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APPLICATIONS OF KATO'S INEQUALITY FOR n-TUPLES OF OPERATORS IN HILBERT SPACES, (II)

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ABSTRACT. In this paper, by the use of the famous Kato's inequality for bounded linear operators, we establish some new inequalities for n-tuples of operators and apply them for functions of normal operators defined by power series as well as for some norms and numerical radii that arise in multivariate Operator Theory. They provide a natural continuation of the results in previous paper with (I) in the title.

1. INTRODUCTION

In 1952, Kato [18] proved the following generalization of Schwarz inequality:

$$|\langle Tx, y\rangle|^2 \leq \langle (T^*T)^{\alpha} x, x\rangle \left\langle (TT^*)^{1-\alpha} y, y\right\rangle, \tag{1.1}$$

for any $x, y \in H$, $\alpha \in [0,1]$ and T is a bounded linear operator on H.

Utilizing the operator modulus notation, we can write (1.1) as follows

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle. \tag{1.2}$$

For related results to the Kato's inequality, see [7]-[16], [18]-[24] and [32].

In the recent paper [4], by employing Kato's inequality (1.2), Dragomir, Cho and Kim established the following results for sequences of bonded linear operators on complex Hilbert spaces:

Theorem 1. *Let $(T_1, ..., T_n) \in \mathcal{B}(H) \times ... \times \mathcal{B}(H) := \mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ an n-tuple of nonnegative weights not all of them equal to zero, then*

$$\sum_{j=1}^{n} p_j |\langle T_j x, y\rangle| \leq \left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j|^{2\alpha} + |T_j|^{2(1-\alpha)}}{2} \right) x, x \right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n} p_j \left(\frac{|T_j^*|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right) y, y \right\rangle^{1/2} \tag{1.3}$$

for any $\alpha \in [0,1]$ and any $x, y \in H$,

and

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Theorem 2. *With the assumptions in Theorem 1, we have*

$$\begin{aligned}
(1.4)\qquad & \sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right|^2 \\
& \leq \frac{1}{2}\sum_{j=1}^{n} p_j \left(\|T_j x\|^{2\alpha}\left\|T_j^* y\right\|^{2(1-\alpha)} + \left\|T_j^* y\right\|^{2\alpha}\|T_j x\|^{2(1-\alpha)}\right) \\
& \leq \frac{1}{2}\left[\left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2\right)^{1-\alpha}\right. \\
& \left. + \left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{1-\alpha}\left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2\right)^{\alpha}\right] \\
& \leq \frac{1}{2}\sum_{j=1}^{n} p_j \left(\|T_j x\|^2 + \left\|T_j^* y\right\|^2\right)
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

For various related results, see the papers [1], [25]-[29] and [5]-[6].

Motivated by the above results, we establish in this paper more inequalities for n-tuples of bounded linear operators that can be obtained from the Kato's result (1.2) and apply them for functions of normal operators defined by power series as well as for some norms and numerical radii that can be associated with this n-tuples of bonded linear operators on Hilbert spaces. The paper is a natural continuation [4].

2. Some Inequalities for n-Tuples of Operators

The following result holds:

Theorem 3. *Let $(T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ an n-tuple of nonnegative weights not all of them equal to zero, then*

$$\begin{aligned}
(2.1)\qquad \left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{T_j + T_j^*}{2}\right) x, y\right\rangle\right| & \leq \sum_{j=1}^{n} p_j \left|\left\langle \frac{T_j + T_j^*}{2} x, y\right\rangle\right| \\
& \leq \sum_{j=1}^{n} p_j \left[\frac{\left|\langle T_j x, y\rangle\right| + \left|\left\langle T_j^* x, y\right\rangle\right|}{2}\right] \\
& \leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{\left|T_j\right|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2}\right] x, x\right\rangle^{1/2} \\
& \times \left\langle \sum_{j=1}^{n} p_j \left[\frac{\left|T_j\right|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2}\right] y, y\right\rangle^{1/2}
\end{aligned}$$

for any $\alpha \in [0,1]$ *and, in particular, for* $\alpha = \frac{1}{2}$

$$\left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{T_j + T_j^*}{2}\right) x, y\right\rangle\right| \leq \sum_{j=1}^{n} p_j \left|\left\langle \frac{T_j + T_j^*}{2} x, y\right\rangle\right| \tag{2.2}$$
$$\leq \sum_{j=1}^{n} p_j \left[\frac{|\langle T_j x, y\rangle| + \left|\langle T_j^* x, y\rangle\right|}{2}\right]$$
$$\leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] x, x\right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] y, y\right\rangle^{1/2}$$

for any $x, y \in H$.

Proof. The first two inequalities are obvious by the properties of the modulus.

Utilising Kato's inequality we have

$$|\langle T_j x, y\rangle| \leq \left\langle |T_j|^{2\alpha} x, x\right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y\right\rangle^{1/2} \tag{2.3}$$

and, by replacing x with y we have

$$|\langle T_j y, x\rangle| \leq \left\langle |T_j|^{2\alpha} y, y\right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2(1-\alpha)} x, x\right\rangle^{1/2}$$

i.e.,

$$\left|\langle T_j^* x, y\rangle\right| \leq \left\langle \left|T_j^*\right|^{2(1-\alpha)} x, x\right\rangle^{1/2} \left\langle |T_j|^{2\alpha} y, y\right\rangle^{1/2} \tag{2.4}$$

for any $j \in \{1, ..., n\}$ and $x, y \in H$.

Adding the inequalities (2.3) and (2.4) and utilizing the elementary inequality

$$ab + cd \leq \left(a^2 + c^2\right)^{1/2} \left(b^2 + d^2\right)^{1/2}, a, b, c, d \geq 0$$

we get

$$|\langle T_j x, y\rangle| + \left|\langle T_j^* x, y\rangle\right| \leq \left\langle |T_j|^{2\alpha} x, x\right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y\right\rangle^{1/2} \tag{2.5}$$
$$+ \left\langle \left|T_j^*\right|^{2(1-\alpha)} x, x\right\rangle^{1/2} \left\langle |T_j|^{2\alpha} y, y\right\rangle^{1/2}$$
$$\leq \left\langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] x, x\right\rangle^{1/2} \times \left\langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}\right] y, y\right\rangle^{1/2}$$

for any $j \in \{1, ..., n\}$ and $x, y \in H$.

