Received 20/08/12

Reverses of Schwarz Inequality in Inner Product Spaces with Applications

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Abstract. New reverses of the Schwarz inequality in complex inner products spaces with applications for bounded linear operators are given. Some Grüss' type inequalities and their applications for numerical radius and the operator norm are provided as well.

1. Introduction

The following reverse of the celebrated *Schwarz inequality* in the complex inner product space H equiped with the inner product $\langle \cdot, \cdot \rangle$ holds:

Let $x, y \in H$ and $\gamma, \Gamma \in \mathbb{C}$ such that either

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0 \tag{1.1}$$

or, equivalently,

$$\left\| x - \frac{\gamma + \Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|y\| . \tag{1.2}$$

Then we have the inequality

$$\begin{aligned} & \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \\ & \leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 - \left| \langle x, y \rangle - \frac{\gamma + \Gamma}{2} \|y\|^2 \right|^2 \\ & \leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 . \end{aligned} \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible.

The inequality (1.3) was obtained for the first time in 2003, see [**5**], for $y = e, \|e\| = 1$ as a particular case of a reverse for the Bessel inequality. In the present form, the inequality (1.3) was stated in [**11**, p. 19].

This reverse of Schwarz inequality can be employed to obtain a refinement of the *Grüss inequality* as follows [**5**] (see also [**11**, p. 90]):

Let $e, x, y \in H$ with $\|e\| = 1$ and $\varphi, \Phi, \delta, \Delta \in \mathbb{C}$ such that either

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0, \operatorname{Re} \langle \Delta e - y, y - \delta e \rangle \geq 0 \tag{1.4}$$

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Inner product, Schwarz inequality, Grüss inequality, Bounded linear operators,

or, equivalently,

$$\left\| x - \frac{\varphi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \left\| y - \frac{\delta + \Delta}{2} e \right\| \leq \frac{1}{2} |\Delta - \delta|. \tag{1.5}$$

Then

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{4} |\Phi - \varphi| |\Delta - \delta| - \left| \langle x, e \rangle - \frac{\varphi + \Phi}{2} \right| \left| \langle y, e \rangle - \frac{\delta + \Delta}{2} \right| \\ & \leq \frac{1}{4} |\Phi - \varphi| |\Delta - \delta|. \end{aligned} \tag{1.6}$$

The constant $\frac{1}{4}$ is best possible.

The inequality between the first and last term in (1.6) was obtained in 1999 in [**4**].

Following [**6**], we can define for the complex numbers α, β and the bounded linear operator A the following transform

$$C_{\alpha,\beta}(A) := (A^* - \overline{\alpha} I)(\beta I - A), \tag{1.7}$$

where by A^* we denote the adjoint of A.

We list some properties of the transform $C_{\alpha,\beta}(\cdot)$ that are useful in the following [**6**]:

(i) For any $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$ we have:

$$C_{\alpha,\beta}(I) = (1 - \overline{\alpha})(\beta - 1) I, \qquad C_{\alpha,\alpha}(A) = -(\alpha I - A)^* (\alpha I - A), \tag{1.8}$$

$$C_{\alpha,\beta}(\gamma A) = |\gamma|^2 C_{\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}}(A) \quad \text{for each } \gamma \in \mathbb{C} \setminus \{0\}, \tag{1.9}$$

$$[C_{\alpha,\beta}(A)]^* = C_{\beta,\alpha}(A) \tag{1.10}$$

and

$$C_{\overline{\beta},\overline{\alpha}}(A^*) - C_{\alpha,\beta}(A) = A^* A - A A^*. \tag{1.11}$$

(ii) The operator $A \in B(H)$ is normal if and only if $C_{\overline{\beta},\overline{\alpha}}(A^*) = C_{\alpha,\beta}(A)$ for each $\alpha, \beta \in \mathbb{C}$.

(iii) If $A \in B(H)$ is invertible and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, then

$$\left(A^{-1}\right)^* C_{\alpha,\beta}(A) A^{-1} = \overline{\alpha} \beta C_{\frac{1}{\alpha}, \frac{1}{\beta}}\left(A^{-1}\right). \tag{1.12}$$

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re} \langle Ty, y \rangle \geq 0$ for any $y \in H$.

The following simple characterization result is useful in the following [**6**]:

For $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$ the following statements are equivalent:

(i) The transform $C_{\alpha,\beta}(A)$ is accretive;

(ii) The transform $C_{\overline{\alpha},\overline{\beta}}(A^*)$ is accretive;

(iii) We have the norm inequality

$$\left\| A - \frac{\alpha + \beta}{2} I \right\| \leq \frac{1}{2} |\beta - \alpha|. \tag{1.13}$$

For two bounded linear operators $A, B \in B(H)$ and the vector $x \in H, \|x\| = 1$ define the functional

$$G(A, B; x) := \langle Ax, Bx \rangle - \langle Ax, x \rangle \langle x, Bx \rangle.$$

Utilising the inequality (1.6), the following result concerning operator inequalities of Grüss type may be stated :

Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then

$$\begin{aligned} &|G(A, B; x)| \\ &\leq \frac{1}{4}|\beta - \alpha||\delta - \gamma| - \left|\left\langle\left(A - \frac{\alpha+\beta}{2}I\right)x, x\right\rangle\right|\left|\left\langle\left(B - \frac{\gamma+\delta}{2}I\right)x, x\right\rangle\right| \\ &\leq \frac{1}{4}|\beta - \alpha||\delta - \gamma|, \end{aligned} \tag{1.14}$$

for any $x \in H, \|x\| = 1$.

