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COMPARING TWO INTEGRAL MEANS FOR ABSOLUTELY CONTINUOUS FUNCTIONS WHOSE ABSOLUTE VALUE OF THE DERIVATIVE ARE CONVEX AND APPLICATIONS

DAH-YAN HWANG[1] AND SILVESTRU SEVER DRAGOMIR[2,3]

Abstract. Some new estimates for the difference between the integral mean of a function and its mean over a subinterval are established and new applications for special means and probability density functions are also given.

1. Introduction

The classical Ostrowski type integral inequality [1] stipulates a bound for the difference between a function evaluated at an interior point and the average of the function over an interval. That is,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x)\right| \le \left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$, where $f' \in L_\infty(a,b)$, that is,

$$\|f'\|_\infty = ess \sup_{t\in[a,b]} |f'(t)| < \infty,$$

and $f : [a,b] \to R$ is a differentiable function on (a,b). Here, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For various results and generalizations concerning Ostrowski's inequality, see [2-13] and the references therein.

In [14], Barnett et al. compared the difference of two integral means as in the following Theorem 1 in which the function has the first derivative bounded where is defined. The obtained results are also generalizations of (1.1) and have been applied to probability density functions, special means, Jeffreys divergence in Information Theory and the sampling of continuous streams in Statistics.

Theorem 1. *Let $f : [a,b] \to R$ be an absolutely continuous function with the property that $f' \in L_\infty[a,b]$. Then, for $a \le x < y \le b$, we have the inequality*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\le \left\{\frac{1}{4}+\left[\frac{\frac{a+b}{2}-\frac{x+y}{2}}{b-a-y+x}\right]^2\right\}(b-a-y+x)\|f'\|_\infty \\ &\le \frac{1}{2}(b-a-y+x)\|f'\|_\infty. \end{aligned} \tag{1.2}$$

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The constant $\frac{1}{4}$ *is best possible in the first inequality and* $\frac{1}{2}$ *is best in the second one.*

The class of s-convex function in the second sense, usually denoted by K_s^2, was introduced by Hudzik and Maligranda [15]. This class is defined in the following way, $f : [0,\infty) \to R$ is said to be *s-convex function in the second sense* if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0,\infty), \alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. For example, the function $f : [0, 1] \to [0, 1]$ defined by $f(t) = t^s, s \in (0, 1]$, is a s-convex function in the second sense. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [16], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense. Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense and $a, b \in [0,\infty)$ with $a < b$. If $f \in L[a, b]$, then the following inequality holds:

$$2^{s-1} f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant is the best possible in the second inequality (1.3).

Recently, Alomari et al. [17] have established some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are s-convex functions in the second sense.

The main purpose of this article, is to establish some new results related to the inequality (1.2) for the functions whose absolute value of the first derivatives are convex. In section 3, the corresponding versions in the case that the power of the absolute value of the first derivative is s-convex in the second sense are obtained. Applying the obtained results, some new inequalities for special means and the probability density functions will be also given in section 4 and section 5, respectively.

For convenience, we denote

$$\begin{aligned} A &= \frac{(y-x)(b-a-y+x)}{b-a},\ B = \frac{(x-a)(y-x)}{b-a}, \\ I(a,b,x,y) &= \frac{2(x-a)^2}{b-a} + \frac{6(x-a)^2(y-x)}{(b-a)(b-a-y+x)} - \frac{2(x-a)^3(y-x)}{(b-a)(b-a-y+x)^2} \\ &\quad - \frac{3(x-a)(y-x)}{b-a} + \frac{(y-x)(b-a-y+x)}{b-a} \\ J(a,b,x,y) &= \frac{2(x-a)^3(y-x)}{(b-a)(b-a-y+x)^2} - \frac{3(x-a)2(y-x)}{(b-a)} \\ &\quad + \frac{2(b-a-y+x)(y-x)}{(b-a)} - \frac{2(b-y)^2}{b-a} \end{aligned}$$

where $a \le x < y \le b$.

