Received 15/07/12

POWER SERIES INEQUALITIES RELATED TO YOUNG'S INEQUALITY AND APPLICATIONS

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ABSTRACT. On utilising a refimenent and a reverse of Young's inequality, in this paper, we establish new inequalities for functions defined by power series with positive coefficients, which improve the famous Hölder's inequality for power series. Some applications for special functions such as polylogarithm, hypergeometric and Bessel functions are presented as well.

1. Introduction

The classical Young inequality for two scalars is the $\nu-$weighted arithmetic-geometric mean inequality, which is a fundamental relation between two nonnegative real numbers. This inequality says that for a, b are positive real numbers and $0 \leq \nu \leq 1$,

$$a^{\nu} b^{1-\nu} \leq \nu a + (1-\nu) b \tag{1.1}$$

with equality if and only if $a = b$. If p, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then the inequality (1.1) can be written as

$$xy \leq \frac{x^q}{q} + \frac{y^p}{p} \tag{1.2}$$

for any x, $y \geq 0$. In this form, the inequality (1.1) was used to prove the celebrated Hölder inequality, see (1.9) below.

The inequalities (1.1) and (1.2) have been refined by several authors (see [**3**], [**4**], [**9**], [**10**], [**11**], [**13**], [**16**] and the references cited therein). For instance, Hirzallah and Kittaneh [**13**] obtained the refinement of the scalar Young's inequality (1.1) as follows:

$$[\nu a + (1-\nu) b]^2 - \left(a^{\nu} b^{1-\nu}\right)^2 \geq r^2 (a-b)^2 \tag{1.3}$$

for $a, b \geq 0$, $0 \leq \nu \leq 1$ and $r = \min\{\nu, 1-\nu\}$. Notice that, in [**16**] Kittaneh and Manasrah provided the refinement of the Young inequality (1.1) in the following form,

$$\nu a + (1-\nu) b - a^{\nu} b^{1-\nu} \geq r \left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Young's inequality, Hölder's inequality, Power series, Special functions.

for $a,b\geq 0$, $0\leq\nu\leq 1$.

For all $x,y\geq 0$ and $1<p\leq 2$ with $\frac{1}{p}+\frac{1}{q}=1$, Aldaz [**3**] proved the following inequality,

$$\frac{1}{q}\left(x^{\frac{q}{2}}-y^{\frac{p}{2}}\right)^2\leq\frac{x^q}{q}+\frac{y^p}{p}-xy\leq\frac{1}{p}\left(x^{\frac{q}{2}}-y^{\frac{p}{2}}\right)^2, \tag{1.5}$$

which provided a refinement and a reverse of the Young's inequality (1.2). The inequality (1.5) can be written in the same notation as above when we change the variables in (1.5) as $a=x^q, b=y^p, \nu=\frac{1}{q}$ and $1-\nu=\frac{1}{p}$, i.e.,

$$2\nu\left(\frac{a+b}{2}-\sqrt{ab}\right)\leq\nu a+(1-\nu)\,b-a^{\nu}b^{1-v}\leq 2\,(1-\nu)\left(\frac{a+b}{2}-\sqrt{ab}\right) \tag{1.6}$$

for any $a,b\geq 0$ and $\nu\in\left[0,\frac{1}{2}\right]$. This inequality (1.6) has appeared in [**4**] (also in [**5**], [**9**]).

A generalization of the Young inequality (1.1) was given by Furuichi in [**11**], that is

$$\sum_{j=1}^{n}p_ja_j-\prod_{j=1}^{n}a_j^{p_j}\geq np_{\min}\left(\frac{1}{n}\sum_{j=1}^{n}a_j-\prod_{j=1}^{n}a_j^{1/n}\right) \tag{1.7}$$

for $a_j,p_j\geq 0$, $j\in\{1,2,\ldots,n\}$ with $\sum_{j=1}^{n}p_j=1$ and $p_{\min}=\min\{p_1,p_2,\ldots,p_n\}$. The equality holds in (1.7) if and only if $a_1=a_2=\ldots=a_n$. Note that, for $n=2$, the inequality (1.7) reduces to (1.4).

Other generalizations of Young's inequality can be found in [**1**] and [**2**]. See also [**8**], [**12**], [**17**], [**18**] and the references cited therein for some improvements and their recent advances on Young's inequality.

If, now we consider an analytic function defined by power series

$$f(x)=\sum_{n=0}^{\infty}a_nx^n \tag{1.8}$$

with positive coefficients and convergent on the interval $(0,R)$ and utilizing the weighted version of Hölder's inequality, namely

$$\sum_{k=1}^{n}p_ka_kb_k\leq\left(\sum_{k=1}^{n}p_ka_k^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}p_kb_k^q\right)^{\frac{1}{q}} \tag{1.9}$$

where $p_k,a_k,b_k\geq 0$, $k\in\{1,2,\ldots,n\}$ and $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we can state that

$$f\,(xy)=\sum_{n=0}^{\infty}a_nx^ny^n\leq\left(\sum_{n=0}^{\infty}a_nx^{pn}\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}a_ny^{qn}\right)^{\frac{1}{q}}=f^{\frac{1}{p}}\,(x^p)\,f^{\frac{1}{q}}\,(y^q) \tag{1.10}$$

for $x,y>0$ with $xy,x^p,y^q<R$.

In [**14**], the authors provided some related results to the Hölder's inequality (1.10) via Young's inequality (1.1) for the functions defined by complex power series with real coefficients. In this paper, we refine the Young's inequality (1.1) and (1.4), and utilizing these results, we derive new inequalities for functions defined by real power series with positive coefficients and convergent on the interval $(0, R)$, which improve the Hölder type inequality (1.10) for power series. In particular, we improve the refinements of Hölder's type inequalities from the the paper [**14**] for the real power series with positive coefficients. Some applications for fundamental functions of interest are also presented.

2. Some inequalities via a refinement of Young's inequality

Before we state our results, we first refine and provide a reverse for the Young's inequality as follows.

Lemma 1. *For any $a, b \geq 0$ and $\nu \in [0,1]$, we have*

$$2\min\{\nu, 1-\nu\}\left(\frac{a+b}{2}-\sqrt{ab}\right) \leq \nu a+(1-\nu)b-a^{\nu}b^{1-\nu} \leq 2\max\{\nu, 1-\nu\}\left(\frac{a+b}{2}-\sqrt{ab}\right). \tag{2.1}$$

Proof. We recall the following result obtained by Dragomir in [**8**] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \\ & \leq \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right) \\ & \leq n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned} \tag{2.2}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

We notice that Furuichi's result (1.7) is a particular case of (2.2) applied for the convex function $f(t) = \exp(t)$ and denoting $\exp(x_j)$ as a_j for $j \in \{1,\dots,n\}$.