Multiplying the inequalities (2.5) by $p_j \geq 0$ and then summing over j from 1 to n and utilizing the weighted Cauchy-Buniakowski-Schwarz inequality we have

$$\begin{aligned}
(2.6)\qquad & \sum_{j=1}^{n} p_j \left[|\langle T_j x, y\rangle| + \left|\langle T_j^* x, y\rangle\right| \right] \\
& \leq \sum_{j=1}^{n} p_j \left\langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)} \right] x, x \right\rangle^{1/2} \left\langle \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)} \right] y, y \right\rangle^{1/2} \\
& \leq \left\langle \sum_{j=1}^{n} p_j \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)} \right] x, x \right\rangle^{1/2} \\
& \times \left\langle \sum_{j=1}^{n} p_j \left[|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)} \right] y, y \right\rangle^{1/2}
\end{aligned}$$

for $x, y \in H$, which is equivalent with the third inequality in (2.1). □

Remark 1. *The particular case $y = x$ is of interest for providing numerical radii inequalities and can be stated as:*

$$\begin{aligned}
(2.7)\qquad \left| \left\langle \sum_{j=1}^{n} p_j \left(\frac{T_j + T_j^*}{2} \right) x, x \right\rangle \right| & \leq \sum_{j=1}^{n} p_j \left| \left\langle \frac{T_j + T_j^*}{2} x, x \right\rangle \right| \\
& \leq \sum_{j=1}^{n} p_j \left| \langle T_j x, x \rangle \right| \\
& \leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] x, x \right\rangle
\end{aligned}$$

for any $\alpha \in [0, 1]$ and, for $\alpha = \frac{1}{2}$,

$$\begin{aligned}
(2.8)\qquad \left| \left\langle \sum_{j=1}^{n} p_j \left(\frac{T_j + T_j^*}{2} \right) x, x \right\rangle \right| & \leq \sum_{j=1}^{n} p_j \left| \left\langle \frac{T_j + T_j^*}{2} x, x \right\rangle \right| \\
& \leq \sum_{j=1}^{n} p_j \left| \langle T_j x, x \rangle \right| \\
& \leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + \left|T_j^*\right|}{2} \right] x, x \right\rangle
\end{aligned}$$

for any $x \in H$.

The case of unitary vectors provides more refinements as follows:

Remark 2. *With the assumptions in Theorem 3, we have*

$$\sum_{j=1}^{n} p_j \left[\frac{|\langle T_j x, y\rangle| + \left|\left\langle T_j^* x, y\right\rangle\right|}{2} \right] \tag{2.9}$$
$$\leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] x, x \right\rangle^{1/2}$$
$$\times \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] y, y \right\rangle^{1/2}$$
$$\leq \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] \right)^{1/2} x, x \right\rangle$$
$$\times \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] \right)^{1/2} y, y \right\rangle$$
$$\leq \frac{1}{2} \left[\left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] \right)^{1/2} x, x \right\rangle^{2} \right.$$
$$\left. + \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] \right)^{1/2} y, y \right\rangle^{2} \right]$$
$$\leq \frac{1}{2} \left[\left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] x, x \right\rangle \right.$$
$$\left. + \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right] y, y \right\rangle \right]$$

for any $\alpha \in [0, 1]$ *and, in particular,*

$$\sum_{j=1}^{n} p_j \left[\frac{|\langle T_j x, y\rangle| + \left|\left\langle T_j^* x, y\right\rangle\right|}{2} \right] \tag{2.10}$$
$$\leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + \left|T_j^*\right|}{2} \right] x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + \left|T_j^*\right|}{2} \right] y, y \right\rangle^{1/2}$$
$$\leq \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + \left|T_j^*\right|}{2} \right] \right)^{1/2} x, x \right\rangle \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + \left|T_j^*\right|}{2} \right] \right)^{1/2} y, y \right\rangle$$

$$\leq \frac{1}{2}\left[\left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right]\right)^{1/2} x, x\right\rangle^2 + \left\langle \left(\sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right]\right)^{1/2} y, y\right\rangle^2\right]$$

$$\leq \frac{1}{2}\left[\left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] x, x\right\rangle + \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j| + |T_j^*|}{2}\right] y, y\right\rangle\right]$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

The proofs follow by utilizing the Hölder-McCarthy inequalities $\langle P^r x, x\rangle \leq \langle Px, x\rangle^r$ and $\langle Px, x\rangle^s \leq \langle P^s x, x\rangle$ that hold for the positive operator P, for $r \in (0,1)$, $s \in [1, \infty)$ and $x \in H$ with $\|x\| = 1$. The details are omitted.

In order to employ the above result in obtaining some inequalities for functions of normal operators defined by power series, we need the following version of (2.1).

Remark 3. *If we write the inequality (2.1) for the normal operators* $N_j, j \in \{1, ..., n\}$ *then we get*

$$\left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{N_j + N_j^*}{2}\right) x, y\right\rangle\right| \leq \sum_{j=1}^{n} p_j \left|\left\langle \frac{N_j + N_j^*}{2} x, y\right\rangle\right| \tag{2.11}$$
$$\leq \sum_{j=1}^{n} p_j \left[\frac{|\langle N_j x, y\rangle| + |\langle N_j^* x, y\rangle|}{2}\right]$$
$$\leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|N_j|^{2\alpha} + |N_j|^{2(1-\alpha)}}{2}\right] x, x\right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n} p_j \left[\frac{|N_j|^{2\alpha} + |N_j|^{2(1-\alpha)}}{2}\right] y, y\right\rangle^{1/2}$$

for any $\alpha \in [0,1]$ *and, in particular, for* $\alpha = \frac{1}{2}$

$$\left|\left\langle \sum_{j=1}^{n} p_j \left(\frac{N_j + N_j^*}{2}\right) x, y\right\rangle\right| \leq \sum_{j=1}^{n} p_j \left|\left\langle \frac{N_j + N_j^*}{2} x, y\right\rangle\right| \tag{2.12}$$
$$\leq \sum_{j=1}^{n} p_j \left[\frac{|\langle N_j x, y\rangle| + |\langle N_j^* x, y\rangle|}{2}\right]$$
$$\leq \left\langle \sum_{j=1}^{n} p_j |N_j| x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |N_j| y, y\right\rangle^{1/2}$$

for any $x, y \in H$.

