y, y\right\rangle \leq \left\langle |T_j|^2 x, x\right\rangle^{1-\alpha} \left\langle \left|T_j^*\right|^2 y, y\right\rangle^{\alpha} \tag{2.15}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1, j \in \{1, ..., n\}$ and $\alpha \in (0,1)$.

If we add (2.12) with (2.13) and make use of (2.14) and (2.15), we deduce

$$2\left|\langle T_j x, y\rangle\right|^2 \leq \left\langle |T_j|^2 x, x\right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y\right\rangle^{1-\alpha} + \left\langle \left|T_j^*\right|^2 y, y\right\rangle^{\alpha} \left\langle |T_j|^2 x, x\right\rangle^{1-\alpha} \tag{2.16}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1, j \in \{1, ..., n\}$ and $\alpha \in (0,1)$.

Now, if we multiply (2.16) with $p_j \geq 0$, sum over j from 1 to n we get

$$\begin{aligned} 2\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right|^2 \leq & \sum_{j=1}^{n} p_j \left\langle |T_j|^2 x, x\right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y\right\rangle^{1-\alpha} \\ & + \sum_{j=1}^{n} p_j \left\langle \left|T_j^*\right|^2 y, y\right\rangle^{\alpha} \left\langle |T_j|^2 x, x\right\rangle^{1-\alpha} \end{aligned} \tag{2.17}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and $\alpha\in(0,1)$.

Since $\left\langle |T_j|^2 x,x\right\rangle = \|T_jx\|^2$ and $\left\langle |T_j^*|^2 y,y\right\rangle = \|T_j^*y\|^2$, $j\in\{1,...,n\}$, then we get from (2.17) the first inequality in (2.11).

Now, on making use of the weighted Hölder discrete inequality

$$\sum_{j=1}^n p_ja_jb_j \le \left(\sum_{j=1}^n p_ja_j^p\right)^{1/p}\left(\sum_{j=1}^n p_jb_j^q\right)^{1/q}, p,q>1, \frac{1}{p}+\frac{1}{q}=1,$$

where $(a_1,...,a_n),(b_1,...,b_n)\in\mathbb{R}_+^n$, we also have

$$\sum_{j=1}^n p_j\|T_jx\|^{2\alpha}\left\|T_j^*y\right\|^{2(1-\alpha)} \le \left(\sum_{j=1}^n p_j\|T_jx\|^2\right)^{\alpha}\left(\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2\right)^{1-\alpha}$$

and

$$\sum_{j=1}^n p_j\left\|T_j^*y\right\|^{2\alpha}\|T_jx\|^{2(1-\alpha)} \le \left(\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2\right)^{\alpha}\left(\sum_{j=1}^n p_j\|T_jx\|^2\right)^{1-\alpha}.$$

Summing these two inequalities we deduce the second inequality in (2.11).

Finally, on utilizing the Hölder's inequality

$$ab+cd\le(a^p+c^p)^{1/p}(b^q+d^q)^{1/q}, a,b,c,d\ge0$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left(\sum_{j=1}^n p_j\|T_jx\|^2\right)^{\alpha}\left(\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2\right)^{1-\alpha}+\left(\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2\right)^{\alpha}\left(\sum_{j=1}^n p_j\|T_jx\|^2\right)^{1-\alpha}$$
$$\le\left(\sum_{j=1}^n p_j\|T_jx\|^2+\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2\right)^{\alpha}\left(\sum_{j=1}^n p_j\|T_jx\|^2+\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2\right)^{1-\alpha}$$
$$=\sum_{j=1}^n p_j\|T_jx\|^2+\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2.$$

and the proof is concluded. □

Remark 4. *For $\alpha=\frac{1}{2}$ we get from (2.11) that*

$$\sum_{j=1}^n p_j|\langle T_jx,y\rangle|^2 \tag{2.18}$$
$$\le\sum_{j=1}^n p_j\|T_jx\|\left\|T_j^*y\right\|\le\left(\sum_{j=1}^n p_j\|T_jx\|^2\right)^{1/2}\left(\sum_{j=1}^n p_j\left\|T_j^*y\right\|^2\right)^{1/2}$$
$$\le\frac{1}{2}\sum_{j=1}^n p_j\left(\|T_jx\|^2+\left\|T_j^*y\right\|^2\right)$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

3. Inequalities for Functions of Normal Operators

Now, by the help of power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n\geq 0$, then $f_A=f$.