For $n = 2$, we deduce from (2.2) that

$$\begin{aligned} & 2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \\ & \leq \nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y] \\ & \leq 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{2.3}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

If we take $\Phi(x) = \exp(x)$, then we get from (2.3)

$$\begin{aligned} & 2\min\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right] \\ & \leq \nu\exp(x)+(1-\nu)\exp(y)-\exp[\nu x+(1-\nu)y] \\ & \leq 2\max\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{2.4}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Further, denote $\exp(x) = a$, $\exp(y) = b$ with $a, b > 0$, then from (2.4) we obtain the desired result (2.1). □

From the refinement of the Young's inequality (2.1), we have the following corollary:

Corollary 1. *For any $x, y \geq 0$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned} &2\min\left\{\frac{1}{q}, \frac{1}{p}\right\}\left(\frac{x^q + y^p}{2} - x^{\frac{q}{2}} y^{\frac{p}{2}}\right) \\ &\leq \frac{x^q}{q} + \frac{y^p}{p} - xy \\ &\leq 2\max\left\{\frac{1}{q}, \frac{1}{p}\right\}\left(\frac{x^q + y^p}{2} - x^{\frac{q}{2}} y^{\frac{p}{2}}\right). \end{aligned} \tag{2.5}$$

Proof. The proof follows by choosing $a = x^q$, $b = y^p$, $\nu = \frac{1}{q}$, $1 - \nu = \frac{1}{p}$ in Lemma 1. □

Remark 1. *The first inequality in (2.1) provides the Kittaneh and Manasrah result in (1.4) as well as a reverse of that result. It is also more general than the Aldaz result (1.5) since no restriction on p is assumed.*

First, utilizing the inequality (1.1) for the real power series with positive coefficients, the following result holds.

Theorem 1. *Let $f(x) = \sum_{n=0}^{\infty} p_n x^n$ be a power series with positive coefficients and convergent on $(0, R)$. Then for $\nu \in [0,1]$, $x, y \geq 0$ such that $y, xy, x^\nu y, x^{1-\nu} y \in (0, R)$, we have*

$$f(x^\nu y) f\left(x^{1-\nu} y\right) \leq f(xy) f(y). \tag{2.6}$$

Proof. The proof follows by choosing in (1.1) $a = x^j$, $b = x^k$, $j, k \in \{0, 1, \ldots, n\}$. Thus, we have

$$x^{\nu j} x^{(1-\nu)k} \leq \nu x^j + (1-\nu) x^k \tag{2.7}$$

for any $x \geq 0$ and $\nu \in [0,1]$.

If now we multiply this inequality (2.7) with $p_j y^j p_k y^k \geq 0$, $y \in (0, R)$ and summing over j and k from 0 to n, then we get

$$\begin{aligned} &\sum_{j=0}^{n} p_j (x^\nu y)^j \sum_{k=0}^{n} p_k \left(x^{1-\nu} y\right)^k \\ &\leq \nu \sum_{j=0}^{n} p_j (xy)^j \sum_{k=0}^{n} p_k y^k + (1-\nu) \sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k (xy)^k. \end{aligned} \tag{2.8}$$

Since all the series whose partial sums are involved in inequality (2.8) are convergent on the interval $(0, R)$ by taking the limit as $n \to \infty$ in (2.8), we deduce the desired inequality (2.6). □

Remark 2. *(a) If $xy = z$ in (2.6), then we have*

$$f\left(y^{\nu}z^{1-\nu}\right)f\left(y^{1-\nu}z^{\nu}\right)\leq f(y)f(z) \tag{2.9}$$

for $y, z, y^{\nu}z^{1-\nu}, y^{1-\nu}z^{\nu} \in (0,R)$ and $\nu \in [0,1]$.

(b) If $y = x$ in (2.6), then we also have

$$f\left(x^{1+\nu}\right)f\left(x^{2-\nu}\right)\leq f(x^2)f(x) \tag{2.10}$$

for $x, x^2, x^{1+\nu}, x^{2-\nu} \in (0,R)$ and $\nu \in [0,1]$.

Some applications of the inequality (2.9) for particular functions of interest are as follows:

(1) If we apply the inequality (2.9) for the function $f(x)=\dfrac{1}{1-x}=\sum\limits_{n=0}^{\infty}x^n$, $x\in(0,1)$, then we get

$$(1-y)(1-z)\leq\left(1-y^{\nu}z^{1-\nu}\right)\left(1-y^{1-\nu}z^{\nu}\right) \tag{2.11}$$

for $y, z, y^{\nu}z^{1-\nu}, y^{1-\nu}z^{\nu} \in (0,1)$ and $\nu \in [0,1]$.

(2) If we consider the function $f(x)=\ln\left(\dfrac{1}{1-x}\right)=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n}$, $x\in(0,1)$ and applying the inequality (2.9), then we get

$$\ln\left(1-y^{\nu}z^{1-\nu}\right)\ln\left(1-y^{1-\nu}z^{\nu}\right)\leq\ln(1-y)\ln(1-z) \tag{2.12}$$

for $y, z, y^{\nu}z^{1-\nu}, y^{1-\nu}z^{\nu} \in (0,1)$ and $\nu \in [0,1]$.

Next, based on refinement and a reverse of the Young's inequality (2.1), we prove the following inequality.

Theorem 2. *Let $f(x)$ be as in Theorem 1. Then, one has the inequality*

$$\begin{aligned}&2\min\{\nu,1-\nu\}\left[f(xy)f(y)-f^2\left(x^{\frac{1}{2}}y\right)\right]\\&\leq f(xy)f(y)-f(x^{\nu}y)f\left(x^{1-\nu}y\right)\\&\leq 2\max\{\nu,1-\nu\}\left[f(xy)f(y)-f^2\left(x^{\frac{1}{2}}y\right)\right]\end{aligned} \tag{2.13}$$

for $x, y \geq 0$ such that $xy, y, x^{\frac{1}{2}}y, x^{\nu}y, x^{1-\nu}y \in (0,R)$ and $\nu \in [0,1]$.