The following results involving quadratics also holds:

Theorem 4. *Let* $(T_1,...,T_n) \in \mathcal{B}^{(n)}(H)$ *be an* n*-tuple of bounded linear operators on the Hilbert space* $(H;\langle\cdot,\cdot\rangle)$ *and* $(p_1,...,p_n) \in \mathbb{R}_+^{*n}$ *an* n*-tuple of nonnegative weights not all of them equal to zero, then*

$$
\begin{aligned}
(2.13)\qquad & \sum_{j=1}^{n} p_j \left[|\langle T_j x, y\rangle|^2 + \left|\langle T_j^* x, y\rangle\right|^2 \right] \\
& \leq \sum_{j=1}^{n} p_j \left[\|T_j x\|^{2\alpha} \left\|T_j^* y\right\|^{2(1-\alpha)} + \|T_j y\|^{2\alpha} \left\|T_j^* x\right\|^{2(1-\alpha)} \right] \\
& \leq \left(\sum_{j=1}^{n} p_j \|T_j x\|^2 \right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2 \right)^{1-\alpha} \\
& + \left(\sum_{j=1}^{n} p_j \|T_j y\|^2 \right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left\|T_j^* x\right\|^2 \right)^{1-\alpha} \\
& \leq \left(\sum_{j=1}^{n} p_j \left[\|T_j x\|^2 + \|T_j y\|^2 \right] \right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left[\left\|T_j^* y\right\|^2 + \left\|T_j^* x\right\|^2 \right] \right)^{1-\alpha}
\end{aligned}
$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and* $\alpha \in [0,1]$.

Proof. We must prove the inequalities only in the case $\alpha \in (0,1)$, since the case $\alpha = 0$ or $\alpha = 1$ follows directly from the corresponding case of Kato's inequality.

Utilising Kato's inequality we have

$$
(2.14)\qquad |\langle T_j x, y\rangle|^2 \leq \left\langle |T_j|^{2\alpha} x, x \right\rangle \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y \right\rangle
$$

and, by replacing x with y we have

$$
(2.15)\qquad \left|\langle T_j^* x, y\rangle\right|^2 \leq \left\langle \left|T_j^*\right|^{2(1-\alpha)} x, x \right\rangle \left\langle |T_j|^{2\alpha} y, y \right\rangle
$$

for any $j \in \{1,...,n\}$ and $x, y \in H$.

By Hölder-McCarthy inequality $\langle P^r x, x\rangle \leq \langle Px, x\rangle^r$ for $r \in (0,1)$ and $x \in H$ with $\|x\| = 1$ we also have

$$
(2.16)\qquad \left\langle |T_j|^{2\alpha} x, x \right\rangle \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y \right\rangle \leq \left\langle |T_j|^2 x, x \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha}
$$

and

$$
(2.17)\qquad \left\langle \left|T_j^*\right|^{2(1-\alpha)} x, x \right\rangle \left\langle |T_j|^{2\alpha} y, y \right\rangle \leq \left\langle |T_j|^2 y, y \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 x, x \right\rangle^{1-\alpha}
$$

for any $j \in \{1,...,n\}$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

We then obtain by summation

$$
\begin{aligned}
(2.18)\qquad & |\langle T_j x, y\rangle|^2 + \left|\langle T_j^* x, y\rangle\right|^2 \\
& \leq \left\langle |T_j|^2 x, x \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha} + \left\langle |T_j|^2 y, y \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 x, x \right\rangle^{1-\alpha}
\end{aligned}
$$

for any $j \in \{1,...,n\}$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, if we multiply (2.18) with $p_j \geq 0$, sum over j from 1 to n we get

$$\sum_{j=1}^{n} p_j \left[\left|\langle T_j x, y\rangle\right|^2 + \left|\langle T_j^* x, y\rangle\right|^2 \right] \tag{2.19}$$
$$\leq \sum_{j=1}^{n} p_j \left\langle \left|T_j\right|^2 x, x \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha}$$
$$+ \sum_{j=1}^{n} p_j \left\langle \left|T_j\right|^2 y, y \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 x, x \right\rangle^{1-\alpha}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in (0,1)$.

Since $\left\langle \left|T_j\right|^2 x, x \right\rangle = \|T_j x\|^2, \left\langle \left|T_j^*\right|^2 y, y \right\rangle = \left\|T_j^* y\right\|^2, \left\langle \left|T_j\right|^2 y, y \right\rangle = \|T_j y\|^2$ and $\left\langle \left|T_j^*\right|^2 x, x \right\rangle = \left\|T_j^* x\right\|^2$ $j \in \{1, ..., n\}$, then we get from (2.19) the first part of (2.13).

Now, on making use of the weighted Hölder discrete inequality

$$\sum_{j=1}^{n} p_j a_j b_j \leq \left(\sum_{j=1}^{n} p_j a_j^p\right)^{1/p} \left(\sum_{j=1}^{n} p_j b_j^q\right)^{1/q}, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

where $(a_1, ..., a_n), (b_1, ..., b_n) \in \mathbb{R}_+^n$, we also have

$$\sum_{j=1}^{n} p_j \|T_j x\|^{2\alpha} \left\|T_j^* y\right\|^{2(1-\alpha)} \leq \left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^{2(1-\alpha)}\right)^{1-\alpha}$$

and

$$\sum_{j=1}^{n} p_j \|T_j y\|^{2\alpha} \left\|T_j^* x\right\|^{2(1-\alpha)} \leq \left(\sum_{j=1}^{n} p_j \|T_j y\|^2\right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left\|T_j^* x\right\|^2\right)^{1-\alpha}.$$

Summing these two inequalities we deduce the second inequality in (2.13).