For other similar results see [**6**]. For more results on Schwarz's inequality on inner product spaces, see [**7**]-[**20**], [**23**]-[**31**] and the references therein.

Motivated by the above results we establish in this paper some new reverses of the Schwarz inequality as well as some Grüss' type inequalities in inner product spaces and apply them for bounded linear operators to generalise the results due to Bernstein in [**1**]. We also obtain some new numerical range and norm inequalities. Applications for discrete and integral inequalities are also provided.

2. Vector Inequalities

We start with the following result that provides an invariant property for the constant in the Schwarz inequality.

Lemma 1 (One parameter identity). *For any $x, y \in H$ and any $\lambda \in \mathbb{C}$ we have the equality*

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 = \|x - \lambda y\|^2\|y\|^2 - |\langle x - \lambda y, y\rangle|^2. \tag{2.1}$$

Proof. By the properties of inner product we have for any $y, x \in H$ and any $\lambda \in \mathbb{C}$ that

$$\begin{aligned} &\|x - \lambda y\|^2\|y\|^2 - |\langle x - \lambda y, y\rangle|^2 \\ &= \left(\|x\|^2 - 2\operatorname{Re}\left(\overline{\lambda}\langle x, y\rangle\right) + |\lambda|^2\|y\|^2\right)\|y\|^2 - \left|\langle x, y\rangle - \lambda\|y\|^2\right|^2 \\ &= \|x\|^2\|y\|^2 - 2\|y\|^2\operatorname{Re}\left(\overline{\lambda}\langle x, y\rangle\right) + |\lambda|^2\|y\|^4 \\ &- |\langle y, x\rangle|^2 + 2\|y\|^2\operatorname{Re}\left(\overline{\lambda}\langle x, y\rangle\right) - |\lambda|^2\|y\|^4 \\ &= \|y\|^2\|x\|^2 - |\langle y, x\rangle|^2. \end{aligned}$$

□

The following inequality holds:

Corollary 1. *For any $x, y \in H$ and any $\lambda \in \mathbb{C}$ we have the inequality*

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \|x - \lambda y\|^2\|y\|^2. \tag{2.2}$$

The equality holds in (2.2) if and only if $\langle x, y\rangle = \lambda\|y\|^2$.

Remark 1. *We observe that, if*

$$\left\|x - \frac{\gamma+\Gamma}{2}y\right\| \leq \frac{1}{2}|\Gamma - \gamma|\|y\|,$$

then by choosing $\lambda := \frac{\gamma+\Gamma}{2}$ *in (2.1), we get*

$$\begin{aligned} \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 &= \left\| x - \frac{\gamma+\Gamma}{2} y \right\|^2 \|y\|^2 - \left| \left\langle x - \frac{\gamma+\Gamma}{2} y, y \right\rangle \right|^2 \\ &\leq \frac{1}{2} |\Gamma - \gamma|^2 \|y\|^4 - \left| \left\langle x - \frac{\gamma+\Gamma}{2} y, y \right\rangle \right|^2 \end{aligned} \tag{2.3}$$

which provides a simpler proof for the inequality (1.3) than the one given in [**5**].

Lemma 2 (Two parameters identity). *For any* $x, y \in H$ *and any* $\lambda, \delta \in \mathbb{C}$ *we have the equality*

$$\begin{aligned} &\left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \right] |\delta - \lambda|^2 \\ &= \|x - \lambda y\|^2 \|x - \delta y\|^2 - |\langle x - \lambda y, x - \delta y\rangle|^2 . \end{aligned} \tag{2.4}$$

Proof. Denote $z := x - \lambda y$. Utilising the inner product properties, we have

$$\begin{aligned} |\langle x - \lambda y, x - \delta y\rangle|^2 &= |\langle z, x - \delta y\rangle|^2 = |\langle z, x - \lambda y + (\lambda - \delta) y\rangle|^2 \\ &= |\langle z, z + (\lambda - \delta) y\rangle|^2 = \left| \|z\|^2 + \overline{(\lambda - \delta)} \langle z, y\rangle \right|^2 \\ &= \|z\|^4 + 2 \|z\|^2 \operatorname{Re}\left((\lambda - \delta) \overline{\langle z, y\rangle} \right) + |\lambda - \delta|^2 |\langle z, y\rangle|^2 \\ &= \|z\|^4 + 2 \|z\|^2 \operatorname{Re}\left((\lambda - \delta) \overline{\langle z, y\rangle} \right) + |\lambda - \delta|^2 \|z\|^2 \|y\|^2 \\ &\quad - |\lambda - \delta|^2 \left[\|z\|^2 \|y\|^2 - |\langle z, y\rangle|^2 \right] . \end{aligned} \tag{2.5}$$