Proof. We use the inequality (2.1) for $a = x^j$, $b = x^k$, $j, k \in \{0,1,\ldots,n\}$ to get

$$\begin{aligned}&2\min\{\nu,1-\nu\}\left(\frac{x^j+x^k}{2}-x^{\frac{j}{2}}x^{\frac{k}{2}}\right)\\&\leq\nu x^j+(1-\nu)x^k-x^{\nu j}x^{(1-\nu)k}\\&\leq 2\max\{\nu,1-\nu\}\left(\frac{x^j+x^k}{2}-x^{\frac{j}{2}}x^{\frac{k}{2}}\right).\end{aligned} \tag{2.14}$$

for any $x, y \geq 0$ and $\nu \in [0,1]$.

Then, multiplying the inequality (2.14) with $p_j y^j p_k y^k \geq 0$ and summing over j and k from 0 to n, we have

$$\begin{aligned} (2.15) \qquad & 2t\left[\frac{1}{2}\left(\sum_{j=0}^{n} p_j x^j y^j \sum_{k=0}^{n} p_k y^k + \sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k x^k y^k\right)\right. \\ & \left.-\sum_{j=0}^{n} p_j x^{\frac{j}{2}} y^j \sum_{k=0}^{n} p_k x^{\frac{k}{2}} y^k\right] \\ & \leq \nu \sum_{j=0}^{n} p_j x^j y^j \sum_{k=0}^{n} p_k y^k + (1-\nu) \sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k x^k y^k \\ & -\sum_{j=0}^{n} p_j x^{\nu j} y^j \sum_{k=0}^{n} p_k x^{(1-\nu)k} y^k \\ & \leq 2T\left[\frac{1}{2}\left(\sum_{j=0}^{n} p_j x^j y^j \sum_{k=0}^{n} p_k y^k + \sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k x^k y^k\right)\right. \\ & \left.-\sum_{j=0}^{n} p_j x^{\frac{j}{2}} y^j \sum_{k=0}^{n} p_k x^{\frac{k}{2}} y^k\right] \end{aligned}$$

where $t = \min\{\nu, 1-\nu\}$ and $T = \max\{\nu, 1-\nu\}$.

Since all the series whose partial sums are involved in inequality (2.15) are convergent on the interval $(0, R)$ by taking the limit as $n \to \infty$ in (2.15), we deduce the desired result (2.13). □

Remark 3. *(a) If $xy = z$ in (2.13), then we have*

$$\begin{aligned} (2.16) \qquad & 2\min\{\nu, 1-\nu\}\left[f(y)f(z) - f^2\left(\sqrt{yz}\right)\right] \\ & \leq f(y)f(z) - f\left(y^v z^{1-\nu}\right) f\left(y^{1-\nu} z^{\nu}\right) \\ & \leq 2\max\{\nu, 1-\nu\}\left[f(y)f(z) - f^2\left(\sqrt{yz}\right)\right] \end{aligned}$$

for $y, z, z^{\nu} y^{1-\nu}, z^{1-\nu} y^{v} \in (0, R)$ and $\nu \in [0,1]$. This result provides somehow a symmetric form for (2.13) and has some nice applications as well, see (2.18).

(b) If $y = x$ in (2.13), then we also have

$$\begin{aligned} (2.17) \qquad & 2\min\{\nu, 1-\nu\}\left[f(x)f\left(x^2\right) - f^2\left(x^{\frac{3}{2}}\right)\right] \\ & \leq f(x)f(x^2) - f\left(x^{1+\nu}\right) f\left(x^{2-\nu}\right) \\ & \leq 2\max\{\nu, 1-\nu\}\left[f(x)f\left(x^2\right) - f^2\left(x^{\frac{3}{2}}\right)\right]. \end{aligned}$$

for $x, x^2, x^{\frac{3}{2}}, x^{1+\nu}, x^{2-\nu} \in (0, R)$ and $\nu \in [0,1]$.

Now, if we consider the function $f(x) = \exp(x)$, $x \in \mathbb{R}$ and applying the inequality (2.16), then we get

$$\begin{aligned} (2.18) \qquad & 2\min\{\nu, 1-\nu\}\left[\exp(y+z) - \exp\left(2\sqrt{yz}\right)\right] \\ & \leq \exp(y+z) - \exp\left(y^v z^{1-\nu} + y^{1-\nu} z^{\nu}\right) \\ & \leq 2\max\{\nu, 1-\nu\}\left[\exp(y+z) - \exp\left(2\sqrt{yz}\right)\right] \end{aligned}$$

for any $y, z \geq 0$ and $\nu \in [0,1]$.

The second improvement of the Hölder's inequality (1.10) via a refinement and a reverse of the Young's inequality (2.5) is incorporated in the following results.

THEOREM 3. *Let $f(x)$ be as in Theorem 1. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y \geq 0$ such that xy, x^q, y^p, $x^{\frac{q}{2}} y^{\frac{p}{2}} \in (0, R)$, then*

$$2t \left[\frac{1}{2} \left[f(x^q) + f(y^p) \right] - f\left(x^{\frac{q}{2}} y^{\frac{p}{2}} \right) \right] \leq \frac{1}{q} f(x^q) + \frac{1}{p} f(y^p) - f(xy) \leq 2T \left[\frac{1}{2} \left[f(x^q) + f(y^p) \right] - f\left(x^{\frac{q}{2}} y^{\frac{p}{2}} \right) \right] \tag{2.19}$$

where $t = \min\left\{\frac{1}{q}, \frac{1}{p}\right\}$ and $T = \max\left\{\frac{1}{q}, \frac{1}{p}\right\}$.

PROOF. If we choose $x = x^j$, $y = y^j$, $j \in \{0, 1, 2, \ldots, n\}$, then we have from (2.5)

$$2t \left(\frac{x^{qj} + y^{pj}}{2} - x^{\frac{q}{2}j} y^{\frac{p}{2}j} \right) \leq \frac{x^{qj}}{q} + \frac{y^{pj}}{p} - (xy)^j \leq 2T \left(\frac{x^{qj} + y^{pj}}{2} - x^{\frac{q}{2}j} y^{\frac{p}{2}j} \right) \tag{2.20}$$

for any $x, y \geq 0$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If we multiply this inequality (2.20) with positive quantities p_j, $j \in \{0, 1, 2, \ldots, n\}$ and summing over j from 0 to n, then we get

$$2t \left(\frac{1}{2} \left[\sum_{j=0}^{n} p_j x^{qj} + \sum_{j=0}^{n} p_j y^{pj} \right] - \sum_{j=0}^{n} p_j \left(x^{\frac{q}{2}} y^{\frac{p}{2}} \right)^j \right) \leq \frac{1}{q} \sum_{j=0}^{n} p_j x^{qj} + \frac{1}{p} \sum_{j=0}^{n} p_j y^{pj} - \sum_{j=0}^{n} p_j (xy)^j \leq 2T \left(\frac{1}{2} \left[\sum_{j=0}^{n} p_j x^{qj} + \sum_{j=0}^{n} p_j y^{pj} \right] - \sum_{j=0}^{n} p_j \left(x^{\frac{q}{2}} y^{\frac{p}{2}} \right)^j \right). \tag{2.21}$$