Finally, on utilizing the Hölder inequality

$$ab + cd \leq (a^p + c^p)^{1/p} (b^q + d^q)^{1/q}, a, b, c, d \geq 0$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left(\sum_{j=1}^{n} p_j \|T_j x\|^2\right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2\right)^{1-\alpha}$$
$$+ \left(\sum_{j=1}^{n} p_j \|T_j y\|^2\right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left\|T_j^* x\right\|^2\right)^{1-\alpha}$$
$$\leq \left(\sum_{j=1}^{n} p_j \|T_j x\|^2 + \sum_{j=1}^{n} p_j \|T_j y\|^2\right)^{\alpha} \left(\sum_{j=1}^{n} p_j \left\|T_j^* y\right\|^2 + \sum_{j=1}^{n} p_j \left\|T_j^* x\right\|^2\right)^{1-\alpha}$$

and the proof is completed. □

Remark 4. *Utilizing the elementary inequality for complex numbers*

$$\left|\frac{z+w}{2}\right|^2 \le \frac{|z|^2+|w|^2}{2}, z, w \in \mathbb{C}$$

we have

$$\sum_{j=1}^n p_j \left[\left|\left\langle\left(\frac{T_j+T_j^*}{2}\right)x, y\right\rangle\right|^2\right] \le \sum_{j=1}^n p_j \left[\frac{|\langle T_j x, y\rangle|^2 + \left|\left\langle T_j^* x, y\right\rangle\right|^2}{2}\right] \tag{2.20}$$

and by the weighted arithmetic mean-geometric mean inequality

$$a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha) b, a, b \ge 0, \alpha \in [0,1]$$

we also have

$$\begin{aligned} &\left(\sum_{j=1}^n p_j \left[\|T_j x\|^2 + \|T_j y\|^2\right]\right)^\alpha \left(\sum_{j=1}^n p_j \left[\left\|T_j^* y\right\|^2 + \left\|T_j^* x\right\|^2\right]\right)^{1-\alpha} \\ &\le \alpha \sum_{j=1}^n p_j \left[\|T_j x\|^2 + \|T_j y\|^2\right] + (1-\alpha) \sum_{j=1}^n p_j \left[\left\|T_j^* y\right\|^2 + \left\|T_j^* x\right\|^2\right]. \end{aligned} \tag{2.21}$$

If we choose $\alpha = \frac{1}{2}$ and use (2.4), (2.20) and (2.22) we derive

$$\begin{aligned} &\sum_{j=1}^n p_j \left[\left|\left\langle\left(\frac{T_j+T_j^*}{2}\right)x, y\right\rangle\right|^2\right] \\ &\le \sum_{j=1}^n p_j \left[\frac{|\langle T_j x, y\rangle|^2 + \left|\left\langle T_j^* x, y\right\rangle\right|^2}{2}\right] \\ &\le \frac{1}{2} \sum_{j=1}^n p_j \left[\|T_j x\| \left\|T_j^* y\right\| + \|T_j y\| \left\|T_j^* x\right\|\right] \\ &\le \frac{1}{2} \left(\sum_{j=1}^n p_j \|T_j x\|^2\right)^{1/2} \left(\sum_{j=1}^n p_j \left\|T_j^* y\right\|\right)^{1/2} \\ &+ \frac{1}{2} \left(\sum_{j=1}^n p_j \|T_j y\|^2\right)^{1/2} \left(\sum_{j=1}^n p_j \left\|T_j^* x\right\|^2\right)^{1/2} \\ &\le \left(\sum_{j=1}^n p_j \left[\frac{\|T_j x\|^2 + \|T_j y\|^2}{2}\right]\right)^{1/2} \left(\sum_{j=1}^n p_j \left[\frac{\left\|T_j^* y\right\|^2 + \left\|T_j^* x\right\|^2}{2}\right]\right)^{1/2} \\ &\le \sum_{j=1}^n p_j \left[\frac{\|T_j x\|^2 + \|T_j y\|^2 + \left\|T_j^* y\right\|^2 + \left\|T_j^* x\right\|^2}{4}\right] \\ &= \frac{1}{2}\left[\sum_{j=1}^n p_j \left\langle \frac{|T_j|^2 + \left|T_j^*\right|^2}{2} x, x\right\rangle + \sum_{j=1}^n p_j \left\langle \frac{|T_j|^2 + \left|T_j^*\right|^2}{2} y, y\right\rangle\right] \end{aligned} \tag{2.22}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Remark 5. *The case of normal operators N_j, $j \in \{1, ..., n\}$ is of interest for functions of operators and maybe stated as follows:*

$$
\begin{aligned}
&\sum_{j=1}^{n} p_j \left[\left| \left\langle \left(\frac{N_j + N_j^*}{2} \right) x, y \right\rangle \right|^2 \right] \\
&\leq \sum_{j=1}^{n} p_j \left[\frac{\left|\langle N_j x, y\rangle\right|^2 + \left|\langle N_j^* x, y\rangle\right|^2}{2} \right] \\
&\leq \frac{1}{2} \sum_{j=1}^{n} p_j \left[\|N_j x\|^{2\alpha} \|N_j y\|^{2(1-\alpha)} + \|N_j y\|^{2\alpha} \|N_j x\|^{2(1-\alpha)} \right] \\
&\leq \frac{1}{2} \left(\sum_{j=1}^{n} p_j \|N_j x\|^2 \right)^{\alpha} \left(\sum_{j=1}^{n} p_j \|N_j y\|^2 \right)^{1-\alpha} \\
&+ \frac{1}{2} \left(\sum_{j=1}^{n} p_j \|N_j y\|^2 \right)^{\alpha} \left(\sum_{j=1}^{n} p_j \|N_j x\|^2 \right)^{1-\alpha} \\
&\leq \frac{1}{2} \sum_{j=1}^{n} p_j \left[\|N_j x\|^2 + \|N_j y\|^2 \right]
\end{aligned} \tag{2.23}
$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$.

3. Inequalities for Functions of Normal Operators

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$, then $f_A = f$.

As some natural examples that are useful for applications, we can point out that, if

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z}, \ z \in D(0,1); \\
g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \ z \in \mathbb{C}; \\
h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \ z \in \mathbb{C}; \\
l(z) &= \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \ z \in D(0,1);
\end{aligned} \tag{3.1}
$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \ z \in D(0,1); \tag{3.2}$$

$$g_A(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \ z \in \mathbb{C};$$

$$h_A(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \ z \in \mathbb{C};$$

$$l_A(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ z \in D(0,1).$$

The following result is a functional inequality for normal operators that can be obtained from (2.1).

Theorem 5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If N is a normal operator on the Hilbert space H and for $\alpha \in [0,1]$ we have that $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then we have the inequalities*

$$\left| \left\langle \left(\frac{f(N) + f(N^*)}{2} \right) x, y \right\rangle \right| \le \left\langle \left(\frac{f_A\left(|N|^{2\alpha}\right) + f\left(|N|^{2(1-\alpha)}\right)}{2} \right) x, x \right\rangle^{1/2} \times \left\langle \left(\frac{f_A\left(|N|^{2\alpha}\right) + f\left(|N|^{2(1-\alpha)}\right)}{2} \right) y, y \right\rangle^{1/2} \tag{3.3}$$

for any $x, y \in H$.