As some natural examples that are useful for applications, we can point out that, if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\ z\in D(0,1); \tag{3.1}$$
$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C};$$
$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ z\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1); \tag{3.2}$$
$$g_A(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$
$$h_A(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$
$$l_A(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).$$

The following result is a functional inequality for normal operators that can be obtained from (2.1).

Theorem 5. *Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If N is a normal operator on the Hilbert space H and for $\alpha\in(0,1)$ we have that $\|N\|^{2\alpha},\|N\|^{2(1-\alpha)}<R$, then we have the inequalities*

$$|\langle f(N)x,y\rangle|\leq\frac{1}{2}\left\langle\left[f_A\left(|N|^{2\alpha}\right)+f_A\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \times\left\langle\left[f_A\left(|N|^{2\alpha}\right)+f_A\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2} \tag{3.3}$$

for any $x,y\in H$.

In particular, if $\|N\|<R$, then

$$|\langle f(N)x,y\rangle|\leq\langle f_A(|N|)x,x\rangle^{1/2}\langle f_A(|N|)y,y\rangle^{1/2} \tag{3.4}$$

for any $x,y\in H$.

Proof. If N is a normal operator, then for any $j \in \mathbb{N}$ we have that

$$\left|N^{j}\right|^{2}=\left(N^{*} N\right)^{j}=|N|^{2 j}.$$

Now, utilizing the inequality (2.12) we can write that

$$\begin{aligned}
&\left|\left\langle\sum_{j=0}^{n} a_{j} N^{j} x, y\right\rangle\right| \\
&\leq \sum_{j=0}^{n}\left|a_{j}\right|\left|\left\langle N^{j} x, y\right\rangle\right| \\
&\leq\left\langle\sum_{j=0}^{n}\left|a_{j}\right|\left(\frac{|N|^{2 j \alpha}+|N|^{2 j(1-\alpha)}}{2}\right) x, x\right\rangle^{1 / 2} \\
&\times\left\langle\sum_{j=0}^{n}\left|a_{j}\right|\left(\frac{|N|^{2 j \alpha}+|N|^{2 j(1-\alpha)}}{2}\right) y, y\right\rangle^{1 / 2}
\end{aligned} \tag{3.5}$$

for any $x, y \in H$ and $n \in \mathbb{N}$.

Since $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then it follows that the series $\sum_{j=0}^{\infty}\left|a_{j}\right|\left(|N|^{2 \alpha}\right)^{j}$ and $\sum_{j=0}^{\infty}\left|a_{j}\right|\left(|N|^{2(1-\alpha)}\right)^{j}$ are absolute convergent in $\mathcal{B}(H)$, and by taking the limit over $n \rightarrow \infty$ in (3.5) we deduce the desired result (3.3). □

Remark 5. *With the assumptions in Theorem 5, if we take the supremum over $y \in H, \|y\|=1$, then we get the vector inequality*

$$\begin{aligned}
\|f(N) x\| \leq & \frac{1}{2}\left\langle\left[f_{A}\left(|N|^{2 \alpha}\right)+f_{A}\left(|N|^{2(1-\alpha)}\right)\right] x, x\right\rangle^{1 / 2} \\
& \times\left\|f_{A}\left(|N|^{2 \alpha}\right)+f_{A}\left(|N|^{2(1-\alpha)}\right)\right\|
\end{aligned} \tag{3.6}$$

for any $x \in H$, which in its turn produces the norm inequality

$$\|f(N)\| \leq \frac{1}{2}\left\|f_{A}\left(|N|^{2 \alpha}\right)+f_{A}\left(|N|^{2(1-\alpha)}\right)\right\| \tag{3.7}$$

for any $\alpha \in [0,1]$.

Moreover, if we take $y = x$ in (3.3), then we have

$$|\langle f(N) x, x\rangle| \leq \frac{1}{2}\left\langle\left[f_{A}\left(|N|^{2 \alpha}\right)+f_{A}\left(|N|^{2(1-\alpha)}\right)\right] x, x\right\rangle \tag{3.8}$$

for any $x \in H$, which, by taking the supremum over $x \in H, \|x\|=1$ generates the numerical radius inequality

$$w(f(N)) \leq \frac{1}{2} w\left[f_{A}\left(|N|^{2 \alpha}\right)+f_{A}\left(|N|^{2(1-\alpha)}\right)\right] \tag{3.9}$$

for any $\alpha \in [0,1]$.