Observe also that

$$\begin{aligned} &\|z\|^4 + 2 \|z\|^2 \operatorname{Re}\left((\lambda - \delta) \overline{\langle z, y\rangle} \right) + |\lambda - \delta|^2 \|z\|^2 \|y\|^2 \\ &= \|z\|^2 \left[\|z\|^2 + 2 \operatorname{Re}\left(\overline{(\lambda - \delta) \overline{\langle z, y\rangle}} \right) + |\lambda - \delta|^2 \|y\|^2 \right] \\ &= \|z\|^2 \left[\|z\|^2 + 2 \operatorname{Re}\left(\langle z, (\lambda - \delta) y\rangle \right) + |\lambda - \delta|^2 \|y\|^2 \right] \\ &= \|z\|^2 \|z + (\lambda - \delta) y\|^2 = \|x - \lambda y\|^2 \|x - \delta y\|^2 . \end{aligned} \tag{2.6}$$

On making use of (2.5) and (2.6), we deduce the desired result (2.4). □

Corollary 2. *For any* $x, y \in H$ *and any* $\lambda, \delta \in \mathbb{C}$ *with* $\delta \neq \lambda$ *we have the inequality*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{|\delta - \lambda|^2} \|x - \lambda y\|^2 \|x - \delta y\|^2 . \tag{2.7}$$

The equality holds in (2.7) if and only if $x - \lambda y \perp x - \delta y$.

Remark 2. *We have from (2.7) the following symmetric inequality*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \|x - y\|^2 \|x + y\|^2 , \tag{2.8}$$

for any $x, y \in H$. *The equality holds in (2.8) if and only if* $x - y \perp x + y$.

In the case of real spaces, if we define the angle $\varphi \in [0, \pi]$ *between the vectors* x *and* y *by*

$$\cos \varphi := \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

then

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 = \|x\|^2 \|y\|^2 \sin^2 \varphi$$

and by (2.8) we get the inequality

$$\|x\| \|y\| \sin \varphi \leq \frac{1}{2} \|x - y\| \|x + y\| , \tag{2.9}$$

which has a simple geometrical meaning, namely, the area of the parallelogram generated by the vectors x *and* y *is less than half of the product of the length of its diagonals. The equality holds iff the diagonals are orthogonal.*

The following reverse of Schwarz inequality may be stated as well:

Corollary 3. *Let* $x, y \in H$ *and* $\gamma, \Gamma, \varphi, \Phi \in \mathbb{C}$ *with* $\Gamma \neq \gamma$, $\Phi \neq \varphi$, $\gamma + \Gamma \neq \varphi + \Phi$ *and such that either*

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Phi y - x, x - \varphi y \rangle \geq 0 \tag{2.10}$$

or, equivalently,

$$\left\| x - \frac{\gamma + \Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|$$

and

$$\left\| x - \frac{\varphi + \Phi}{2} y \right\| \leq \frac{1}{2} |\Phi - \varphi| \|y\| .$$

Then we have

$$\begin{aligned} & \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \\ & \leq \frac{1}{|\gamma + \Gamma - \varphi - \Phi|^2} \left[\frac{|\Gamma - \gamma|^2 |\Phi - \varphi|^2}{4} \|y\|^4 \right. \\ & \left. - \left| \left\langle x - \frac{\gamma + \Gamma}{2} y, x - \frac{\varphi + \Phi}{2} y \right\rangle \right|^2 \right] \\ & \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2 |\Phi - \varphi|^2}{|\gamma + \Gamma - \varphi - \Phi|^2} \|y\|^4 . \end{aligned} \tag{2.11}$$

We are able now to state the following Grüss type result:

Theorem 1. *Let* $x, y, e \in H$ *with* $\|e\| = 1$. *Then for any* $\lambda, \delta, \gamma, \eta \in \mathbb{C}$ *with* $\lambda \neq \delta$ *and* $\gamma \neq \eta$ *we have the inequality*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{|\lambda - \delta| |\gamma - \eta|} (\|x - \lambda e\| \|x - \delta e\| \|y - \gamma e\| \|y - \eta e\| \\ & - |\langle x - \lambda e, x - \delta e \rangle| |\langle y - \gamma e, y - \eta e \rangle|) \\ & \leq \frac{1}{|\lambda - \delta| |\gamma - \eta|} \|x - \lambda e\| \|x - \delta e\| \|y - \gamma e\| \|y - \eta e\| . \end{aligned} \tag{2.12}$$

Proof. Applying Schwarz's inequality for the vectors $x-\langle x,e\rangle e$ and $y-\langle y,e\rangle e$ and taking into account that

$$\langle x-\langle x,e\rangle e, y-\langle y,e\rangle e\rangle = \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle,$$

$$\|x-\langle x,e\rangle e\|^2 = \|x\|^2 - |\langle x,e\rangle|^2$$

and

$$\|y-\langle y,e\rangle e\|^2 = \|y\|^2 - |\langle y,e\rangle|^2$$

we have the inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \le \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} \tag{2.13}$$

for any $x,y,e\in H$ with $\|e\|=1$.

From (2.4) we also have that

$$\begin{aligned}&\left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2} \\ &= \frac{1}{|\delta-\lambda|}\left(\|x-\lambda e\|^2\|x-\delta e\|^2 - |\langle x-\lambda e, x-\delta e\rangle|^2\right)^{1/2}\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} \\ &= \frac{1}{|\gamma-\eta|}\left(\|y-\gamma e\|^2\|y-\eta e\|^2 - |\langle y-\gamma e, y-\eta e\rangle|^2\right)^{1/2}\end{aligned} \tag{2.15}$$

for any $x,y,e\in H$ with $\|e\|=1$ and for any $\lambda,\delta,\gamma,\eta\in\mathbb{C}$ with $\lambda\neq\delta$ and $\gamma\neq\eta$.