Since all the series whose partial sums are involved in inequality (2.21) are convergent on the interval $(0, R)$ by taking the limit as $n \to \infty$ in (2.21), we deduce the desired inequality (2.19). □

COROLLARY 2. *If $y = x$ in (2.19), then for any $x^2, x^q, x^p, x^{\frac{pq}{2}} \in (0, R)$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$2t \left[\frac{1}{2} \left[f(x^q) + f(x^p) \right] - f\left(x^{\frac{pq}{2}} \right) \right] \leq \frac{1}{q} f(x^q) + \frac{1}{p} f(x^p) - f\left(x^2\right) \leq 2T \left[\frac{1}{2} \left[f(x^q) + f(x^p) \right] - f\left(x^{\frac{pq}{2}} \right) \right]. \tag{2.22}$$

Some applications of the inequality (2.22) for particular functions of interest are as follows:

(1) If we apply the inequality (2.22) for the function $f(x) = \frac{1}{1-x}$, $x \in (0,1)$, then we get

$$t\left[\frac{1}{1-x^q} + \frac{1}{1-x^p} - \frac{2}{1-x^{\frac{pq}{2}}}\right] \leq \frac{1}{q\left(1-x^q\right)} + \frac{1}{p\left(1-x^p\right)} - \frac{1}{1-x^2} \leq T\left[\frac{1}{1-x^q} + \frac{1}{1-x^p} - \frac{2}{1-x^{\frac{pq}{2}}}\right] \tag{2.23}$$

for $x^2, x^q, x^p, x^{\frac{pq}{2}} \in (0,1)$, $p > 1$.

(2) If we apply the inequality (2.22) for the function $f(x) = \ln\left(\frac{1}{1-x}\right)$, $x \in (0,1)$, then we get

$$\left(\frac{\left(1-x^{\frac{pq}{2}}\right)^2}{\left(1-x^q\right)\left(1-x^p\right)}\right)^t \leq \frac{1-x^2}{\left(1-x^q\right)^{\frac{1}{q}}\left(1-x^p\right)^{\frac{1}{p}}} \leq \left(\frac{\left(1-x^{\frac{pq}{2}}\right)^2}{\left(1-x^q\right)\left(1-x^p\right)}\right)^T, \tag{2.24}$$

for $x^2, x^q, x^p, x^{\frac{pq}{2}} \in (0,1)$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(3) If we consider the function $f(x) = \sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$, $x \in \mathbb{R}$ and applying the inequality (2.22), then we get

$$t\left[\sinh\left(x^q\right) + \sinh\left(x^p\right) - 2\sinh\left(x^{\frac{pq}{2}}\right)\right] \tag{2.25}$$

$$\leq \frac{1}{q}\sinh\left(x^q\right) + \frac{1}{p}\sinh\left(x^p\right) - \sinh\left(x^2\right) \leq T\left[\sinh\left(x^q\right) + \sinh\left(x^p\right) - 2\sinh\left(x^{\frac{pq}{2}}\right)\right],$$

for any $x \geq 0$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Similar result can be obtained for $\cosh(x)$ as well.

Further, we utilize the inequality (2.5) to improve the results from ([**14**]), giving the refinements and the reverses of the Hölder's inequality for two functions defined by power series with positive coefficients. First, the following result holds.

Theorem 4. *Let $f(x) = \sum_{n=0}^{\infty} p_n x^n$ and $g(x) = \sum_{n=0}^{\infty} q_n x^n$ be two power series with positive coefficients and convergent on $(0, R)$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y \geq 0$ such that xy, x^p, x^q, y^p, $y^q, x^{\frac{q}{2}}y^{\frac{p}{2}}, x^{\frac{p}{2}}y^{\frac{q}{2}}, (xy)^{\frac{q}{2}}, (xy)^{\frac{p}{2}}, xy^{q-1}, xy^{p-1} \in$*

$(0,R)$, *then*

$$t\left[f(x^q)g(y^q)+g(x^p)f(y^p)-2f\left(x^{\frac{q}{2}}y^{\frac{p}{2}}\right)g\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \leq \frac{1}{q}f(x^q)g(y^q)+\frac{1}{p}g(x^p)f(y^p)-f(xy)\,g(xy) \leq T\left[f(x^q)g(y^q)+g(x^p)f(y^p)-2f\left(x^{\frac{q}{2}}y^{\frac{p}{2}}\right)g\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \tag{2.26}$$

and

$$t\left[f(x^q)g(y^p)+g(x^p)f(y^q)-2f(x^{\frac{q}{2}}y^{\frac{q}{2}})g(x^{\frac{p}{2}}y^{\frac{p}{2}})\right] \leq \frac{1}{q}f(x^q)g(y^p)+\frac{1}{p}g(x^p)f(y^q)-f(xy^{q-1})g(xy^{p-1}) \leq T\left[f(x^q)g(y^p)+g(x^p)f(y^q)-2f(x^{\frac{q}{2}}y^{\frac{q}{2}})g(x^{\frac{p}{2}}y^{\frac{p}{2}})\right]. \tag{2.27}$$

PROOF. If we choose in (2.5) $x=x^jy^k, y=x^ky^j, j,k\in\{0,1,2\dots,n\}$, then we have

$$2t\left(\frac{x^{qj}y^{qk}+x^{pk}y^{pj}}{2}-x^{\frac{q}{2}j}y^{\frac{p}{2}j}x^{\frac{p}{2}k}y^{\frac{q}{2}k}\right) \leq \frac{1}{q}\left(x^{qj}y^{qk}\right)+\frac{1}{p}\left(x^{pk}y^{pj}\right)-(xy)^j\,(xy)^k \leq 2T\left(\frac{x^{qj}y^{qk}+x^{pk}y^{pj}}{2}-x^{\frac{q}{2}j}y^{\frac{p}{2}j}x^{\frac{p}{2}k}y^{\frac{q}{2}k}\right) \tag{2.28}$$

for any $x,y\geq 0$ and $p>1$ with $\frac{1}{q}+\frac{1}{p}=1$.