2. The function $|f'|$ Is Convex

Let $f : [a,b] \to R$ be an absolutely continuous function and $a \le x < y \le b$. Denote $K_{x,y} : [a,b] \to R$, the kernel given by

$$K_{x,y}(s) = \begin{cases} \frac{a-s}{b-a}, & \text{if } s \in [a,x], \\ \frac{s-x}{y-x} + \frac{a-s}{b-a}, & \text{if } s \in (x,y), \\ \frac{b-s}{b-a}, & \text{if } s \in [y,b]. \end{cases}$$

The following lemma plays an important role in this article.

Lemma 1. *Let $f : [a,b] \to R$ be an absolutely continuous function and $a \le x < y \le b$. Then we have the identity*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du \\ &= \frac{-(x-a)^2}{b-a}\int_0^1 tf'((1-t)a+tx)dt \\ &+ \int_0^1 \left(\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right)f'((1-t)x+ty)dt \\ &+ \frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt. \end{aligned} \tag{2.1}$$

Proof. Using the following identity given in [14],

$$\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du = \int_a^b K_{x,y}(s)f'(s)ds,$$

and by suitable substitution of variables, we have the identity (2.1). □

Theorem 2. *Let $f : [a,b] \to R$ be an absolutely continuous function and $|f'|$ is convex on $[a,b]$. Then, for $a \le x < y \le b$, we have the inequality*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\le \frac{1}{6}\left[\frac{(x-a)^2}{b-a}|f'(a)| + I(a,b,x,y)|f'(x)| + J(a,b,x,y)|f'(y)| \right.\\ &\left. + \frac{(b-y)^2}{b-a}|f'(b)|\right]. \end{aligned} \tag{2.2}$$

Proof. By Lemma 1, we obtain

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\le \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\ &+ \int_0^1 \left|\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right| \cdot \big|f'((1-t)x+ty)\big|dt \\ &+ \frac{(b-y)^2}{b-a}\int_0^1 (1-t)\big|f'((1-t)y+tb)\big|dt. \end{aligned} \tag{2.3}$$

Using the convexity of $|f'|$, we get

$$
\begin{aligned}
(2.4)\qquad & \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\
& \le \frac{(x-a)^2}{b-a}\int_0^1 \Big[t(1-t)\,|f'\,(a)|+t^2|f'(x)|\Big]dt \\
& = \frac{(x-a)^2}{6(b-a)}|f'(a)|+\frac{(x-a)^2}{3(b-a)}|f'(x)|,
\end{aligned}
$$

$$
\begin{aligned}
(2.5)\qquad & \int_0^1 \left|\frac{(y-x)(b-a-y+x)}{b-a}t-\frac{(x-a)(y-x)}{b-a}\right|\cdot\big|f'((1-t)x+ty)\big|dt \\
& \le \int_0^{B/A}(B-At)\big[(1-t)|f'(x)|+t|f'(y)|\big]dt \\
& + \int_{B/A}^1 (At-B)\big[(1-t)|f'(x)|+t|f'(y)|\big]dt \\
& = \left(\frac{B^2}{A}-\frac{B^3}{3A^2}-\frac{B}{2}+\frac{A}{6}\right)|f'(x)|+\left(\frac{B^3}{3A^2}-\frac{B}{2}+\frac{A}{3}\right)|f'(y)|
\end{aligned}
$$

and

$$
\begin{aligned}
(2.6)\qquad & \frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt \\
& \le \frac{(b-y)^2}{b-a}\int_0^1 \big[(1-t)^2|f'(y)|+t(1-t)|f'(b)|\big]dt \\
& = \frac{(b-y)^2}{3(b-a)}|f'(y)|+\frac{(b-y)^2}{6(b-a)}|f'(b)|.
\end{aligned}
$$

By combining inequalities (2.3), (2.4), (2.5) and (2.6), we have the inequality (2.2). This completes the proof of Theorem 2. □