Now, if we multiply (2.14) with (2.15) we get

$$\begin{aligned}&\left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} \\ &\le \frac{1}{|\delta-\lambda|\,|\gamma-\eta|} \\ &\times\left(\|x-\lambda e\|^2\|x-\delta e\|^2 - |\langle x-\lambda e, x-\delta e\rangle|^2\right)^{1/2} \\ &\times\left(\|y-\gamma e\|^2\|y-\eta e\|^2 - |\langle y-\gamma e, y-\eta e\rangle|^2\right)^{1/2}.\end{aligned} \tag{2.16}$$

Further, if we use the elementary inequality

$$\left(a^2-b^2\right)^{1/2}\left(c^2-d^2\right)^{1/2} \le ac-bd$$

that holds for $a\ge b\ge 0$ and $c\ge d\ge 0$, we also have

$$\begin{aligned}&\left(\|x-\lambda e\|^2\|x-\delta e\|^2 - |\langle x-\lambda e, x-\delta e\rangle|^2\right)^{1/2} \\ &\times\left(\|y-\gamma e\|^2\|y-\eta e\|^2 - |\langle y-\gamma e, y-\eta e\rangle|^2\right)^{1/2} \\ &\le (\|x-\lambda e\|\,\|x-\delta e\|\,\|y-\gamma e\|\,\|y-\eta e\| \\ &-|\langle x-\lambda e, x-\delta e\rangle|\,|\langle y-\gamma e, y-\eta e\rangle|).\end{aligned} \tag{2.17}$$

Finally, on making use of the inequalities (2.13), (2.16) and (2.17) we deduce the desired result (2.12). □

Remark 3. *We have from (2.12) the following symmetric inequality:*

$$\begin{aligned} &|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \\ &\leq \frac{1}{4}\left(\|x-e\|\,\|x+e\|\,\|y-e\|\,\|y+e\|\right. \\ &\left.-\left|\langle x-e,x+e\rangle\right|\left|\langle y-e,y+e\rangle\right|\right) \\ &\leq \frac{1}{4}\|x-e\|\,\|x+e\|\,\|y-e\|\,\|y+e\|, \end{aligned} \tag{2.18}$$

for any $x,y,e\in H$ with $\|e\|=1$.

For the complex parameters $\gamma,\Gamma\in\mathbb{C}$ with $\Gamma\neq\gamma$ and $e\in H$ with $\|e\|=1$, consider the nonempty set

$$S(\gamma,\Gamma;e) := \left\{x\in H : \left\|x-\frac{\gamma+\Gamma}{2}e\right\| \leq \frac{1}{2}|\Gamma-\gamma|\right\}.$$

With this notation we have the following result:

Corollary 4. *Let $\gamma_k,\Gamma_k,\varphi_k,\Phi_k\in\mathbb{C}$ with $\Gamma_k\neq\gamma_k$, $\Phi_k\neq\varphi_k$, $\gamma_k+\Gamma_k\neq\varphi_k+\Phi_k$ where $k=1,2$. If $e\in H$ with $\|e\|=1$, $x\in S(\gamma_1,\Gamma_1;e)\cap S(\varphi_1,\Phi_1;e)$ and $y\in S(\gamma_2,\Gamma_2;e)\cap S(\varphi_2,\Phi_2;e)$, where the intersections are assumed to be nonempty, then we have*

$$\begin{aligned} &|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \\ &\leq \frac{1}{|\gamma_1+\Gamma_1-\varphi_1-\Phi_1|\,|\gamma_2+\Gamma_2-\varphi_2-\Phi_2|} \\ &\cdot\left[\frac{|\Gamma_1-\gamma_1|\,|\Phi_1-\varphi_1|\,|\Gamma_2-\gamma_2|\,|\Phi_2-\varphi_2|}{4}\right. \\ &\left.-\left|\left\langle x-\frac{\gamma_1+\Gamma_1}{2}e, x-\frac{\Phi_1+\varphi_1}{2}e\right\rangle\right|\left|\left\langle y-\frac{\gamma_2+\Gamma_2}{2}e, y-\frac{\Phi_2+\varphi_2}{2}e\right\rangle\right|\right] \\ &\leq \frac{1}{4}\cdot\frac{|\Gamma_1-\gamma_1|\,|\Phi_1-\varphi_1|\,|\Gamma_2-\gamma_2|\,|\Phi_2-\varphi_2|}{|\gamma_1+\Gamma_1-\varphi_1-\Phi_1|\,|\gamma_2+\Gamma_2-\varphi_2-\Phi_2|}. \end{aligned} \tag{2.19}$$

3. Applications for Linear Operators

We can emply the above results to get some inequalities for operators as follows.