Multiplying this inequality (2.28) with $p_jq_k\geq 0$ and summing over j and k from 0 to n, we get

$$2t\left(\frac{1}{2}\left[\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{qk}+\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{pj}\right]-\sum_{j=0}^{n}p_jx^{\frac{q}{2}j}y^{\frac{p}{2}j}\sum_{k=0}^{n}q_kx^{\frac{p}{2}k}y^{\frac{q}{2}k}\right) \leq \frac{1}{q}\left(\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{qk}\right)+\frac{1}{p}\left(\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{pj}\right)-\sum_{j=0}^{n}p_j\,(xy)^j\sum_{k=0}^{n}q_k\,(xy)^k \leq 2T\left(\frac{1}{2}\left[\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{qk}+\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{pj}\right]-\sum_{j=0}^{n}p_jx^{\frac{q}{2}j}y^{\frac{p}{2}j}\sum_{k=0}^{n}q_kx^{\frac{p}{2}k}y^{\frac{q}{2}k}\right) \tag{2.29}$$

where $p > 1$, $\frac{1}{q} + \frac{1}{p} = 1$.

Further, if we choose in (2.5) $x = \frac{x^j}{y^j}$, $y = \frac{x^k}{y^k}$, $y \neq 0$, $j, k \in \{0, 1, 2, \ldots, n\}$ and repeating the same method as above, then we get

$$\begin{aligned}
(2.30) \qquad & 2t\left(\frac{1}{2}\left[\sum_{j=0}^{n} p_j x^{qj} \sum_{k=0}^{n} q_k y^{pk} + \sum_{k=0}^{n} q_k x^{pk} \sum_{j=0}^{n} p_j y^{qj}\right]\right. \\
& \left. - \sum_{j=0}^{n} p_j x^{\frac{q}{2}j} y^{\frac{q}{2}j} \sum_{k=0}^{n} q_k x^{\frac{p}{2}k} y^{\frac{p}{2}k}\right) \\
& \leq \frac{1}{q} \sum_{j=0}^{n} p_j x^{qj} \sum_{k=0}^{n} q_k y^{pk} + \frac{1}{p} \sum_{k=0}^{n} q_k x^{pk} \sum_{j=0}^{n} p_j y^{qj} \\
& - \sum_{j=0}^{n} p_j x^j y^{(q-1)j} \sum_{k=0}^{n} q_k x^k y^{(p-1)k} \\
& \leq 2T\left(\frac{1}{2}\left[\sum_{j=0}^{n} p_j x^{qj} \sum_{k=0}^{n} q_k y^{pk} + \sum_{k=0}^{n} q_k x^{pk} \sum_{j=0}^{n} p_j y^{qj}\right]\right. \\
& \left. - \sum_{j=0}^{n} p_j x^{\frac{q}{2}j} y^{\frac{q}{2}j} \sum_{k=0}^{n} q_k x^{\frac{p}{2}k} y^{\frac{p}{2}k}\right)
\end{aligned}$$

where $p > 1$, $\frac{1}{q} + \frac{1}{p} = 1$.

Since all the series whose partial sums are involved in inequalities (2.29) and (2.30) convergent on the interval $(0, R)$ by taking the limit as $n \to \infty$ in (2.29) and (2.30) respectively, we deduce the desired inequalities (2.26) and (2.27). □

Corollary 3. *If $g(x) = f(x)$ in (2.26) and (2.27), then we have*

$$\begin{aligned}
(2.31) \qquad & t\left[f(x^q)f(y^q) + f(x^p)f(y^p) - 2f\left(x^{\frac{q}{2}} y^{\frac{p}{2}}\right) f\left(x^{\frac{p}{2}} y^{\frac{q}{2}}\right)\right] \\
& \leq \frac{1}{q} f(x^q)f(y^q) + \frac{1}{p} f(x^p)f(y^p) - f^2(xy) \\
& \leq T\left[f(x^q)f(y^q) + f(x^p)f(y^p) - 2f\left(x^{\frac{q}{2}} y^{\frac{p}{2}}\right) f\left(x^{\frac{p}{2}} y^{\frac{q}{2}}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
(2.32) \qquad & t\left[f(x^q)f(y^p) + f(x^p)f(y^q) - 2f(x^{\frac{q}{2}} y^{\frac{q}{2}}) f(x^{\frac{p}{2}} y^{\frac{p}{2}})\right] \\
& \leq \frac{1}{q} f(x^q)f(y^p) + \frac{1}{p} f(x^p)f(y^q) - f(xy^{q-1})f(xy^{p-1}) \\
& \leq T\left[f(x^q)f(y^p) + f(x^p)f(y^q) - 2f(x^{\frac{q}{2}} y^{\frac{q}{2}}) f(x^{\frac{p}{2}} y^{\frac{p}{2}})\right]
\end{aligned}$$

.

respectively, for any $x, y \geq 0$ such that $xy, x^p, x^q, y^p, y^q \in (0, R)$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The above results (2.31) and (2.32) have some natural applications for particular functions of interest. For example, if we apply the inequality (2.31) and (2.32) for the function $f(x) = \exp(x)$, $x \in \mathbb{R}$, then we get the following inequalities

$$\begin{aligned} &t\left[\exp(x^q+y^q)+\exp(x^p+y^p)-2\exp\left(x^{\frac{q}{2}}y^{\frac{p}{2}}+x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \\ &\le \frac{1}{q}\exp(x^q+y^q)+\frac{1}{p}\exp(x^p+y^p)-\exp(2xy) \\ &\le T\left[\exp(x^q+y^q)+\exp(x^p+y^p)-2\exp\left(x^{\frac{q}{2}}y^{\frac{p}{2}}+x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} &t\left[\exp(x^q+y^p)+\exp(x^p+y^q)-2\exp(x^{\frac{q}{2}}y^{\frac{q}{2}}+x^{\frac{p}{2}}y^{\frac{p}{2}})\right] \\ &\le \frac{1}{q}\exp(x^q+y^p)+\frac{1}{p}\exp(x^p+y^q)-\exp(xy^{q-1}+xy^{p-1}) \\ &\le T\left[\exp(x^q+y^p)+\exp(x^p+y^q)-2\exp(x^{\frac{q}{2}}y^{\frac{q}{2}}+x^{\frac{p}{2}}y^{\frac{p}{2}})\right]. \end{aligned} \tag{2.34}$$

respectively, for any $x \ge 0, y > 0$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