Remark 1. *By simple computation and the definition of* $\|f'\|_\infty$*, we have*

$$
\begin{aligned}
& \frac{1}{6}\left[\frac{(x-a)^2}{b-a}|f'(a)|+I(a,b,x,y)|f'(x)|+J(a,b,x,y)|f'(y)|+\frac{(b-y)^2}{b-a}|f'(b)\right] \\
& = \frac{(x-a)^2}{b-a}\int_0^1 t\big[(1-t)|f'(a)|+t|f'(x)|\big]dt \\
& + \int_0^1 \left|\frac{(y-x)(b-a-y+x)}{b-a}t-\frac{(x-a)(y-x)}{b-a}\right|\cdot\big[(1-t)|f'(x)|+t|f'(y)|\big]dt \\
& + \frac{(b-y)^2}{b-a}\int_0^1 (1-t)\big[(1-t)|f'(y)|+t|f'(b)|\big]dt \\
& \le \left[\frac{(x-a)^2}{b-a}\int_0^1 tdt+\int_0^1\left|\frac{(y-x)(b-a-y+x)}{b-a}t-\frac{(x-a)(y-x)}{b-a}\right|dt\right. \\
& \left.+\frac{(b-y)^2}{b-a}\int_0^1 (1-t)dt\right]\cdot\|f'\|_\infty
\end{aligned}
$$

$$= \left[\frac{1}{b-a}\int_a^x (s-a)ds + \int_x^y \left|\frac{s-x}{y-x} + \frac{a-x}{b-a}\right| ds + \frac{1}{b-a}\int_y^b (b-s)ds\right] \cdot \|f'\|_\infty$$
$$= \left\{\frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{x+y}{2}}{b-a-y+x}\right]^2\right\}(b-a-y+x) \cdot \|f'\|_\infty,$$

where the last identity had been given in the proofs of Theorem 2.2 in [14]*. Therefore, for strict convex functions, we have the bound in* (2.2) *is smaller than the one in* (1.2)*.*

If we set $y = x + h$ with $x + h \in (a, b)$, then by (2.2) *we get*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{h}\int_x^{x+h} f(t)dt\right|$$
$$\leq \frac{1}{6}\left[\frac{(x-a)^2}{b-a}|f'(a)| + I(a,b,x,x+h)|f'(x)|\right.$$
$$\left. + J(a,b,x,x+h)|f'(x+h)| + \frac{(b-x-h)^2}{b-a}|f'(b)|\right]$$

Now, letting $h \to 0^+$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x)\right| \tag{2.7}$$
$$\leq \left(\frac{M + 2|f'(x)|}{3}\right)(b-a)\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]$$

where $M = max\{|f'(a)|, |f'(b)|\}$.

We note that the bound in (2.7) is better than the one in (1.1) for the strict convex function.

3. The Function $|f'|^q$ is s-Convex

In this section, some bounds for the difference between the integral means of a function compared to its mean over a subinterval for the s-convex function $|f'|^q$ are given.

Theorem 3. *Let $f : I \subseteq [0, \infty) \to R$ be an absolutely continuous on $\mathring{I}$ (the interior of I) such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1], p, q > 1, 1/p + 1/q = 1$, and $f' \in L_\infty[a, b]$, then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \tag{3.1}$$
$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\|f'\|_\infty$$
$$\cdot \left[\frac{(x-a)^2}{b-a} + \left(\frac{B^{p+1} + (A-B)^{p+1}}{A}\right)^{\frac{1}{p}} + \frac{(b-y)^2}{b-a}\right]$$

for $a \leq x < y \leq b$.

Proof. By Lemma 1 and using the Hölder inequality, we obtain that

$$
\begin{aligned}
(3.2)\qquad & \left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\
&+ \int_0^1 \left|\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right| \cdot |f'((1-t)x+ty)|dt \\
&+ \frac{(b-y)^2}{b-a}\int_0^1 (1-t)|f'((1-t)y+tb)|dt \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)a+tx)|^q\,dt\right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 \left|\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right|^p dt\right)^{\frac{1}{p}} \\
&\cdot \left(\int_0^1 |f'((1-t)x+ty)|^q\,dt\right)^{\frac{1}{q}} \\
&+ \frac{(b-y)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)y+tb)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Now, using the s-convexity of $|f'|^q$ and $f' \in L_\infty[a,b]$, we get

$$
\begin{aligned}
(3.3)\qquad \int_0^1 |f'((1-t)a+tx)|^q\,dt &\le \int_0^1 \left[(1-t)^s\,|f'(a)|^q + t^s\,|f'(x)|^q\right]dt \\
&= \frac{1}{s+1}\left(|f'(a)|^q + |f'(x)|^q\right) \le \frac{2\|f'\|_\infty^q}{s+1},
\end{aligned}
$$