Proof. If N is a normal operator, then for any $j \in \mathbb{N}$ we have that

$$\left|N^j\right|^2 = (N^* N)^j = |N|^{2j}.$$

Utilising the inequality (2.11) we have

$$\begin{aligned}(3.4)\qquad &\left|\left\langle \sum_{j=0}^{n} a_j \left(\frac{N^j + (N^*)^j}{2}\right) x, y\right\rangle\right| \\ &\le \sum_{j=0}^{n} |a|_j \left|\left\langle \frac{N^j + (N^*)^j}{2} x, y\right\rangle\right| \\ &\le \sum_{j=0}^{n} |a|_j \left[\frac{\left|\left\langle N^j x, y\right\rangle\right| + \left|\left\langle (N^*)^j x, y\right\rangle\right|}{2}\right] \\ &\le \left\langle \sum_{j=0}^{n} |a|_j \left[\frac{\left(|N|^{2\alpha}\right)^j + \left(|N|^{2(1-\alpha)}\right)^j}{2}\right] x, x\right\rangle^{1/2} \\ &\times \left\langle \sum_{j=1}^{n} |a|_0 \left[\frac{\left(|N|^{2\alpha}\right)^j + \left(|N|^{2(1-\alpha)}\right)^j}{2}\right] y, y\right\rangle^{1/2}\end{aligned}$$

for any $\alpha \in [0,1]$, $n \in \mathbb{N}$ and any $x, y \in H$.

Since $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then it follows that the series $\sum_{j=0}^{\infty} |a_j| \left(|N|^{2\alpha}\right)^j$ and $\sum_{j=0}^{\infty} |a_j| \left(|N|^{2(1-\alpha)}\right)^j$ are absolute convergent in $\mathcal{B}(H)$, and by taking the limit over $n \to \infty$ in (3.4) we deduce the desired result (3.3). □

Remark 6. *With the assumptions in Theorem 5, if we take the supremum over $y \in H, \|y\| = 1$, then we get the vector inequality*

$$(3.5)\qquad \left\|\left(\frac{f(N) + f(N^*)}{2}\right) x\right\| \le \frac{1}{2}\left\langle \left(f_A\left(|N|^{2\alpha}\right) + f\left(|N|^{2(1-\alpha)}\right)\right) x, x\right\rangle^{1/2} \times \left\|f_A\left(|N|^{2\alpha}\right) + f\left(|N|^{2(1-\alpha)}\right)\right\|$$

for any $x \in H$, which in its turn produces the norm inequality

$$(3.6)\qquad \left\|\frac{f(N) + f(N^*)}{2}\right\| \le \frac{1}{2}\left\|f_A\left(|N|^{2\alpha}\right) + f\left(|N|^{2(1-\alpha)}\right)\right\|$$

for any $\alpha \in [0,1]$.

Moreover, if we take $y = x$ in (3.3), then we have

$$(3.7)\qquad \left|\left\langle \frac{f(N) + f(N^*)}{2} x, x\right\rangle\right| \le \frac{1}{2}\left\langle \left[f_A\left(|N|^{2\alpha}\right) + f\left(|N|^{2(1-\alpha)}\right)\right] x, x\right\rangle$$

for any $x \in H$, which, by taking the supremum over $x \in H, \|x\| = 1$ generates the numerical radius inequality

$$(3.8)\qquad w\left(\frac{f(N) + f(N^*)}{2}\right) \le \frac{1}{2} w\left[f_A\left(|N|^{2\alpha}\right) + f\left(|N|^{2(1-\alpha)}\right)\right]$$

for any $\alpha \in [0,1]$.

Making use of the examples in (3.1) and (3.2) we can state the vector inequalities:

$$\begin{aligned}
&\left|\left\langle\left[\frac{\ln\left(1_H+N\right)^{-1}+\ln\left(1_H+N^*\right)^{-1}}{2}\right]x,y\right\rangle\right| \\
&\leq \frac{1}{2}\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]x,x\right\rangle^{1/2} \\
&\times\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]y,y\right\rangle^{1/2},
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\left|\left\langle\left[\frac{\left(1_H+N\right)^{-1}+\left(1_H+N^*\right)^{-1}}{2}\right]x,y\right\rangle\right| \\
&\leq \frac{1}{2}\left\langle\left[\left(1_H-|N|^{2\alpha}\right)^{-1}+\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]x,x\right\rangle^{1/2} \\
&\times\left\langle\left[\ln\left(1_H-|N|^{2\alpha}\right)^{-1}+\ln\left(1_H-|N|^{2(1-\alpha)}\right)^{-1}\right]y,y\right\rangle^{1/2},
\end{aligned} \tag{3.10}$$

for any $x,y\in H$ and $\|N\|<1$.

We also have the inequalities

$$\begin{aligned}
&\left|\left\langle\left[\frac{\sin(N)+\sin(N^*)}{2}\right]x,y\right\rangle\right| \\
&\leq \frac{1}{2}\left\langle\left[\sinh\left(|N|^{2\alpha}\right)+\sinh\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \\
&\times\left\langle\left[\sinh\left(|N|^{2\alpha}\right)+\sinh\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\left|\left\langle\left[\frac{\cos(N)+\cos(N^*)}{2}\right]x,y\right\rangle\right| \\
&\leq \frac{1}{2}\left\langle\left[\cosh\left(|N|^{2\alpha}\right)+\cosh\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2} \\
&\times\left\langle\left[\cosh\left(|N|^{2\alpha}\right)+\cosh\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}
\end{aligned} \tag{3.12}$$

for any $x,y\in H$ and N a normal operator.

If we utilize the following function as power series representations with nonnegative coefficients:

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\qquad z\in D(0,1);\tag{3.13}$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\qquad z\in D(0,1);$$

$$\tanh^{-1}(z)=\sum_{n=0}^{\infty}\frac{1}{2n+1}z^{2n-1},\qquad z\in D(0,1);$$

$${}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is the *Gamma function*, then we can state the following vector inequalities:

$$\begin{aligned}&\left|\left\langle\left[\frac{\exp(N)+\exp(N^*)}{2}\right]x,y\right\rangle\right|\\&\leq\frac{1}{2}\left\langle\left[\exp\left(|N|^{2\alpha}\right)+\exp\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\&\times\left\langle\left[\exp\left(|N|^{2\alpha}\right)+\exp\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned}\tag{3.14}$$

for any $x,y\in H$ and N a normal operator.