THEOREM 5. *Let $f(x)$ and $g(x)$ be as in Theorem 4. Then one has the inequality*

$$\begin{aligned} &t\left[f(x^p)g(y^q)+g(x^p)f(y^q)-2g\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)f\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \\ &\le \frac{1}{q}f(x^p)g(y^q)+\frac{1}{p}g(x^p)f(y^q)-f\left(x^{p-1}y^{q-1}\right)g(xy) \\ &\le T\left[f(x^p)g(y^q)+g(x^p)f(y^q)-2g\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)f\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} &t\left[g\left(x^2\right)f\left(y^q\right)+f\left(x^p\right)g\left(y^2\right)-2f\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)g(xy)\right] \\ &\le \frac{1}{q}g\left(x^2\right)f\left(y^q\right)+\frac{1}{p}f\left(x^p\right)g\left(y^2\right)-f(xy)\,g\left(x^{\frac{2}{q}}y^{\frac{2}{p}}\right) \\ &\le T\left[g\left(x^2\right)f\left(y^q\right)+f\left(x^p\right)g\left(y^2\right)-2f\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)g(xy)\right]. \end{aligned} \tag{2.36}$$

PROOF. Also, if we choose in (2.5) $x = \frac{y^k}{y^j}$, $y = \frac{x^k}{x^j}$, $x, y \neq 0$ and $x = x^{\frac{2}{q}k}y^j$, $y = x^j y^{\frac{2}{p}k}$, $j, k \in \{0, 1, 2, \ldots, n\}$, then we have the following inequalities:

$$\begin{aligned} &t\left(x^{pj}y^{qk}+x^{pk}y^{qj}-2x^{\frac{p}{2}k}y^{\frac{q}{2}k}x^{\frac{p}{2}j}y^{\frac{q}{2}j}\right) \\ &\le \frac{1}{q}x^{pj}y^{qk}+\frac{1}{p}x^{pk}y^{qj}-x^{(p-1)j}y^{(q-1)j}x^k y^k \\ &\le T\left(x^{pj}y^{qk}+x^{pk}y^{qj}-2x^{\frac{p}{2}k}y^{\frac{q}{2}k}x^{\frac{p}{2}j}y^{\frac{q}{2}j}\right) \end{aligned} \tag{2.37}$$

and

$$t\left(x^{2k}y^{qj}+x^{pj}y^{2k}-2x^{\frac{p}{2}j}y^{\frac{q}{2}j}x^{k}y^{k}\right) \tag{2.38}$$
$$\leq \frac{1}{q}x^{2k}y^{qj}+\frac{1}{p}x^{pj}y^{2k}-x^{j}y^{j}x^{\frac{2}{q}k}y^{\frac{2}{p}k}$$
$$\leq T\left(x^{2k}y^{qj}+x^{pj}y^{2k}-2x^{\frac{p}{2}j}y^{\frac{q}{2}j}x^{k}y^{k}\right)$$

respectively, for any $x, y \geq 0$ and $p > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Now, repeating the same method as Theorem 4, we obtain the desired inequalities (2.35) and (2.36). □

Corollary 4. *If $g(x) = f(x)$ in (2.35) and (2.36), then we have*

$$2t\left[f(x^p)f(y^q)-f^2\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \leq f(x^p)f(y^q)-f\left(x^{p-1}y^{q-1}\right)f(xy) \tag{2.39}$$
$$\leq 2T\left[f(x^p)f(y^q)-f^2\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right]$$

and

$$t\left[f\left(x^2\right)f\left(y^q\right)+f\left(x^p\right)f\left(y^2\right)-2f\left(xy\right)f\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \tag{2.40}$$
$$\leq \frac{1}{q}f\left(x^2\right)f\left(y^q\right)+\frac{1}{p}f\left(x^p\right)f\left(y^2\right)-f\left(xy\right)f\left(x^{\frac{2}{q}}y^{\frac{2}{p}}\right)$$
$$\leq T\left[f\left(x^2\right)f\left(y^q\right)+f\left(x^p\right)g\left(y^2\right)-2f\left(xy\right)f\left(x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right].$$

The above inequalities (2.39) and (2.40) also provide some natural applications for particular functions of interest. We give here some examples as follows:

(1) If we apply the inequality (2.35) for the function $f(x) = \sinh(x)$ and $g(x) = \cosh(x)$, $x \in \mathbb{R}$, then we get

$$t\left[\sinh(x^p+y^q)-\sinh\left(2x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \tag{2.41}$$
$$\leq \frac{1}{q}\sinh(x^p)\cosh(y^q)+\frac{1}{p}\cosh(x^p)\sinh(y^q)$$
$$-\sinh\left(x^{p-1}y^{q-1}\right)\cosh\left(xy\right)$$
$$\leq T\left[\sinh(x^p+y^q)-\sinh\left(2x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right]$$

for any $x, y \geq 0$ and $p > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

(2) Further, if we consider the function $f(x) = \exp(x)$, $x \in \mathbb{R}$ and applying the inequality (2.39), then we get

$$2t\left[\exp(x^p+y^q)-\exp\left(2x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right] \tag{2.42}$$
$$\leq \exp(x^p+y^q)-\exp\left(x^{p-1}y^{q-1}+xy\right)$$
$$\leq 2T\left[\exp(x^p+y^q)-\exp\left(2x^{\frac{p}{2}}y^{\frac{q}{2}}\right)\right]$$

for any $x, y \geq 0$ and $p > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Finally, the following result holds.

THEOREM 6. *Let $f(x)$ and $g(x)$ be as in Theorem 4. Then one has the inequality*

$$\begin{aligned} & t\left[f\left(x^{p}\right) g\left(y^{2}\right)+g\left(x^{2}\right) f\left(y^{q}\right)-2 f\left(x^{\frac{p}{2}} y^{\frac{q}{2}}\right) g(x y)\right] \\ & \leq \frac{1}{q} f\left(x^{p}\right) g\left(y^{2}\right)+\frac{1}{p} g\left(x^{2}\right) f\left(y^{q}\right)-f\left(x^{p-1} y^{q-1}\right) g\left(x^{\frac{2}{p}} y^{\frac{2}{q}}\right) \\ & \leq T\left[f\left(x^{p}\right) g\left(y^{2}\right)+g\left(x^{2}\right) f\left(y^{q}\right)-2 f\left(x^{\frac{p}{2}} y^{\frac{q}{2}}\right) g(x y)\right] \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} & t\left[f\left(x^{2}\right) g\left(y^{q}\right)+g\left(x^{2}\right) f\left(y^{p}\right)-2 f\left(x y^{\frac{p}{2}}\right) g\left(x y^{\frac{q}{2}}\right)\right] \\ & \leq \frac{1}{q} f\left(x^{2}\right) g\left(y^{q}\right)+\frac{1}{p} g\left(x^{2}\right) f\left(y^{p}\right)-f\left(x^{\frac{2}{q}} y\right) g\left(x^{\frac{2}{p}} y\right) \\ & \leq T\left[f\left(x^{2}\right) g\left(y^{q}\right)+g\left(x^{2}\right) f\left(y^{p}\right)-2 f\left(x y^{\frac{p}{2}}\right) g\left(x y^{\frac{q}{2}}\right)\right] . \end{aligned} \tag{2.44}$$

PROOF. Again, the proof follows by using the same method as in Theorem 4 on choosing in (2.5) as $x=\frac{y^{\frac{2}{q} k}}{y^{j}}$, $y=\frac{x^{\frac{2}{p} k}}{x^{j}}$, $x, y \neq 0$ and $x=x^{\frac{2}{q} j} y^{k}$, $y=x^{\frac{2}{p} k} y^{j}$, $j, k \in\{0,1,2, \ldots, n\}$ respectively. □

COROLLARY 5. *If $g(x)=f(x)$ in (2.43) and (2.44), then we have*

$$\begin{aligned} & t\left[f\left(x^{p}\right) f\left(y^{2}\right)+f\left(x^{2}\right) f\left(y^{q}\right)-2 f(x y) f\left(x^{\frac{p}{2}} y^{\frac{q}{2}}\right)\right] \\ & \leq \frac{1}{q} f\left(x^{p}\right) f\left(y^{2}\right)+\frac{1}{p} f\left(x^{2}\right) f\left(y^{q}\right)-f\left(x^{p-1} y^{q-1}\right) f\left(x^{\frac{2}{p}} y^{\frac{2}{q}}\right) \\ & \leq T\left[f\left(x^{p}\right) f\left(y^{2}\right)+f\left(x^{2}\right) f\left(y^{q}\right)-2 f(x y) f\left(x^{\frac{p}{2}} y^{\frac{q}{2}}\right)\right] \end{aligned} \tag{2.45}$$

and

$$\begin{aligned} & t\left[f\left(x^{2}\right)\left[f\left(y^{q}\right)+f\left(y^{p}\right)\right]-2 f\left(x y^{\frac{p}{2}}\right) f\left(x y^{\frac{q}{2}}\right)\right] \\ & \leq f\left(x^{2}\right)\left[\frac{1}{q} f\left(y^{q}\right)+\frac{1}{p} f\left(y^{p}\right)\right]-f\left(x^{\frac{2}{q}} y\right) f\left(x^{\frac{2}{p}} y\right) \\ & \leq T\left[f\left(x^{2}\right)\left[f\left(y^{q}\right)+f\left(y^{p}\right)\right]-2 f\left(x y^{\frac{p}{2}}\right) f\left(x y^{\frac{q}{2}}\right)\right] . \end{aligned} \tag{2.46}$$

3. Applications for special functions

In this section, we provide some inequalities for special functions such as polylogarithm, hypergeometric and modified Bessel functions for the first kind by utilizing the inequality (2.9). Before that, we recall here some basic concepts of these functions that will be used in the sequal.

The *polylogarithm* $Li_n(x)$, also known as the de Jonquières function is the function defined by

$$Li_n(x)=\sum_{k=1}^{\infty} \frac{x^k}{k^n} \tag{3.1}$$

for all values of order n and $x \in (0,1)$. When $n = 1$, the polylogarithm involves the ordinary logarithm, i.e., $Li_1(x) = \ln\left(\frac{1}{1-x}\right)$, $x \in (0,1)$ and for $n = 2$,

$$Li_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} \tag{3.2}$$

is known as the *dilogarithm* or *Spence's function.*

The polylogarithm (3.1) reduces to the ratio of a polynomial in x for some integer values of n. For instance

$$Li_0(x) = \frac{x}{1-x}, \qquad Li_{-1}(x) = \frac{x}{(1-x)^2},$$
$$Li_{-2}(x) = \frac{x(x+1)}{(1-x)^3}, \quad Li_{-3}(x) = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

The *hypergeometric functions* ${}_2F_1(a,b;c;x)$ are defined by the *Gauss series*, that is,

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^n}{k!} \tag{3.3}$$

for a,b,c are real numbers with $c \neq 0,-1,-2,\ldots$ and $x \in (0,1)$, while the $(t)_k$ is known as *Pochhammer* symbol which is defined by

$$(t)_k = \begin{cases} 1 & \text{if } k = 0, \\ t(t+1)\cdots(t+k-1) & \text{if } k = 1,2,3,\ldots \end{cases}.$$

Some of the basic properties of the hypergeometric functions are

$$\begin{aligned} {}_2F_1(a,b;c;x) &= {}_2F_1(b,a;c;x), \\ {}_2F_1(a,b;c;x) &= (1-x)^{c-a-b}\ {}_2F_1(c-a,c-b;c;x), \\ {}_2F_1(a,b;c;x) &= \frac{1}{(1-x)^a}\ {}_2F_1\left(c,c-b;c;\frac{x}{x-1}\right), \\ {}_2F_1(a,b;b;x) &= \frac{x}{(1-x)^a}, \\ c\,{}_2F_1'(a,b;b;x) &= ab\,{}_2F_1(a+1,b+1;c+1;x). \end{aligned}$$

Hypergeometric functions (3.3) with particular values of a,b and c reduce to elementary functions, such as

$$\begin{aligned} {}_2F_1(1,1;1;x) &= {}_2F_1(1,2;2;x) = \frac{1}{1-x}, \\ {}_2F_1(1,2;1;x) &= \frac{1}{(1-x)^2}, \\ {}_2F_1(1,1;2;x) &= \frac{1}{x}\ln\left(\frac{1}{1-x}\right), \\ {}_2F_1(1,1;2;-x) &= \frac{1}{x}\ln(1+x). \end{aligned}$$

Further, the *Bessel functions* of the first kind, denoted as $J_\alpha(x)$ are defined by the power series

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\alpha+k)!} \left(\frac{x}{2}\right)^{2k+\alpha} \tag{3.4}$$

for any $\alpha, x \in \mathbb{R}$ such that $x \in (0,1)$. If x is replaced by the arguments $\pm ix$, then from (3.4) we have

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{k=0}^{\infty} \frac{1}{k!\,(\alpha+k)!} \left(\frac{x}{2}\right)^{2k+\alpha} \tag{3.5}$$

for any $\alpha, x \in \mathbb{R}$ such that $x \in (0,1)$. These functions (3.5) are called the *modified Bessel functions* of the first kind.

In the following, we state some particular values of $I_\alpha(x)$ for $\alpha \in \mathbb{R}$ and $x > 0$.

$$\begin{aligned} I_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \sinh(x), \quad I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cosh(x), \\ I_{\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \left(\cosh(x) - \frac{\sinh(x)}{x}\right), \\ I_{-\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \left(\sinh(x) - \frac{\cosh(x)}{x}\right). \end{aligned}$$

It is clearly seen that from (3.1), (3.3) and (3.5), that is, $Li_n(x)$, ${}_2F_1(a,b;c;x)$ and $I_\alpha(x)$ are power series functions with positive coefficients and convergent on the interval $(0,1)$. Therefore, all the results in the above section hold true. For instance, from (2.9) we have the following inequalities.

Corollary 6. *If $Li_n(x)$ is the polylogarithm function, then we have*

$$Li_n\left(y^\nu z^{1-\nu}\right) Li_n\left(y^{1-\nu} z^\nu\right) \le Li_n(y) Li_n(z) \tag{3.6}$$

for $y, z, y^\nu z^{1-\nu}, y^{1-\nu} z^\nu \in (0,1)$, $\nu \in [0,1]$ and $n \in \mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$.

In particular, if $n = 0$ in (3.6), then we get the inequality (2.11). Also, if $n = 1$ in (3.6), then we get the inequality (2.12). Further, we obtain the following inequality by choosing $n = 2$ in (3.6)

$$Li_2\left(x^\nu y^{1-\nu}\right) Li_2\left(x^{1-\nu} y^\nu\right) \le Li_2(x) Li_2(y) \tag{3.7}$$

for $y, z, y^\nu z^{1-\nu}, y^{1-\nu} z^\nu \in (0,1)$ and $\nu \in [0,1]$, where $Li_2(x)$ is the dilogarithm function which is difined in (3.2).

Corollary 7. *If ${}_2F_1(a,b;c;x)$ is the hypergeometric function, the we have*

$${}_2F_1\left(a,b;c;y^\nu z^{1-\nu}\right) {}_2F_1\left(a,b;c;y^{1-\nu} z^\nu\right) \le {}_2F_1(a,b;c;y)\, {}_2F_1(a,b;c;z) \tag{3.8}$$

for $y, z, y^\nu z^{1-\nu}, y^{1-\nu} z^\nu \in (0,1)$ and $\nu \in [0,1]$.

In particular, if we choose $a = b = c = 1$, then the inequality (3.8) reduces to (2.11). In fact, the inequality (3.8) reduces to (2.11) for any $a, b, c \in \mathbb{R}$ such that $c = b \neq 0, -1, -2, \ldots$.

Corollary 8. *If $I_\alpha(x)$ is the modified Bessel function for the first kind, then we have*

$$I_\alpha\left(y^\nu z^{1-\nu}\right) I_\alpha\left(y^{1-\nu} z^\nu\right) \le I_\alpha(y) I_\alpha(z) \tag{3.9}$$

for $y, z, y^\nu z^{1-\nu}, y^{1-\nu} z^\nu \in (0,1)$ and $\nu \in [0,1]$.

In particular, if $\alpha = 0$, then from (3.9) we get

$$I_0\left(y^\nu z^{1-\nu}\right) I_0\left(y^{1-\nu} z^\nu\right) \le I_0(y) I_0(z) \tag{3.10}$$

for $y, z, y^\nu z^{1-\nu}, y^{1-\nu} z^\nu \in (0,1)$ and $\nu \in [0,1]$, where $I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{4^k (k!)^2}$.

If in (3.9) we choose $\alpha = \frac{1}{2}$, *the we obtain*

$$\sinh\left(y^{\nu}z^{1-\nu}\right)\sinh\left(y^{1-\nu}z^{\nu}\right) \leq \sinh(y)\sinh(z) \tag{3.11}$$

for $y, z, y^{\nu}z^{1-\nu}, y^{1-\nu}z^{\nu} \in (0,1)$ *and* $\nu \in [0,1]$.

If we take $\alpha = \frac{3}{2}$, *the we get from (3.9)*

$$\begin{aligned} &\left[y^{\nu}z^{1-\nu}\cosh\left(y^{\nu}z^{1-\nu}\right) - \sinh\left(y^{\nu}z^{1-\nu}\right)\right] \\ &\times\left[y^{1-\nu}z^{\nu}\cosh\left(y^{1-\nu}z^{\nu}\right) - \sinh\left(y^{1-\nu}z^{\nu}\right)\right] \\ &\leq \left[y\cosh(y) - \sinh(y)\right]\left[z\cosh(z) - \sinh(z)\right] \end{aligned} \tag{3.12}$$

for $y, z, y^{\nu}z^{1-\nu}, y^{1-\nu}z^{\nu} \in (0,1)$ *and* $\nu \in [0,1]$.

Remark 4. *For* $\alpha = -\frac{1}{2}$ *and* $-\frac{3}{2}$ *in (3.9), we get the dual results, namely*

$$\cosh\left(y^{\nu}z^{1-\nu}\right)\cosh\left(y^{1-\nu}z^{\nu}\right) \leq \cosh(y)\cosh(z) \tag{3.13}$$

and

$$\begin{aligned} &\left[y^{\nu}z^{1-\nu}\sinh\left(y^{\nu}z^{1-\nu}\right) - \cosh\left(y^{\nu}z^{1-\nu}\right)\right] \\ &\times\left[y^{1-\nu}z^{\nu}\sinh\left(y^{1-\nu}z^{\nu}\right) - \cosh\left(y^{1-\nu}z^{\nu}\right)\right] \\ &\leq \left[y\sinh(y) - \cosh(y)\right]\left[z\sinh(z) - \cosh(z)\right] \end{aligned} \tag{3.14}$$

respectively, for $y, z, y^{\nu}z^{1-\nu}, y^{1-\nu}z^{\nu} \in (0,1)$ *and* $\nu \in [0,1]$.

Other inequalities connected to these special functions for further reading can be found in the literature, see [**6**], [**7**], [**15**], [**19**], [**20**] and the references cited therein.

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