$$
\begin{aligned}
(3.4)\qquad \int_0^1 |f'((1-t)x+ty)|^q\,dt &\le \int_0^1 \left[(1-t)^s\,|f'(x)|^q + t^s\,|f'(y)|^q\right]dt \\
&= \frac{1}{s+1}\left(|f'(x)|^q + |f'(y)|^q\right) \le \frac{2\|f'\|_\infty^q}{s+1}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.5)\qquad \int_0^1 |f'((1-t)y+tb)|^q\,dt &\le \int_0^1 \left[(1-t)^s\,|f'(y)|^q + t^s\,|f'(b)|^q\right]dt \\
&= \frac{1}{s+1}\left(|f'(y)|^q + |f'(b)|^q\right) \le \frac{2\|f'\|_\infty^q}{s+1},
\end{aligned}
$$

since, by simple computation, we have

$$(3.6)\qquad \int_0^1 t^p dt = \frac{1}{p+1},$$

$$(3.7)\qquad \int_0^1 (1-t)^p dt = \frac{1}{p+1}$$

and

$$\begin{aligned}(3.8)\qquad &\int_0^1\left|\frac{(y-x)(b-a-y+x)}{b-a}t-\frac{(x-a)(y-x)}{b-a}\right|^p dt\\ &\le\int_0^{B/A}(B-At)^p dt+\int_{B/A}^1(At-B)^p dt\\ &=\frac{1}{p+1}\left[\frac{B^{p+1}+(A-B)^{p+1}}{A}\right].\end{aligned}$$

Combining inequalities (3.2)-(3.8), we have the inequality (3.1). This completes the proofs of Theorem 3. □

Remark 2. *If we set $y=x+h$ with $x+h\in(a,b)$, then by (3.1), we get*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{h}\int_x^{x+h}f(u)du\right|\\ &\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\|f'\|_\infty\left[\frac{(x-a)^2}{b-a}\right.\\ &\left.+\frac{h}{b-a}\left(\frac{(x-a)^{p+1}+(b-x-h)^{p+1}}{b-a-h}\right)^{\frac{1}{p}}+\frac{(b-x-h)^2}{b-a}\right].\end{aligned}$$

Now, letting $h\to 0^+$, we have

$$\begin{aligned}(3.9)\qquad &\left|\frac{1}{b-a}\int_a^b f(x)dx-f(x)dx\right|\\ &\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\|f'\|_\infty\left[\frac{(x-a)^2+(b-y)^2}{b-a}\right].\end{aligned}$$

We note that the inequality (3.9) is the inequality (2.2) given in [17].

Theorem 4. *Let $f:I\subseteq[0,\infty)\to R$ be an absolutely continuous on $\mathring{I}$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is s-concave in the second sense on $[a,b]$ for some fixed $s\in(0,1], p,q>1, 1/p+1/q=1$, then we have the inequality*

$$\begin{aligned}(3.10)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{y-x}\int_x^y f(u)du\right|\\ &\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}2^{\frac{(s-1)}{q}}\left[\frac{(x-a)^2}{b-a}\left|f'\left(\frac{a+x}{2}\right)\right|\right.\\ &+\left(\frac{B^{p+1}+(A-B)^{p+1}}{A}\right)^{\frac{1}{p}}\left|f'\left(\frac{x+y}{2}\right)\right|\\ &\left.+\frac{(b-y)^2}{b-a}\left|f'\left(\frac{y+b}{2}\right)\right|\right]\end{aligned}$$

for $a\le x<y\le b$.