If $\|N\|<1$, then we also have the inequalities

$$\begin{aligned}&\left|\left\langle\left[\frac{\ln\left(\frac{1_H+N}{1_H-N}\right)+\ln\left(\frac{1_H+N^*}{1_H-N^*}\right)}{2}\right]x,y\right\rangle\right|\\&\leq\frac{1}{2}\left\langle\left[\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)+\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right]x,x\right\rangle^{1/2}\\&\times\left\langle\left[\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)+\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right]y,y\right\rangle^{1/2}\end{aligned}\tag{3.15}$$

$$\begin{aligned}&\left|\left\langle\left[\frac{\tanh^{-1}(N)+\tanh^{-1}(N^*)}{2}\right]x,y\right\rangle\right|\\&\leq\frac{1}{2}\left\langle\left[\tanh^{-1}\left(|N|^{2\alpha}\right)+\tanh^{-1}\left(|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\&\times\left\langle\left[\tanh^{-1}\left(|N|^{2\alpha}\right)+\tanh^{-1}\left(|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&\left|\left\langle\left[\frac{{}_2F_1(\alpha,\beta,\gamma,N)+{}_2F_1(\alpha,\beta,\gamma,N^*)}{2}\right]x,y\right\rangle\right|\\&\leq\frac{1}{2}\left\langle\left[{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2\alpha}\right)+{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2(1-\alpha)}\right)\right]x,x\right\rangle^{1/2}\\&\times\left\langle\left[{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2\alpha}\right)+{}_2F_1\left(\alpha,\beta,\gamma,|N|^{2(1-\alpha)}\right)\right]y,y\right\rangle^{1/2}\end{aligned}\tag{3.17}$$

for any $x, y \in H$.

From a different perspective, we also have:

Theorem 6. *With the assumption of Theorem 5 and if N is a normal operator on the Hilbert space H and $z \in \mathbb{C}$ such that $\|N\|^2, |z|^2 < R$, then we have the inequalities*

$$\begin{aligned}(3.18)\qquad &\left|\left\langle\left(\frac{f(zN)+f(zN^*)}{2}\right)x,y\right\rangle\right|^2\\ &\le \frac{1}{2}f_A\left(|z|^2\right)\left[\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{\alpha}\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{1-\alpha}\right.\\ &\left.+\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{\alpha}\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{1-\alpha}\right]\\ &\le \frac{1}{2}f_A\left(|z|^2\right)\left[\left\langle f_A\left(|N|^2\right)x,x\right\rangle+\left\langle f_A\left(|N|^2\right)y,y\right\rangle\right]\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

In particular, for $\alpha = \frac{1}{2}$ we have

$$\begin{aligned}(3.19)\qquad &\left|\left\langle\left(\frac{f(zN)+f(zN^*)}{2}\right)x,y\right\rangle\right|^2\\ &\le f_A\left(|z|^2\right)\left\langle f_A\left(|N|^2\right)x,x\right\rangle^{1/2}\left\langle f_A\left(|N|^2\right)y,y\right\rangle^{1/2}\\ &\le \frac{1}{2}f_A\left(|z|^2\right)\left[\left\langle f_A\left(|N|^2\right)x,x\right\rangle+\left\langle f_A\left(|N|^2\right)y,y\right\rangle\right]\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. If we use the third and the fourth inequality in (2.23) we have

$$\begin{aligned}(3.20)\qquad &\sum_{j=0}^{n}|a_j|\left[\left|\left\langle\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2\right]\\ &\le \frac{1}{2}\left(\sum_{j=0}^{n}|a_j|\left\|N^jx\right\|^2\right)^{\alpha}\left(\sum_{j=0}^{n}|a_j|\left\|N^jy\right\|^2\right)^{1-\alpha}\\ &+\frac{1}{2}\left(\sum_{j=0}^{n}|a_j|\left\|N^jy\right\|^2\right)^{\alpha}\left(\sum_{j=0}^{n}|a_j|\left\|N^jx\right\|^2\right)^{1-\alpha}\\ &\le \frac{1}{2}\sum_{j=0}^{n}|a_j|\left[\left\|N^jx\right\|^2+\left\|N^jy\right\|^2\right]\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Since N is a normal operator on the Hilbert space H, then

$$\left\|N^jx\right\|^2 = \left\langle\left|N^j\right|^2x,x\right\rangle = \left\langle|N|^{2j}x,x\right\rangle$$

for any $j \in \{0,...,n\}$ and for any $x \in H$ with $\|x\| = 1$.

Then from (3.20) we get

$$\sum_{j=0}^{n}|a_j|\left[\left|\left\langle\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2\right] \tag{3.21}$$
$$\leq\frac{1}{2}\left(\left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j}\,x,x\right\rangle\right)^{\alpha}\left(\left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j}\,y,y\right\rangle\right)^{1-\alpha}$$
$$+\frac{1}{2}\left(\left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j}\,y,y\right\rangle\right)^{\alpha}\left(\left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j}\,x,x\right\rangle\right)^{1-\alpha}$$
$$\leq\frac{1}{2}\left[\left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j}\,x,x\right\rangle+\left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j}\,y,y\right\rangle\right]$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and $\alpha\in[0,1]$.

By the weighted Cauchy-Buniakowski-Schwarz inequality we also have

$$\left|\left\langle\sum_{j=0}^{n}a_jz^j\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2 \tag{3.22}$$
$$\leq\sum_{j=0}^{n}|a_j|\,|z|^{2j}\sum_{j=0}^{n}|a_j|\left[\left|\left\langle\left(\frac{N^j+(N^*)^j}{2}\right)x,y\right\rangle\right|^2\right]$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Now, since the series $\sum_{j=0}^{\infty}a_jz^jN^j$, $\sum_{j=0}^{\infty}a_jz^j\,(N^*)^j$, $\sum_{j=0}^{\infty}|a_j|\,|z|^{2j}$, $\sum_{j=0}^{\infty}|a_j|\,|N|^{2j}$ are convergent, then by (3.21) and (3.22) on letting $n\to\infty$, we deduce the desired result (3.18). □

Similar inequalities for some particular functions of interest can be stated. However, the details are left to the interested reader.