Making use of the examples in (3.1) and (3.2) we can state the vector inequalities:

$$\begin{aligned}(3.10)\qquad & \left|\left\langle \ln\left(1_H+N\right)^{-1}x,y\right\rangle\right| \\ & \leq \frac{1}{2}\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]x,x\right\rangle^{1/2} \\ & \times\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]y,y\right\rangle^{1/2},\end{aligned}$$

and

$$\begin{aligned}(3.11)\qquad & \left|\left\langle \left(1_H+N\right)^{-1}x,y\right\rangle\right| \\ & \leq \frac{1}{2}\left\langle\left[\left(1_H-|N|^{2\alpha}\right)^{-1}+\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]x,x\right\rangle^{1/2} \\ & \times\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]y,y\right\rangle^{1/2},\end{aligned}$$

for any $x,y\in H$ and $\|N\|<1$.

We also have the inequalities

$$\begin{aligned}(3.12)\qquad \left|\left\langle \sin\left(N\right)x,y\right\rangle\right| & \leq \frac{1}{2}\left\langle\left[\sinh\left(|N|^{2\alpha}\right)+\sinh\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \\ & \times\left\langle\left[\sinh\left(|N|^{2\alpha}\right)+\sinh\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned}$$

and

$$\begin{aligned}(3.13)\qquad \left|\left\langle \cos\left(N\right)x,y\right\rangle\right| & \leq \frac{1}{2}\left\langle\left[\cosh\left(|N|^{2\alpha}\right)+\cosh\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \\ & \times\left\langle\left[\cosh\left(|N|^{2\alpha}\right)+\cosh\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned}$$

for any $x,y\in H$ and N a normal operator.

If we utilize the following function as power series representations with nonnegative coefficients:

$$\begin{aligned}(3.14)\qquad \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) & = \sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \qquad z\in D(0,1); \\ \sin^{-1}(z) & = \sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\left(2n+1\right)n!}z^{2n+1}, \qquad z\in D(0,1); \\ \tanh^{-1}(z) & = \sum_{n=0}^{\infty}\frac{1}{2n+1}z^{2n-1}, \qquad z\in D(0,1); \\ {}_2F_1(\alpha,\beta,\gamma,z) & = \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n, \alpha,\beta,\gamma>0, \\ & z\in D(0,1);\end{aligned}$$

where Γ is the *Gamma function*, then we can state the following vector inequalities:

$$|\langle \exp(N)\,x,y\rangle| \le \frac{1}{2}\left\langle \left[\exp\left(|N|^{2\alpha}\right)+\exp\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \times \left\langle \left[\exp\left(|N|^{2\alpha}\right)+\exp\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2} \tag{3.15}$$

for any $x,y\in H$ and N a normal operator.

If $\|N\|<1$, then we also have the inequalities

$$\begin{aligned}&\left|\left\langle \ln\left(\frac{1_H+N}{1_H-N}\right)x,y\right\rangle\right| \\ &\le \frac{1}{2}\left\langle\left[\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)+\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right]x,x\right\rangle^{1/2} \\ &\times\left\langle\left[\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)+\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right]y,y\right\rangle^{1/2}\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\left|\left\langle \tanh^{-1}(N)\,x,y\right\rangle\right| \\ &\le \frac{1}{2}\left\langle\left[\tanh^{-1}\left(|N|^{2\alpha}\right)+\tanh^{-1}\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \\ &\times\left\langle\left[\tanh^{-1}\left(|N|^{2\alpha}\right)+\tanh^{-1}\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&|\langle {}_2F_1(\alpha,\beta,\gamma,N)\,x,y\rangle| \\ &\le \frac{1}{2}\left\langle\left[{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2\alpha}\right)+{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \\ &\times\left\langle\left[{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2\alpha}\right)+{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned} \tag{3.18}$$

for any $x,y\in H$.

From a different perspective, we also have:

Theorem 6. *With the assumption of Theorem 5 and if N is a normal operator on the Hilbert space H and $z\in\mathbb{C}$ such that $\|N\|^2,|z|^2<R$, then we have the inequalities*

$$\begin{aligned}|\langle f(zN)\,x,y\rangle|^2 &\le \frac{1}{2}f_A\left(|z|^2\right)\left[\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{\alpha}\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{1-\alpha}\right. \\ &\qquad\left.+\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{1-\alpha}\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{\alpha}\right] \\ &\le \frac{1}{2}f_A\left(|z|^2\right)\left(\left\langle f_A\left(|N|^2\right)x,x\right\rangle+\left\langle f_A\left(|N|^2\right)y,y\right\rangle\right)\end{aligned} \tag{3.19}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and $\alpha\in[0,1]$.

In particular, for $\alpha=\frac{1}{2}$ we have

$$\begin{aligned}|\langle f(zN)\,x,y\rangle|^2 &\le f_A\left(|z|^2\right)\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{1/2}\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{1/2} \\ &\le \frac{1}{2}f_A\left(|z|^2\right)\left(\left\langle f_A\left(|N|^2\right)x,x\right\rangle+\left\langle f_A\left(|N|^2\right)y,y\right\rangle\right)\end{aligned} \tag{3.20}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

Proof. If we use the second and third inequality from (2.11) for powers of operators we have

$$\sum_{j=0}^{n} |a_j| \left|\left\langle N^j x, y\right\rangle\right|^2 \tag{3.21}$$
$$\leq \frac{1}{2}\left[\left(\sum_{j=0}^{n} |a_j| \left\|N^j x\right\|^2\right)^{\alpha} \left(\sum_{j=0}^{n} |a_j| \left\|(N^*)^j y\right\|^2\right)^{1-\alpha} + \left(\sum_{j=0}^{n} |a_j| \left\|N^j x\right\|^2\right)^{1-\alpha} \left(\sum_{j=0}^{n} |a_j| \left\|(N^*)^j y\right\|^2\right)^{\alpha}\right]$$
$$\leq \frac{1}{2}\sum_{j=0}^{n} |a_j| \left(\left\|N^j x\right\|^2 + \left\|(N^*)^j y\right\|^2\right)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Since N is a normal operator on the Hilbert space H, then

$$\left\|N^j x\right\|^2 = \left\langle \left|N^j\right|^2 x, x\right\rangle = \left\langle |N|^{2j} x, x\right\rangle$$

and

$$\left\|(N^*)^j y\right\|^2 = \left\langle \left|(N^*)^j\right|^2 y, y\right\rangle = \left\langle |N^*|^{2j} y, y\right\rangle = \left\langle |N|^{2j} y, y\right\rangle$$

for any $j \in \{0, ..., n\}$ and for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Then from (3.21) we have

$$\sum_{j=0}^{n} |a_j| \left|\left\langle N^j x, y\right\rangle\right|^2 \tag{3.22}$$
$$\leq \frac{1}{2}\left[\left(\left\langle \sum_{j=0}^{n} |a_j| \, |N|^{2j} x, x\right\rangle\right)^{\alpha} \left(\left\langle \sum_{j=0}^{n} |a_j| \, |N|^{2j} y, y\right\rangle\right)^{1-\alpha} + \left(\left\langle \sum_{j=0}^{n} |a_j| \, |N|^{2j} x, x\right\rangle\right)^{1-\alpha} \left(\left\langle \sum_{j=0}^{n} |a_j| \, |N|^{2j} y, y\right\rangle\right)^{\alpha}\right]$$
$$\leq \frac{1}{2}\left(\left\langle \sum_{j=0}^{n} |a_j| \, |N|^{2j} x, x\right\rangle + \left\langle \sum_{j=0}^{n} |a_j| \, |N|^{2j} y, y\right\rangle\right)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

By the weighted Cauchy-Buniakowski-Schwarz inequality we also have

$$\left|\left\langle \sum_{j=0}^{n} a_j z^j N^j x, y\right\rangle\right|^2 \leq \sum_{j=0}^{n} |a_j| \, |z|^{2j} \sum_{j=0}^{n} |a_j| \left|\left\langle N^j x, y\right\rangle\right|^2 \tag{3.23}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, since the series $\sum_{j=0}^{\infty} a_j z^j N^j$, $\sum_{j=0}^{\infty} |a_j| |z|^{2j}$, $\sum_{j=0}^{\infty} |a_j| |N|^{2j}$ are convergent, then by (3.22) and (3.23), on letting $n \to \infty$, we deduce the desired result (3.19). □

Similar inequalities for some particular functions of interest can be stated. However, the details are left to the interested reader.

4. Applications for the Euclidian Norm

In [30], the author has introduced the following norm on the Cartesian product $\mathcal{B}^{(n)}(H) := \mathcal{B}(H) \times \cdots \times \mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H :

$$\|(T_1, \dots, T_n)\|_e := \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \tag{4.1}$$

where $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$ and $\mathbb{B}_n := \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \left| \sum_{j=1}^{n} |\lambda_j|^2 \le 1 \right.\right\}$ is the Euclidean closed ball in $\mathbb{C}^n$.

It is clear that $\|\cdot\|_e$ is a norm on $B^{(n)}(H)$ and for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ we have

$$\|(T_1, \dots, T_n)\|_e = \|(T_1^*, \dots, T_n^*)\|_e,$$

where T_j^* is the adjoint operator of T_j, $j \in \{1, \dots, n\}$. We call this the *Euclidian norm* of an n-tuple of operators $(T_1, \dots, T_n) \in B^{(n)}(H)$.

It has been shown in [30] that the following basic inequality for the Euclidian norm holds true:

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\|^{\frac{1}{2}} \le \|(T_1, \dots, T_n)\|_e \le \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\|^{\frac{1}{2}} \tag{4.2}$$

for any n-tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [30] the author has introduced the *Euclidean operator radius* of an n-tuple of operators $(T_1, \dots, T_n)$ by

$$w_e(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 \right)^{\frac{1}{2}} \tag{4.3}$$

and proved that $w_e(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality:

$$\frac{1}{2} \|(T_1, \dots, T_n)\|_e \le w_e(T_1, \dots, T_n) \le \|(T_1, \dots, T_n)\|_e \tag{4.4}$$

for each n-tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$.

As pointed out in [30], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}} \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\|^{\frac{1}{2}} \le w_e(T_1, \dots, T_n) \le \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\|^{\frac{1}{2}} \tag{4.5}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

In [2], by utilizing the concept of *hypo-Euclidean norm* on H^n we obtained the following representation for the Euclidian norm:

Proposition 1. *For any* $(T_1, \ldots, T_n) \in B^{(n)}(H)$ *we have*

$$\|(T_1, \ldots, T_n)\|_e = \sup_{\|y\|=1, \|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j y, x \rangle|^2 \right)^{\frac{1}{2}}. \tag{4.6}$$

We can state now the following result:

Theorem 7. *For any* $(T_1, \ldots, T_n) \in B^{(n)}(H)$ *we have*

$$\begin{aligned} \|(T_1, \ldots, T_n)\|_e^2 &\le \frac{1}{2} \left[\left(\left\| \sum_{j=1}^{n} |T_j|^2 \right\| \right)^{\alpha} \left(\left\| \sum_{j=1}^{n} |T_j^*|^2 \right\| \right)^{1-\alpha} \right. \\ &\quad \left. + \left(\left\| \sum_{j=1}^{n} |T_j|^2 \right\| \right)^{1-\alpha} \left(\left\| \sum_{j=1}^{n} |T_j^*|^2 \right\| \right)^{\alpha} \right] \\ &\le \frac{1}{2} \left[\left\| \sum_{j=1}^{n} |T_j|^2 \right\| + \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\| \right] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} & w_e^2 (T_1, \ldots, T_n) \\ & \le \frac{1}{2} \left[\sup_{\|x\|=1} \left\{ \left(\left\langle \sum_{j=1}^{n} |T_j|^2 x, x \right\rangle \right)^{\alpha} \left(\left\langle \sum_{j=1}^{n} |T_j^*|^2 x, x \right\rangle \right)^{1-\alpha} \right\} \right. \\ & \left. + \sup_{\|x\|=1} \left\{ \left(\left\langle \sum_{j=1}^{n} |T_j|^2 x, x \right\rangle \right)^{1-\alpha} \left(\left\langle \sum_{j=1}^{n} |T_j^*|^2 x, x \right\rangle \right)^{\alpha} \right\} \right] \\ & \le \frac{1}{2} \left[\left(\left\| \sum_{j=1}^{n} |T_j|^2 \right\| \right)^{\alpha} \left(\left\| \sum_{j=1}^{n} |T_j^*|^2 \right\| \right)^{1-\alpha} \right. \\ & \left. + \left(\left\| \sum_{j=1}^{n} |T_j|^2 \right\| \right)^{1-\alpha} \left(\left\| \sum_{j=1}^{n} |T_j^*|^2 \right\| \right)^{\alpha} \right] \end{aligned} \tag{4.8}$$

for any $\alpha \in [0, 1]$.

Proof. We have from the second inequality in (2.11)

$$\sum_{j=1}^{n}|\langle T_j x, y\rangle|^2 \le \frac{1}{2}\left[\left(\left\langle \sum_{j=1}^{n}|T_j|^2 x, x\right\rangle\right)^{\alpha}\left(\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 y, y\right\rangle\right)^{1-\alpha} + \left(\left\langle \sum_{j=1}^{n}|T_j|^2 x, x\right\rangle\right)^{1-\alpha}\left(\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 y, y\right\rangle\right)^{\alpha}\right] \tag{4.9}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Taking the supremum over $\|x\| = \|y\| = 1$ we have

$$\begin{aligned}
&\|(T_1,\dots,T_n)\|_e^2 \\
&\le \frac{1}{2}\left[\left(\sup_{\|x\|=1}\left\langle \sum_{j=1}^{n}|T_j|^2 x, x\right\rangle\right)^{\alpha}\left(\sup_{\|y\|=1}\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 y, y\right\rangle\right)^{1-\alpha}\right. \\
&\left.+\left(\sup_{\|x\|=1}\left\langle \sum_{j=1}^{n}|T_j|^2 x, x\right\rangle\right)^{1-\alpha}\left(\sup_{\|y\|=1}\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 y, y\right\rangle\right)^{\alpha}\right] \\
&= \frac{1}{2}\left[\left(\left\|\sum_{j=1}^{n}|T_j|^2\right\|\right)^{\alpha}\left(\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\right)^{1-\alpha}\right. \\
&\left.+\left(\left\|\sum_{j=1}^{n}|T_j|^2\right\|\right)^{1-\alpha}\left(\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\right)^{\alpha}\right],
\end{aligned}$$

which proves the first part of (4.7).

The second part follows by the elementary inequality

$$a^{\alpha}b^{1-\alpha} \le \alpha a + (1-\alpha)\, b$$

for $a, b \ge 0$ and $\alpha \in [0,1]$.

The inequality (4.8) follows from (4.9) by taking $y = x$ and then the supremum over $\|x\| = 1$. □

5. Applications for s-1-Norm and s-1-Numerical Radius

Following [3], we consider the *s-p-norm* of the n-tuple of operators $(T_1,\dots,T_n) \in B^{(n)}(H)$ by

$$\|(T_1,\dots,T_n)\|_{s,p} := \sup_{\|y\|=1,\|x\|=1}\left[\left(\sum_{j=1}^{n}|\langle T_j y, x\rangle|^p\right)^{\frac{1}{p}}\right]. \tag{5.1}$$

For $p = 2$ we get

$$\|(T_1,\dots,T_n)\|_{s,2} = \|(T_1,\dots,T_n)\|_e .$$

We are interested in this section in the case $p=1$, namely on the s-1-norm defined by

$$\left\|(T_1,\dots,T_n)\right\|_{s,1} := \sup_{\|y\|=1,\|x\|=1}\sum_{j=1}^{n}\left|\langle T_j y,x\rangle\right|.$$

Since for any $x,y\in H$ we have $\sum_{j=1}^{n}\left|\langle T_j y,x\rangle\right| \geq \left|\left\langle \sum_{j=1}^{n}T_j y,x\right\rangle\right|$, then by the properties of the supremum we get the basic inequality

$$\left\|\sum_{j=1}^{n}T_j\right\| \leq \left\|(T_1,\dots,T_n)\right\|_{s,1} \leq \sum_{j=1}^{n}\left\|T_j\right\|. \tag{5.2}$$

Similarly, we can also consider the *s-p-numerical radius* of the n-tuple of operators $(T_1,\dots,T_n)\in B^{(n)}(H)$ by [3]

$$w_{s,p}(T_1,\dots,T_n) := \sup_{\|x\|=1}\left[\left(\sum_{j=1}^{n}\left|\langle T_j x,x\rangle\right|^p\right)^{\frac{1}{p}}\right], \tag{5.3}$$

which for $p=2$ reduces to the Euclidean operator radius introduced previously.

We observe that the s-p-numerical radius is also a norm on $B^{(n)}(H)$ for $p\geq 1$ and for $p=1$ it satisfies the basic inequality

$$w\left(\sum_{j=1}^{n}T_j\right) \leq w_{s,1}(T_1,\dots,T_n) \leq \sum_{j=1}^{n}w(T_j). \tag{5.4}$$

We can state the following result:

Theorem 8. *For any $(T_1,\dots,T_n)\in B^{(n)}(H)$ we have*

$$\begin{aligned}\left\|(T_1,\dots,T_n)\right\|_{s,1} &\leq \left\|\sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}}{2}\right)\right\|^{1/2} \\ &\quad\times\left\|\sum_{j=1}^{n}\left(\frac{|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)\right\|^{1/2} \\ &\leq \frac{1}{2}\left[\left\|\sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}}{2}\right)\right\|\right. \\ &\quad\left.+\left\|\sum_{j=1}^{n}\left(\frac{|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)\right\|\right]\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}& w_{s,1}(T_1,\dots,T_n) \\ &\leq \left\|\sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}+|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{4}\right)\right\|.\end{aligned} \tag{5.6}$$

Proof. From (2.1) we have

$$\sum_{j=1}^{n}|\langle T_j x,y\rangle| \leq \left\langle \sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}}{2}\right)x,x\right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n}\left(\frac{|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)y,y\right\rangle^{1/2} \tag{5.7}$$

for any $x,y\in H$.

Taking the supremum over $\|y\|=1, \|x\|=1$ in (5.7) we have

$$\begin{aligned}\|(T_1,\dots,T_n)\|_{s,1} &\leq \left[\sup_{\|x\|=1}\left\langle \sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}}{2}\right)x,x\right\rangle\right]^{1/2} \\ &\quad\times \left[\sup_{\|y\|=1}\left\langle \sum_{j=1}^{n}\left(\frac{|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)y,y\right\rangle\right]^{1/2} \\ &= \left\|\sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}}{2}\right)\right\|^{1/2} \\ &\quad\times \left\|\sum_{j=1}^{n}\left(\frac{|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)\right\|^{1/2}\end{aligned}$$

and the first inequality (5.5) is proved.

The second part follows by the arithmetic mean-geometric mean inequality.

Now, if we take $y=x$ in (5.7), then we get

$$\begin{aligned}\sum_{j=1}^{n}|\langle T_j x,x\rangle| &\leq \left\langle \sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}}{2}\right)x,x\right\rangle^{1/2} \\ &\quad\times \left\langle \sum_{j=1}^{n}\left(\frac{|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)x,x\right\rangle^{1/2} \\ &\leq \frac{1}{2}\left\langle \sum_{j=1}^{n}\left(\frac{|T_j|^{2\alpha}+|T_j|^{2(1-\alpha)}+|T_j^*|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right)x,x\right\rangle.\end{aligned}$$

Taking the supremum over $\|x\|=1$ we deduce the desired result (5.6). □

Remark 6. *If we take $\alpha=\frac{1}{2}$ in the first inequality in (5.5), then we deduce*

$$\|(T_1,\dots,T_n)\|_{s,1} \leq \left\|\sum_{j=1}^{n}|T_j|\right\|^{1/2}\left\|\sum_{j=1}^{n}|T_j^*|\right\|^{1/2} \tag{5.8}$$

from where we get the following refinement of the generalized triangle inequality

$$\left\|\sum_{j=1}^{n} T_j\right\| \leq \|(T_1,\ldots,T_n)\|_{s,1} \leq \left\|\sum_{j=1}^{n} |T_j|\right\|^{1/2} \left\|\sum_{j=1}^{n} |T_j^*|\right\|^{1/2}$$
$$\leq \frac{1}{2}\left[\left\|\sum_{j=1}^{n} |T_j|\right\| + \left\|\sum_{j=1}^{n} |T_j^*|\right\|\right] \leq \sum_{j=1}^{n} \|T_j\|.$$

From (5.6) we also have for $\alpha = \frac{1}{2}$ that

$$w_{s,1}(T_1,\ldots,T_n) \leq \left\|\sum_{j=1}^{n}\left(\frac{|T_j| + |T_j^*|}{2}\right)\right\|. \tag{5.9}$$

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