Proof. Using the s-concavity of $|f'|^q$ and the first inequality of (1.3), we obtain

$$\int_0^1 |f'((1-t)a+tx)|^q dt \le 2^{s-1}\Big|f'\big(\frac{a+x}{2}\big)\Big|^q, \tag{3.11}$$

$$\int_0^1 |f'((1-t)x+ty)|^q dt \le 2^{s-1}\Big|f'\big(\frac{x+y}{2}\big)\Big|^q, \tag{3.12}$$

and

$$\int_0^1 |f'((1-t)y+tb)|^q dt \le 2^{s-1}\Big|f'\big(\frac{y+b}{2}\big)\Big|^q. \tag{3.13}$$

By combining inequalities (3.2), (3.6)-(3.8) and (3.11)-(3.13), we have the inequality (3.10). This completes the proofs of Theorem 4. □

Remark 3. *If we set $y = x + h$ with $x + h \in (a, b)$, then by* (3.10)*, we get*

$$\Big|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{h}\int_x^{x+h} f(u)du\Big|$$
$$\le \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}} 2^{\frac{(s-1)}{q}}\Big[\frac{(x-a)^2}{b-a}\Big|f'\big(\frac{a+x}{2}\big)\Big| + \frac{h}{b-a}\Big(\frac{(x-a)^{p+1}+(b-x-h)^{p+1}}{b-a-h}\Big)^{\frac{1}{p}}$$
$$\cdot\Big|f'\big(\frac{2x+h}{2}\big)\Big|^q + \frac{(b-x-h)^2}{b-a}\Big|f'\big(\frac{x+h+b}{2}\big)\Big|\Big].$$

Now, letting $h \to 0^+$, we have

$$\Big|\frac{1}{b-a}\int_a^b f(u)du - f(x)dx\Big| \le \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}} 2^{\frac{(s-1)}{q}}\Big[\frac{(x-a)^2}{b-a}\Big|f'\big(\frac{a+x}{2}\big)\Big| + \frac{(b-x)^2}{b-a}\Big|f'\big(\frac{x+b}{2}\big)\Big|\Big]. \tag{3.14}$$

We note that the inequality (3.14) *is the inequality* (2.4) *given in* [17].

For $s = 1$ in Theorem 4, we have the following corollary.

Corollary 1. *Let $f : I \subseteq [0,\infty) \to R$ be an absolutely continuous on $\mathring{I}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b], p, q > 1, 1/p + 1/q = 1$, then we have the inequality*

$$\Big|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{y-x}\int_x^y f(u)du\Big| \le \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^2}{b-a}\Big|f'\big(\frac{a+x}{2}\big)\Big| + \Big(\frac{B^{p+1}+(A-B)^{p+1}}{A}\Big)^{\frac{1}{p}}\Big|f'\big(\frac{x+y}{2}\big)\Big| + \frac{(b-y)^2}{b-a}\Big|f'\big(\frac{y+b}{2}\big)\Big|\Big] \tag{3.15}$$

for $a \le x < y \le b$.

4. Applications to Special Means

In the following, we shall consider logarithmic, identric and generalized logarithmic means from two positive real numbers. We take

$$\begin{aligned} L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln\beta-\ln\alpha}, \qquad \alpha,\beta\in R^+, \alpha\neq\beta, \\ I(\alpha,\beta) &= \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}} \qquad \alpha,\beta\in R^+, \alpha\neq\beta, \\ L_p(\alpha,\beta) &= \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, \qquad p\in R\setminus\{-1,0\}, \alpha,\beta\in R^+, \alpha\neq\beta, \end{aligned}$$

where R is the field of real numbers.

Proposition 1. *Let $a,b,x,y\in R, 0<a\le x<y\le b$ and $p\in R\setminus\{-1,0\}$. Then we have*

$$\begin{aligned} &\left|L_p^p(a,b)-L_p^p(x,y)\right| \\ &\le \frac{|p|}{6}\left[\frac{(x-a)^2a^{p-1}}{b-a}+I(a,b,x,y)x^{p-1}+J(a,b,x,y)y^{p-1}+\frac{(b-y)^2b^{p-1}}{b-a}\right]. \end{aligned} \tag{4.1}$$

Proof. The proof is immediate from Theorem 2 with $f(x)=x^p, x\in R^+, p\in R\setminus\{-1,0\}$. □

Proposition 2. *Suppose $a,b,x,y\in R$, and $0<a\le x<y\le b$. Then we have*

$$\begin{aligned} &\left|L^{-1}(a,b)-L^{-1}(x,y)\right| \\ &\le \frac{1}{6}\left[\frac{(x-a)^2}{(b-a)a^2}+\frac{I(a,b,x,y)}{x^2}+\frac{J(a,b,x,y)}{y^2}+\frac{(b-y)^2}{(b-a)b^2}\right]. \end{aligned} \tag{4.2}$$