4. Applications for the Euclidian Norm

In [31], the author has introduced the following norm on the Cartesian product $\mathcal{B}^{(n)}(H):=\mathcal{B}(H)\times\cdots\times\mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H :

$$\|(T_1,\dots,T_n)\|_e:=\sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n}\|\lambda_1T_1+\cdots+\lambda_nT_n\|, \tag{4.1}$$

where $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$ and $\mathbb{B}_n:=\left\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n\,\middle|\sum_{j=1}^{n}|\lambda_j|^2\leq1\right\}$ is the Euclidean closed ball in $\mathbb{C}^n$.

It is clear that $\|\cdot\|_e$ is a norm on $B^{(n)}(H)$ and for any $(T_1,\dots,T_n)\in B^{(n)}(H)$ we have

$$\|(T_1,\dots,T_n)\|_e=\|(T_1^*,\dots,T_n^*)\|_e,$$

where T_j^* is the adjoint operator of T_j, $j\in\{1,\dots,n\}$. We call this the *Euclidian norm* of an n-tuple of operators $(T_1,\dots,T_n)\in B^{(n)}(H)$.

It has been shown in [31] that the following basic inequality for the Euclidian norm holds true:

$$\frac{1}{\sqrt{n}}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{\frac{1}{2}} \leq \|(T_1,\dots,T_n)\|_e \leq \left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{\frac{1}{2}} \tag{4.2}$$

for any n-tuple $(T_1,\dots,T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [31] the author has introduced the *Euclidean operator radius* of an n-tuple of operators $(T_1,\dots,T_n)$ by

$$w_e(T_1,\dots,T_n) := \sup_{\|x\|=1}\left(\sum_{j=1}^{n}\left|\langle T_j x, x\rangle\right|^2\right)^{\frac{1}{2}} \tag{4.3}$$

and proved that $w_e(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality:

$$\frac{1}{2}\|(T_1,\dots,T_n)\|_e \leq w_e(T_1,\dots,T_n) \leq \|(T_1,\dots,T_n)\|_e \tag{4.4}$$

for each n-tuple $(T_1,\dots,T_n) \in B^{(n)}(H)$.

As pointed out in [31], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{\frac{1}{2}} \leq w_e(T_1,\dots,T_n) \leq \left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{\frac{1}{2}} \tag{4.5}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

In [2], by utilizing the concept of *hypo-Euclidean norm* on H^n we obtained the following representation for the Euclidian norm:

Proposition 1. *For any $(T_1,\dots,T_n) \in B^{(n)}(H)$ we have*

$$\|(T_1,\dots,T_n)\|_e = \sup_{\|y\|=1,\|x\|=1}\left(\sum_{j=1}^{n}\left|\langle T_j y, x\rangle\right|^2\right)^{\frac{1}{2}}. \tag{4.6}$$

Theorem 7. *For any $(T_1,\dots,T_n) \in B^{(n)}(H)$ we have*

$$\begin{aligned}\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_e^2 &\leq \left\|\sum_{j=1}^{n}\left|T_j\right|^2\right\|^{\alpha}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1-\alpha} \\ &\leq \alpha\left\|\sum_{j=1}^{n}\left|T_j\right|^2\right\| + (1-\alpha)\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\end{aligned} \tag{4.7}$$

and

$$w_e^2\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right) \tag{4.8}$$
$$\leq \sup_{\|x\|=1}\left[\left\langle \sum_{j=1}^{n}|T_j|^2 x,x\right\rangle^{\alpha}\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 x,x\right\rangle^{1-\alpha}\right]$$
$$\leq \begin{cases} \left\|\sum_{j=1}^{n}|T_j|^2\right\|^{\alpha}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1-\alpha} \\ \\ \left\|\alpha\sum_{j=1}^{n}|T_j|^2+(1-\alpha)\sum_{j=1}^{n}\left|T_j^*\right|^2\right\| \end{cases}$$
$$\leq \alpha\left\|\sum_{j=1}^{n}|T_j|^2\right\|+(1-\alpha)\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|$$

for any $\alpha\in[0,1]$.

Proof. Making use of the inequalities (2.13) and (2.20) we have

$$\sum_{j=1}^{n}\left[\left|\left\langle\left(\frac{T_j+T_j^*}{2}\right)x,y\right\rangle\right|^2\right] \tag{4.9}$$
$$\leq \frac{1}{2}\left\langle \sum_{j=1}^{n}|T_j|^2 x,x\right\rangle^{\alpha}\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 y,y\right\rangle^{1-\alpha}$$
$$+\frac{1}{2}\left\langle \sum_{j=1}^{n}|T_j|^2 y,y\right\rangle^{\alpha}\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 x,x\right\rangle^{1-\alpha}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and $\alpha\in[0,1]$.

Taking the supremum over $\|x\|=\|y\|=1$ in (4.9) we get

$$\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_e^2$$
$$\leq \frac{1}{2}\sup_{\|x\|=1}\left\langle \sum_{j=1}^{n}|T_j|^2 x,x\right\rangle^{\alpha}\sup_{\|y\|=1}\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 y,y\right\rangle^{1-\alpha}$$
$$+\frac{1}{2}\sup_{\|y\|=1}\left\langle \sum_{j=1}^{n}|T_j|^2 y,y\right\rangle^{\alpha}\sup_{\|x\|=1}\left\langle \sum_{j=1}^{n}\left|T_j^*\right|^2 x,x\right\rangle^{1-\alpha}$$
$$=\left\|\sum_{j=1}^{n}|T_j|^2\right\|^{\alpha}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1-\alpha}$$

and the inequality (4.7) is proved.

Now, if we take $y=x$ in (4.9) we get

$$\sum_{j=1}^{n}\left[\left|\left\langle\left(\frac{T_j+T_j^*}{2}\right)x,x\right\rangle\right|^2\right] \le \left\langle\sum_{j=1}^{n}|T_j|^2x,x\right\rangle^{\alpha}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2x,x\right\rangle^{1-\alpha} \le \left\langle\left[\alpha\sum_{j=1}^{n}|T_j|^2+(1-\alpha)\sum_{j=1}^{n}\left|T_j^*\right|^2\right]x,x\right\rangle \tag{4.10}$$

for any $x\in H$ with $\|x\|=1$ and $\alpha\in[0,1]$.

Taking the supremum over $\|x\|=1$ in (4.10) we get the desired result □

Remark 7. *In the particular case $\alpha=\frac{1}{2}$ we get*

$$\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_e^2 \le \left\|\sum_{j=1}^{n}|T_j|^2\right\|^{1/2}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1/2} \le \frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|^2\right\|+\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\right] \tag{4.11}$$

and

$$w_e^2\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right) \le \sup_{\|x\|=1}\left[\left\langle\sum_{j=1}^{n}|T_j|^2x,x\right\rangle^{1/2}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^2x,x\right\rangle^{1/2}\right] \le \begin{cases}\left\|\sum_{j=1}^{n}|T_j|^2\right\|^{1/2}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1/2}\\ \left\|\sum_{j=1}^{n}\frac{|T_j|^2+\left|T_j^*\right|^2}{2}\right\|\end{cases} \le \frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|^2\right\|+\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|\right]. \tag{4.12}$$

5. Applications for s-1-Norm and s-1-Numerical Radius

Following [3], we consider the *s-p-norm* of the n-tuple of operators $(T_1,\dots,T_n)\in B^{(n)}(H)$ given by

$$\|(T_1,\dots,T_n)\|_{s,p} := \sup_{\|y\|=1,\|x\|=1}\left[\left(\sum_{j=1}^{n}|\langle T_jy,x\rangle|^p\right)^{\frac{1}{p}}\right]. \tag{5.1}$$

For $p=2$ we get

$$\|(T_1,\dots,T_n)\|_{s,2}=\|(T_1,\dots,T_n)\|_e .$$

We are interested in this section in the case $p=1$, namely on the s-1-norm defined by

$$\|(T_1,\dots,T_n)\|_{s,1}:=\sup_{\|y\|=1,\|x\|=1}\sum_{j=1}^{n}|\langle T_jy,x\rangle| .$$

Since for any $x,y\in H$ we have $\sum_{j=1}^{n}|\langle T_jy,x\rangle|\geq\left|\left\langle\sum_{j=1}^{n}T_jy,x\right\rangle\right|$, then by the properties of the supremum we get the basic inequality

$$\left\|\sum_{j=1}^{n}T_j\right\|\leq\|(T_1,\dots,T_n)\|_{s,1}\leq\sum_{j=1}^{n}\|T_j\| . \tag{5.2}$$

Similarly, we can also consider the *s-p-numerical radius* of the n-tuple of operators $(T_1,\dots,T_n)\in B^{(n)}(H)$ defined by [3]

$$w_{s,p}(T_1,\dots,T_n):=\sup_{\|x\|=1}\left[\left(\sum_{j=1}^{n}|\langle T_jx,x\rangle|^p\right)^{\frac{1}{p}}\right], \tag{5.3}$$

which for $p=2$ reduces to the Euclidean operator radius introduced previously. We observe that the s-p-numerical radius is also a norm on $B^{(n)}(H)$ for $p\geq1$ and for $p=1$ it satisfies the basic inequality

$$w\left(\sum_{j=1}^{n}T_j\right)\leq w_{s,1}(T_1,\dots,T_n)\leq\sum_{j=1}^{n}w(T_j) . \tag{5.4}$$

Theorem 8. *For any $(T_1,\dots,T_n)\in B^{(n)}(H)$ we have*

$$\begin{aligned}\left\|\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)\right\|_{s,1}&\leq\left\|\sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]\right\|\\&\leq\frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|^{2\alpha}\right\|+\left\|\sum_{j=1}^{n}|T_j^*|^{2(1-\alpha)}\right\|\right]\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}w_{s,1}\left(\frac{T_1+T_1^*}{2},\dots,\frac{T_n+T_n^*}{2}\right)&\leq w_{s,1}(T_1,\dots,T_n)\\&\leq\left\|\sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]\right\|\\&\leq\frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|^{2\alpha}\right\|+\left\|\sum_{j=1}^{n}|T_j^*|^{2(1-\alpha)}\right\|\right]\end{aligned} \tag{5.6}$$

for any $\alpha\in[0,1]$.

Proof. Utilizing the inequality (2.1) we have

$$\sum_{j=1}^{n}\left|\left\langle \frac{T_j+T_j^*}{2}x,y\right\rangle\right| \leq \left\langle \sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]x,x\right\rangle^{1/2} \times \left\langle \sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]y,y\right\rangle^{1/2} \tag{5.7}$$

for any $x,y\in H$ and $\alpha\in[0,1]$.

Taking the supremum in (5.7) over $\|x\|=\|y\|=1$ we get the first inequality in (5.5).

The second part follows by the triangle inequality.

By the inequality (2.7) we have

$$\sum_{j=1}^{n}\left|\left\langle \frac{T_j+T_j^*}{2}x,x\right\rangle\right| \leq \sum_{j=1}^{n}|\langle T_jx,x\rangle| \leq \left\langle \sum_{j=1}^{n}\left[\frac{|T_j|^{2\alpha}+|T_j^*|^{2(1-\alpha)}}{2}\right]x,x\right\rangle$$

for any $x\in H$.

Taking the supremum over $\|x\|=1$ we deduce the desired result (5.6). □

Remark 8. *The case $\alpha=\frac{1}{2}$ produces the following chains of inequalities*

$$\left\|\sum_{j=1}^{n}\left(\frac{T_j+T_j^*}{2}\right)\right\| \leq \left\|\left(\frac{T_1+T_1^*}{2},\ldots,\frac{T_n+T_n^*}{2}\right)\right\|_{s,1} \leq \left\|\sum_{j=1}^{n}\left(\frac{|T_j|+|T_j^*|}{2}\right)\right\| \leq \frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|\right\|+\left\|\sum_{j=1}^{n}|T_j^*|\right\|\right] \tag{5.8}$$

and

$$w\left(\sum_{j=1}^{n}\left(\frac{T_j+T_j^*}{2}\right)\right) \leq w_{s,1}\left(\frac{T_1+T_1^*}{2},\ldots,\frac{T_n+T_n^*}{2}\right) \leq w_{s,1}(T_1,\ldots,T_n) \leq \left\|\sum_{j=1}^{n}\left(\frac{|T_j|+|T_j^*|}{2}\right)\right\| \leq \frac{1}{2}\left[\left\|\sum_{j=1}^{n}|T_j|\right\|+\left\|\sum_{j=1}^{n}|T_j^*|\right\|\right]. \tag{5.9}$$

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