Theorem 2. *Let A be a bounded linear operator on the complex Hilbert space H, e a unit eigenvector for A with corresponding eigenvalue λ and, at the same time, an eigenvector for A^* with corresponding eigenvalue μ. Then for any $g\in H$ with $g=\alpha e+f$ and $f\perp e$ we have*

$$\begin{aligned} |\alpha|^2\|(A-\lambda I)g\|^2 &\leq \|g\|^2\|Ag\|^2 - |\langle Ag,g\rangle|^2 \\ &\leq \left(|\alpha|^2+\|f\|^2\right)\|(A-\lambda I)g\|^2. \end{aligned} \tag{3.1}$$

Proof. Applying Lemma 1 we can write the equality

$$\|g\|^2\|Ag\|^2 - |\langle Ag,g\rangle|^2 = \|g\|^2\|Ag-\lambda g\|^2 - |\langle Ag-\lambda g,g\rangle|^2. \tag{3.2}$$

Due to the orthogonality $f\perp e$ and the fact that $\|e\|=1$, we have that

$$\|g\|^2 = |\alpha|^2+\|f\|^2.$$

Since e is an eigenvector for A corresponding eigenvalue λ we have

$$\begin{aligned} Ag-\lambda g &= A(\alpha e_{\lambda}+f)-\lambda(\alpha e_{\lambda}+f) \\ &= \alpha A(e_{\lambda})+Af-\lambda\alpha e_{\lambda}-\lambda f \\ &= \alpha\lambda e_{\lambda}+Af-\lambda\alpha e_{\lambda}-\lambda f = Af-\lambda f. \end{aligned} \tag{3.3}$$

Since e is also an eigenvector for A^{*} corresponding eigenvalue μ we have

$$\begin{aligned} &\langle A(\alpha e+f)-\lambda(\alpha e+f), \alpha e+f\rangle \\ &= \langle Af-\lambda f, \alpha e+f\rangle = \overline{\alpha}\langle Af, e\rangle + \langle Af, f\rangle - \lambda\overline{\alpha}\langle f, e\rangle - \lambda\langle f, f\rangle \\ &= \overline{\alpha}\langle f, A^{*}e\rangle + \langle Af, f\rangle - \lambda\langle f, f\rangle = \overline{\alpha}\langle f, \mu e\rangle + \langle Af, f\rangle - \lambda\langle f, f\rangle \\ &= \langle Af, f\rangle - \lambda\langle f, f\rangle = \langle Af-\lambda f, f\rangle. \end{aligned} \tag{3.4}$$

On making use of the above equalities, we get

$$\begin{aligned} &\|g\|^{2}\|Ag\|^{2}-|\langle Ag, g\rangle|^{2} \\ &= \left(|\alpha|^{2}+\|f\|^{2}\right)\|Af-\lambda f\|^{2}-|\langle Af-\lambda f, f\rangle|^{2}. \end{aligned} \tag{3.5}$$

Since, obviously

$$\begin{aligned} &\left(|\alpha|^{2}+\|f\|^{2}\right)\|Af-\lambda f\|^{2}-|\langle Af-\lambda f, f\rangle|^{2} \\ &\leq \left(|\alpha|^{2}+\|f\|^{2}\right)\|Af-\lambda f\|^{2} \\ &= \left(|\alpha|^{2}+\|f\|^{2}\right)\|Ag-\lambda g\|^{2} \end{aligned} \tag{3.6}$$

then by (3.5) we get the second inequality in (3.1).

Also, by Schwarz's inequality we have

$$\begin{aligned} &\left(|\alpha|^{2}+\|f\|^{2}\right)\|Af-\lambda f\|^{2}-|\langle Af-\lambda f, f\rangle|^{2} \\ &= |\alpha|^{2}\|Af-\lambda f\|^{2}+\|f\|^{2}\|Af-\lambda f\|^{2}-|\langle Af-\lambda f, f\rangle|^{2} \\ &\geq |\alpha|^{2}\|Af-\lambda f\|^{2} \end{aligned} \tag{3.7}$$

and by (3.5) we get the first inequality in (3.1). □

Proposition 1. *Let A be a bounded linear operator on the complex Hilbert space H. Then for any $\lambda, \delta \in \mathbb{C}$ with $\delta \neq \lambda$ we have the inequality*

$$\|Ag\|^{2}\|g\|^{2}-|\langle Ag, g\rangle|^{2} \leq \frac{1}{|\delta-\lambda|^{2}}\|Ag-\lambda g\|^{2}\|Ag-\delta g\|^{2}. \tag{3.8}$$

The equality holds in (3.8) if and only if $Ag-\lambda g \perp Ag-\delta g$.

Proof. Follows by Corollary 2 for the choices $x = Ag$ and $y = g$. □

Remark 4. *We observe that both the first inequality in (3.1) and the inequality (3.8) were obtained in the particular case of a selfadjoint operator A by H.J. Bernstein in 1987, see* [**1**].

4. Inequalities for the Numerical Range

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator A is the subset of the complex numbers $\mathbb{C}$ given by [**21**, p. 1]:

$$W(A)=\{\langle Ax,x\rangle,\ x\in H,\ \|x\|=1\}.$$

The *numerical radius* $w(A)$ of an operator A on H is given by [**21**, p. 8]:

$$w(A)=\sup\{|\lambda|,\lambda\in W(A)\}=\sup\{|\langle Ax,x\rangle|,\|x\|=1\}. \tag{4.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [**21**, p. 9]:

Theorem 3 (Equivalent norm). *For any $A\in B(H)$ one has*

$$w(A)\le\|A\|\le 2w(A). \tag{4.2}$$

The following reverses of the first inequality in (4.2), i.e., upper bounds under appropriate conditions for the bounded linear operator A for the nonnegative difference $\|A\|^2-w^2(A)$ can be obtained:

Proposition 2. *For any $A\in B(H)$ and $\lambda\in\mathbb{C}$ one has*

$$\begin{aligned}0&\le\|A\|^2-w^2(A)\\&\le\|A-\lambda I\|^2-w_i^2(A-\lambda I)\le\|A-\lambda I\|^2,\end{aligned} \tag{4.3}$$

where we denote by $w_i(B):=\inf_{\|x\|=1}|\langle Bx,x\rangle|$ for $B\in B(H)$.

Proof. Applying the equality (2.1) we have

$$\|Ax\|^2-|\langle Ax,x\rangle|^2=\|Ax-\lambda x\|^2-|\langle Ax-\lambda x,x\rangle|^2 \tag{4.4}$$

for any $x\in H,\ \|x\|=1$.

Taking the suppremum over $x\in H,\ \|x\|=1$, we deduce from (4.4)

$$\begin{aligned}\sup_{\|x\|=1}\|Ax\|^2&\le\sup_{\|x\|=1}|\langle Ax,x\rangle|^2\\&\quad+\sup_{\|x\|=1}\left[\|Ax-\lambda x\|^2-|\langle Ax-\lambda x,x\rangle|^2\right]\\&\le\sup_{\|x\|=1}|\langle Ax,x\rangle|^2+\sup_{\|x\|=1}\|Ax-\lambda x\|^2\\&\quad-\inf_{\|x\|=1}|\langle Ax-\lambda x,x\rangle|^2\end{aligned}$$

which produces the desired result (4.3). □

Remark 5. *We observe that, if $A\in B(H)$ and $\alpha,\beta\in\mathbb{C}$ such that the transform $C_{\alpha,\beta}(A)$ is accretive, then we have the chain of inequalities*

$$\begin{aligned}0&\le\|A\|^2-w^2(A)\\&\le\left\|A-\frac{\alpha+\beta}{2}I\right\|^2-w_i^2\left(A-\frac{\alpha+\beta}{2}I\right)\\&\le\left\|A-\frac{\alpha+\beta}{2}I\right\|^2\le\frac{1}{4}|\alpha-\beta|^2.\end{aligned} \tag{4.5}$$

This result was obtained in [**6**, Theorem 3].

Proposition 3. *For any $A \in B(H)$ and $\delta, \lambda \in \mathbb{C}$ with $\delta \neq \lambda$ one has*

$$0 \leq \|A\|^2 - w^2(A) \leq \frac{1}{|\delta - \lambda|^2} \left[\|A - \lambda I\|^2 \|A - \delta I\|^2 - w_i^2 (C_{\delta,\lambda}(A)) \right] \leq \frac{1}{|\delta - \lambda|^2} \|A - \lambda I\|^2 \|A - \delta I\|^2 . \tag{4.6}$$

Proof. Applying the equality (2.4) we have

$$\begin{aligned} \|Ax\|^2 &= |\langle Ax, x\rangle|^2 \\ &+ \frac{1}{|\delta - \lambda|^2} \left[\|Ax - \lambda x\|^2 \|Ax - \delta x\|^2 - |\langle Ax - \lambda x, Ax - \delta x\rangle|^2 \right] \\ &= |\langle Ax, x\rangle|^2 + \frac{1}{|\delta - \lambda|^2} \\ &\times \left[\|(A - \lambda I)x\|^2 \|(A - \delta I)x\|^2 - \left|\left\langle \left(A^* - \overline{\delta} I\right)(A - \lambda I)x, x\right\rangle\right|^2 \right] \end{aligned} \tag{4.7}$$

for any $x \in H$, $\|x\| = 1$.

Taking the suppremum over $x \in H$, $\|x\| = 1$,in (4.7), we deduce the desired result (4.6). □

Remark 6. *We observe that, if $A \in B(H)$ and $\alpha, \beta, \chi, \psi \in \mathbb{C}$ with $\alpha+\beta \neq \chi+\psi$ and such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\chi,\psi}(A)$ are accretive, then we have the chain of inequalities*

$$\begin{aligned} 0 &\leq \|A\|^2 - w^2(A) \\ &\leq \frac{1}{\left|\frac{\chi+\psi}{2} - \frac{\alpha+\beta}{2}\right|^2} \\ &\cdot \left[\left\|A - \frac{\alpha+\beta}{2} I\right\|^2 \left\|A - \frac{\chi+\psi}{2} I\right\|^2 - w_i^2 \left(C_{\frac{\chi+\psi}{2}, \frac{\alpha+\beta}{2}}(A) \right) \right] \\ &\leq \frac{1}{\left|\frac{\chi+\psi}{2} - \frac{\alpha+\beta}{2}\right|^2} \left\|A - \frac{\alpha+\beta}{2} I\right\|^2 \left\|A - \frac{\chi+\psi}{2} I\right\|^2 \\ &\leq \frac{1}{4} \cdot \frac{|\alpha - \beta|^2 |\chi - \psi|^2}{|\chi + \psi - \alpha - \beta|^2}. \end{aligned} \tag{4.8}$$

We recall that the following result for the powers of operators holds:

$$w(A^n) \leq w^n(A) \tag{4.9}$$

for any natural number $n \geq 1$ and a bounded linear operator A.

The case $n = 2$ is of interest and the following reverse can be provided:

Proposition 4. *If $A \in B(H)$ and $\lambda, \delta, \gamma, \eta \in \mathbb{C}$ with $\lambda \neq \delta$ and $\gamma \neq \eta$ then*

$$0 \leq w^2(A) - w\left(A^2\right) \leq \frac{1}{|\lambda - \delta| |\gamma - \eta|} \|A - \lambda I\| \|A - \delta I\| \|A^* - \gamma I\| \|A^* - \eta I\| . \tag{4.10}$$

Proof. If we choose $x = Au, y = A^*u$ with $u \in H$ and $\|u\| = 1$ in the inequality (2.12), then we have

$$\begin{aligned}(4.11)\qquad &\left|\left\langle A^2u,u\right\rangle - \langle Au,u\rangle^2\right| \\ &\leq \frac{1}{|\lambda-\delta|\,|\gamma-\eta|}\left(\|Au-\lambda u\|\,\|Au-\delta u\|\,\|A^*u-\gamma u\|\,\|A^*u-\eta u\|\right. \\ &\left.-\left|\langle Au-\lambda u, Au-\delta u\rangle\right|\left|\langle A^*u-\gamma u, A^*u-\eta u\rangle\right|\right) \\ &\leq \frac{1}{|\lambda-\delta|\,|\gamma-\eta|}\|Au-\lambda u\|\,\|Au-\delta u\|\,\|A^*u-\gamma u\|\,\|A^*u-\eta u\|,\end{aligned}$$

which is an inequality of interest in itself as well.

By the modulus properties we also have

$$(4.12)\qquad |\langle Au,u\rangle|^2 - \left|\left\langle A^2u,u\right\rangle\right| \leq \left|\left\langle A^2u,u\right\rangle - \langle Au,u\rangle^2\right|$$

for any $u \in H$.

By the inequalities (4.11) and (4.12) we have

$$\begin{aligned}(4.13)\quad |\langle Au,u\rangle|^2 &\leq \left|\left\langle A^2u,u\right\rangle\right| \\ &+\frac{1}{|\lambda-\delta|\,|\gamma-\eta|}\|Au-\lambda u\|\,\|Au-\delta u\|\,\|A^*u-\gamma u\|\,\|A^*u-\eta u\|\end{aligned}$$

for any $u \in H$ with $\|u\| = 1$.

Taking the supremum over $\|u\| = 1$ in (4.13), we deduce the desired result (4.10). □

Corollary 5. *If $A \in B(H)$ and $\lambda, \delta \in \mathbb{C}$ with $\lambda \neq \delta$ then*

$$(4.14)\qquad 0 \leq w^2(A) - w\left(A^2\right) \leq \frac{1}{|\lambda-\delta|^2}\|A-\lambda I\|^2\|A-\delta I\|^2.$$

The proof follows by (4.13) on choosing $\gamma = \bar{\lambda}$ and $\eta = \bar{\delta}$.

Remark 7. *We observe that, if $A \in B(H)$ and $\alpha, \beta, \chi, \psi \in \mathbb{C}$ with $\alpha+\beta \neq \chi+\psi$ and such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\chi,\psi}(A)$ are accretive, then we have the chain of inequalities*

$$\begin{aligned}(4.15)\qquad 0 &\leq w^2(A) - w\left(A^2\right) \\ &\leq \frac{1}{\left|\frac{\chi+\psi}{2}-\frac{\alpha+\beta}{2}\right|^2}\left\|A-\frac{\alpha+\beta}{2}I\right\|^2\left\|A-\frac{\chi+\psi}{2}I\right\|^2 \\ &\leq \frac{1}{4}\cdot\frac{|\alpha-\beta|^2\,|\chi-\psi|^2}{|\chi+\psi-\alpha-\beta|^2}.\end{aligned}$$

5. Applications for Discrete and Integral Inequalities

The discrete and integral versions of the Cauchy-Bunyakovsky-Schwarz inequality play an important role in many applications of the Mathematical Analysis, see for instance [**2**], [**3**], [**13**], [**17**], [**18**], [**22**], [**24**], [**26**] and [**28**].

Motivated by these applications we state here the discrete and the integral version for vector-valued functions with values in a Hilbert space of the inequality (2.7). Notice that the case when the Hilbert space is taken to be the field of complex numbers $\mathbb{C}$ with the inner product $\langle z,w\rangle := z\overline{w},\ z,w\in\mathbb{C}$ then we recapture the classical versions of these inequalities.

Let $(K,\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$, $p_j\geq 0,\ j\in\mathbb{N}$ with $\sum_{j=1}^{\infty}p_j=1$. Consider $\ell_{\mathbf{p}}^2(K)$ as the space

$$\ell_{\mathbf{p}}^2(K) := \left\{ x=(x_j)_{j\in\mathbb{N}} \middle| x_j\in K,\ j\in\mathbb{N} \text{ and } \sum_{j=1}^{\infty}p_j\|x_j\|^2<\infty \right\}.$$

It is well known that $\ell_{\mathbf{p}}^2(K)$ endowed with the inner product

$$\langle x,y\rangle_{\mathbf{p}} := \sum_{j=1}^{\infty}p_j\langle x_j,y_j\rangle$$

is a Hilbert space over $\mathbb{K}$. The norm $\|\cdot\|_{\mathbf{p}}$ of $\ell_{\mathbf{p}}^2(K)$ is given by

$$\|x\|_{\mathbf{p}} := \left(\sum_{j=1}^{\infty}p_j\|x_j\|^2\right)^{\frac{1}{2}}.$$

If $x,y\in\ell_{\mathbf{p}}^2(K)$, then the following Cauchy-Bunyakovsky-Schwarz discrete inequality holds true

$$\sum_{j=1}^{\infty}p_j\|x_j\|^2\sum_{j=1}^{\infty}p_j\|y_j\|^2 \geq \left|\sum_{j=1}^{\infty}p_j\langle x_j,y_j\rangle\right|^2 \tag{5.1}$$

with equality iff there exists a $\lambda\in\mathbb{K}$ such that $x_j=\lambda y_j$ for each $j\in\mathbb{N}$.

Making use of the inequality (2.7) we can state the following reverse inequality for (5.1)

$$\begin{aligned} 0 &\leq \sum_{j=1}^{\infty}p_j\|x_j\|^2\sum_{j=1}^{\infty}p_j\|y_j\|^2 - \left|\sum_{j=1}^{\infty}p_j\langle x_j,y_j\rangle\right|^2 \\ &\leq \frac{1}{|\delta-\lambda|^2}\sum_{j=1}^{\infty}p_j\|x_j-\lambda y_j\|^2\sum_{j=1}^{\infty}p_j\|x_j-\delta y_j\|^2, \end{aligned} \tag{5.2}$$

where $\lambda,\delta\in\mathbb{C}$ with $\lambda\neq\delta$.

In particular, we have

$$\begin{aligned} 0 &\leq \sum_{j=1}^{\infty}p_j\|x_j\|^2\sum_{j=1}^{\infty}p_j\|y_j\|^2 - \left|\sum_{j=1}^{\infty}p_j\langle x_j,y_j\rangle\right|^2 \\ &\leq \frac{1}{4}\sum_{j=1}^{\infty}p_j\|x_j-y_j\|^2\sum_{j=1}^{\infty}p_j\|x_j+y_j\|^2. \end{aligned} \tag{5.3}$$

Moreover, if there exists the positive constants M and N such that

$$\|x_j-y_j\|\leq M \text{ and } \|x_j+y_j\|\leq N \text{ for each } j\in\mathbb{N} \tag{5.4}$$

then we have

$$0 \leq \sum_{j=1}^{\infty}p_j\|x_j\|^2\sum_{j=1}^{\infty}p_j\|y_j\|^2 - \left|\sum_{j=1}^{\infty}p_j\langle x_j,y_j\rangle\right|^2 \leq \frac{1}{4}M^2N^2. \tag{5.5}$$

Assume that $(K;\langle\cdot,\cdot\rangle)$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho:[a,b]\subset\mathbb{R}\to[0,\infty)$ is a Lebesgue integrable function with $\int_a^b\rho(t)\,dt=1$, then we may consider the space $L_\rho^2([a,b];K)$ of all functions$f:[a,b]\to K$, that are Bochner measurable and $\int_a^b\rho(t)\|f(t)\|^2\,dt<\infty$. It is well known that $L_\rho^2([a,b];K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f,g\rangle_\rho:=\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt$$

and generating the norm

$$\|f\|_\rho:=\left(\int_a^b\rho(t)\|f(t)\|^2\,dt\right)^{\frac{1}{2}},$$

is a Hilbert space over $\mathbb{K}$.

The following integral inequality is known in the literature as the Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_a^b\rho(t)\|f(t)\|^2\,dt\int_a^b\rho(t)\|g(t)\|^2\,dt\geq\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt\right|^2, \tag{5.6}$$

provided $f,g\in L_\rho^2([a,b];K)$.

The case of equality holds in (5.6) iff there exists a constant $\lambda\in\mathbb{K}$ such that $f(t)=\lambda g(t)$ for almost every (a.e.) $t\in[a,b]$.

Making use of the inequality (2.7) we can state the following reverse inequality for (5.6)

$$\begin{aligned}0&\leq\int_a^b\rho(t)\|f(t)\|^2\,dt\int_a^b\rho(t)\|g(t)\|^2\,dt-\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt\right|^2\\&\leq\frac{1}{|\delta-\lambda|^2}\int_a^b\rho(t)\|f(t)-\lambda g(t)\|^2\,dt\int_a^b\rho(t)\|f(t)-\delta g(t)\|^2\,dt,\end{aligned} \tag{5.7}$$

where $\lambda,\delta\in\mathbb{C}$ with $\lambda\neq\delta$, and in particular

$$\begin{aligned}0&\leq\int_a^b\rho(t)\|f(t)\|^2\,dt\int_a^b\rho(t)\|g(t)\|^2\,dt-\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt\right|^2\\&\leq\frac{1}{4}\int_a^b\rho(t)\|f(t)-g(t)\|^2\,dt\int_a^b\rho(t)\|f(t)+g(t)\|^2\,dt.\end{aligned} \tag{5.8}$$

If there exist the positive constants P and Q suct that

$$\|f(t)-g(t)\|\leq P\text{ and }\|f(t)+g(t)\|\leq Q\text{ for a.e. }t\in[a,b] \tag{5.9}$$

then

$$\begin{aligned}0&\leq\int_a^b\rho(t)\|f(t)\|^2\,dt\int_a^b\rho(t)\|g(t)\|^2\,dt-\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt\right|^2\\&\leq\frac{1}{4}P^2Q^2.\end{aligned} \tag{5.10}$$

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