Proof. The result follows from Theorem 2 with $f(x)=\frac{1}{x}, x\in R^+$. □

Proposition 3. *Suppose $a,b,x,y\in R$, and $0<a\le x<y\le b$. Then we have*

$$\left|\ln\left[\frac{I(a,b)}{I(x,y)}\right]\right| \le \frac{1}{6}\left[\frac{(x-a)^2}{(b-a)a}+\frac{I(a,b,x,y)}{x}+\frac{J(a,b,x,y)}{y}+\frac{(b-y)^2}{(b-a)b}\right]. \tag{4.3}$$

Proof. The result follows from Theorem 2 with $f(x)=\ln x$. □

Remark 4. *We note that the bounds in* (4.1), (4.2) *and* (4.3) *are better than the ones in* (4.1), (4.2) *and* (4.3) *given in* [14], *respectively.*

Proposition 4. *Let $a,b,x,y\in R, 0<a\le x<y\le b<1, 0<s\le 1$ and $p,q>1, 1/p+1/q=1$. Then we have*

$$\begin{aligned} &\left|L_{(s+q)/q}^{(s+q)/q}(a,b)-L_{(s+q)/q}^{(s+q)/q}(x,y)\right| \\ &\le \frac{s+q}{q}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}} \\ &\cdot\left[\frac{(x-a)^2}{b-a}+\Big(\frac{B^{p+1}+(A-B)^{p+1}}{A}\Big)^{\frac{1}{p}}\right. \\ &\left.+\frac{(b-y)^2}{b-a}\right]. \end{aligned} \tag{4.4}$$

Proof. Taking $f(x) = \frac{q}{s+q}x^{\frac{s+q}{q}}$, for $x \in [0,1], 0 < s \leq 1$ and $q > 1$, we obtain $|f'|^q = x^s \in K_s^2$ for, $x \in [0,1]$ and $\|f'\|_\infty = 1$ for $x \in [0,1]$. By (3.1), we have the desire inequality (4.4). □

Proposition 5. *Let $a, b, x, y \in R, 0 < a \leq x < y \leq b < 1, 0 < \alpha < 1$ and $p, q > 1, 1/p + 1/q = 1$. Then we have*

(4.5)
$$\begin{aligned}&\Big|L_{(\alpha+q)/q}^{(\alpha+q)/q}(a,b) - L_{(\alpha+q)/q}^{(\alpha+q)/q}(x,y)\Big|\\ &\leq \frac{\alpha+q}{q}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^2}{b-a}\Big|f'\Big(\frac{a+x}{2}\Big)\Big| + \Big(\frac{B^{p+1}+(A-B)^{p+1}}{A}\Big)^{\frac{1}{p}}\Big|f'\Big(\frac{x+y}{2}\Big)\Big|\\ &+ \frac{(b-y)^2}{b-a}\Big|f'\Big(\frac{y+b}{2}\Big)\Big|\Big].\end{aligned}$$

Proof. Set $f(x) = \frac{q}{\alpha+q}x^{\frac{\alpha+q}{q}}$, for $x \in R^+, 0 < \alpha \leq 1$ and $q > 1$, we obtain $|f'|^q = x^\alpha$ is concave for $x \in R^+$. By (3.15), we have the desired inequality (4.5), immediately. □

Remark 5. *The inequality* (4.4) *and inequality* (4.5) *are the new type for comparing two generalized logarithmic means.*

5. Applications for Probability Density Functions

In the following, assume that $f : [a,b] \to R^+$ is a probability density function of a certain random variable X and $F : [a,b] \to R^+$, $F(t) = \int_a^t f(x)dx$ is its cumulative distribution function.

Proposition 6. *Let f and F be as above. Then we have*

(5.1)
$$\begin{aligned}&\Big|F(t) - \frac{t-a}{b-a}\Big|\\ &\leq \frac{(b-t)(t-a)}{6(b-a)}\big[(t-a)|f'(a)| + 2(b-a)|f'(t)| + (b-t)|f'(b)|\big],\end{aligned}$$

provided that $|f'|$ is convex.

Proof. Taking $x = a$ and $y = t$ in (2.2), we have the desired inequality. □

Proposition 7. *Let f and F be as above. Then we have*

(5.2)
$$\begin{aligned}&\Big|F(t) - \frac{t-a}{b-a}\Big|\\ &\leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}}\|f'\|_\infty(b-t)(t-a),\end{aligned}$$

provided that $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1], p, q > 1, 1/p + 1/q = 1,$ and $f' \in L_\infty[a,b]$.

Proof. Taking $x = a$ and $y = t$ in (3.1), we have the desired inequality. □

Proposition 8. *Let f and F be as above. Then we have*

$$\left|F(t)-\frac{t-a}{b-a}\right| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{(t-a)(b-t)}{(b-a)}\left[(t-a)\left|f'\left(\frac{a+t}{2}\right)\right|+(b-t)\left|f'\left(\frac{t+b}{2}\right)\right|\right], \tag{5.3}$$

provided that $|f'|^q$ is concave on $[a,b]$, and $p,q>1, 1/p+1/q=1$.

Proof. Taking $x=a$ and $y=t$ in (3.15), we have the desired inequality. □

Remark 6. *The bound in* (5.1) *is better than the one in* (3.1) *given in* [14]*, and* (5.2) *and* (5.3) *are of new type for* (5.1)*.*

Proposition 9. *Let f and F be as above and let*

$$E_t(X)=\int_a^t uf(u)du, t\in[a,b].$$

Then, for $t\in[a,b]$, we have

$$\left|\frac{(b-E(X))(t-a)}{b-a}+E_t(X)-tF(t)\right| \le \frac{(b-t)(t-a)}{6(b-a)}\big[(t-a)|f(a)|+2(b-a)|f(t)|+(b-t)|f(b)|\big], \tag{5.4}$$

provided that $|f|$ is convex.

Proof. Taking $F=f, x=a$ and $y=t$ in (2.2), we get

$$\left|\frac{1}{b-a}\int_a^b F(x)dx-\frac{1}{t-a}\int_a^t F(u)du\right| \le \frac{(b-t)}{6(b-a)}\big[(t-a)|F'(a)|+2(b-a)|F'(t)|+(b-t)|F'(b)|\big]. \tag{5.5}$$

Since

$$\int_a^b F(x)dx=b-E(X)$$

and

$$\int_a^t F(u)du=tF(t)-\int_a^t uf(u)du=tF(t)-E_t(X),$$

thus, by (5.5), we have the desired inequality. □

Similarly, taking $F=f, x=a$ and $y=t$ in (3.1) and (3.15), respectively, we have the following two propositions.

Proposition 10. *Let f, F and $E_t(X)$ be as defined in Proposition 9. Then, for $t\in[a,b]$, we have*

$$\left|\frac{(b-E(X))(t-a)}{b-a}+E_t(X)-tF(t)\right| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\|F'\|_\infty(b-t)(t-a) \tag{5.6}$$

provided that $|f|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, $p,q>1, 1/p+1/q=1$, and $F'\in L_\infty[a,b]$.

Proposition 11. *Let f, F and $E_t(X)$ be as defined in Proposition 9. Then, for $t \in [a,b]$, we have*

$$\Big|\frac{(b-E(X))(t-a)}{b-a}+E_t(X)-tF(t)\Big| \tag{5.7}$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{(t-a)(b-t)}{(b-a)}\Big[(t-a)\Big|f(\frac{a+t}{2})\Big|+(b-t)\Big|f(\frac{t+b}{2})\Big|\Big] \tag{5.8}$$

provided that $|f|^q$ is concave on $[a,b]$, and $p, q > 1, 1/p + 1/q = 1$.

Remark 7. *The bound in* (5.4) *is better than the one in* (3.2) *given in* [14]*, and* (5.6) *and* (5.7) *are of new type for* (3.2) *in* [14].

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[1]Department of Information and Management, Taipei Chengshih University of Science and Technology, No. 2, Xueyuan Rd., Beitou, 112, Taipei, TAIWAN
E-mail address: dyhuang@tpcu.edu.tw

[2]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[3]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa