Received 29/08/12

SOME RESULTS ON COMPARING TWO INTEGRAL MEANS FOR ABSOLUTELY CONTINUOUS FUNCTION AND APPLICATIONS

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ABSTRACT. Some better estimates for the difference between the integral mean of a function and its mean over a subinterval are established. Various applications for special means and probability density functions are also given.

1. Introduction

The classical Ostrowski integral inequality [1] stipulates a bound between a function evaluated at an interior point and the average of the function over an interval. More precisely,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x)\right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$, where $f' \in L_\infty(a,b)$, that is,

$$\|f'\|_\infty = ess \sup_{t\in[a,b]} |f'(t)| < \infty,$$

and $f : [a,b] \to R$ is a differentiable function on (a,b). Here, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

It is worth noticing that this inequality plays a key role in adaptive numerical quadrature rules. For various results and generalizations concerning Ostrowski's inequality, see [2-16] and the references therein.

In [17], Dragomir and Wang introduced the following inequality of Ostrowski-Grüss type. That is,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x) - \left(\frac{a+b}{2} - x\right)\frac{f(b)-f(a)}{b-a}\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.2}$$

for $f : [a,b] \to R$ is a differentiable function on (a,b) and $\gamma \le f'(x) \le \Gamma$ for all $x \in [a,b]$. There are many improvements and refinements of the right hand side of inequality (1.2) in the literature. See for instance [2, 3, 18-20].

On the other hand, in [21], Barnett et al. compared the difference of two integral means as in the following Theorem 1 in which the function has the first derivative bounded where is defined. The results are also a generalization of (1.1) and were applied to probability density functions, special means, Jeffreys divergence in Information Theory and the sampling of continuous streams in Statistics.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D10.

Key words and phrases. Grüss inequality, Ostrowski's inequality, integral means, special mean, probability density function.

Theorem 1. *Let $f : [a,b] \to R$ be an absolutely continuous function with the property that $f' \in L_\infty[a,b]$. Then, for $a \le c < d \le b$, we have the inequality*

$$
\begin{aligned}
(1.3) \qquad & \left| \frac{1}{b-a}\int_a^b f(u)du - \frac{1}{d-c}\int_c^d f(u)du \right| \\
& \le \left\{ \frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{c+d}{2}}{b-a-d+c} \right]^2 \right\} (b-a-d+c)\|f'\|_\infty. \\
& \le \frac{1}{2}(b-a-d+c)\|f'\|_\infty.
\end{aligned}
$$

The constant $\frac{1}{4}$ is best possible in the first inequality and $\frac{1}{2}$ is best in the second one.

The purpose of this article is, by using a variant of the Grüss inequality, to establish some new better inequalities of (1.3) in which f' may not belong to L_∞. Applying these results, some new inequalities for special means and the probability density functions will be also given in Section 3 and Section 4, respectively.

2. Preliminary Lemmas and Main Results

The following lemma is the known Grüss inequality, see [22, p.295].

Lemma 1. *Let $f, g : [a,b] \le R$ be two integrable functions such that $\phi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a,b]$ where $\gamma, \Gamma, \phi, \Phi$ are constants. Then we have the inequality*

$$
\begin{aligned}
& \left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx \right| \\
& \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma).
\end{aligned}
$$

and that constant $\frac{1}{4}$ is the best possible.

Further, Cheng and Sun [23] established the following variant of Grüss's inequality. For extensions in the general case of the Lebesgue integral on measurable spaces, the sharpness of the constant $\frac{1}{2}$ as well as the corresponding discrete version, see [24].

Lemma 2. *Let $f, g : [a,b] \subset R$ be two integrable functions such that $\gamma \le g(x) \le \Gamma$ for all $x \in [a,b]$ where γ and Γ are constants. Then we have the inequality*

$$
\begin{aligned}
& \left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx \right| \\
& \le \frac{1}{2}(\Gamma - \gamma)\frac{1}{(b-a)}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| dx.
\end{aligned}
$$

The following lemma has been obtained by Barnett et al. in [21].

Lemma 3. *Let $f : [a,b] \to R$ be an absolutely continuous function and $a \le c < d \le b$. Denote $K_{c,d} : [a,b] \to R$, the kernel given by*

$$
K_{c,d}(s) = \begin{cases} \frac{a-s}{b-a}, & \text{if } s \in [a,c], \\ \frac{s-c}{d-c} + \frac{a-s}{b-a}, & \text{if } s \in (c,d), \\ \frac{b-s}{b-a}, & \text{if } s \in [d,b]. \end{cases}
$$

Then we have the representation

$$\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{d-c}\int_c^d f(u)du = \int_a^b K_{c,d}(s)f'(s)ds, .$$

The main results are as follows.

Theorem 2. *Let $f : [a,b] \to R$ be an absolutely continuous function such that $\gamma \le f'(x) \le \Gamma$ for all $x \in [a,b]$ where γ and Γ are constants. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{d-c}\int_c^d f(u)du - \frac{b-c-d+a}{2(b-a)}[f(b)-f(a)]\right| \le \frac{1}{4}(b-a+c-d)(\Gamma-\gamma), \tag{2.1}$$

where $a \le c < d \le b$.

Proof. Take $f(x) = K_{c,d}(x)$ and $g(x) = f'(x)$ in Lemma 1. Since $K_{c,d}(c) \le K_{c,d}(x) \le K_{c,d}(d)$ for all $x \in [a,b]$, by Lemma 1, we obtain

$$\left|\frac{1}{b-a}\int_a^b K_{c,d}(x)f'(x)dx - \frac{1}{b-a}\int_a^b K_{c,d}(t)dt\frac{1}{b-a}\int_a^b f'(x)dx\right| \le \frac{1}{4}\big(K_{c,d}(d) - Kc,d(c)\big)(\Gamma-\gamma).$$

Further, by Lemma 3, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{d-c}\int_c^d f(u)du) - \frac{1}{b-a}\int_a^b K_{c,d}(t)dt[f(b)-f(a)]\right| \le \frac{1}{4}(b-a)\big(K_{c,d}(d) - K_{c,d}(c)\big)(\Gamma-\gamma).$$

Now, since

$$\int_a^b K_{c,d}(t)dt = \frac{b-c-d+a}{2}, K_{c,d}(c) = \frac{a-c}{b-a}, K_{c,d}(d) = \frac{b-d}{b-a},$$

by the above inequality we deduce the desire inequality (2.1).

This completes the proof of Theorem 2. □

Remark 1. *The inequality* (2.1) *is a generalization of the inequality* (1.3)*. If we set $d = c+h$ with $c+h \in (a,b)$, then by* (2.1)*, we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{h}\int_c^{c+h} f(u)du - \frac{b-2c-h+a}{2(b-a)}[f(b)-f(a)]\right| \le \frac{1}{4}(b-a+h)(\Gamma-\gamma).$$

Now, letting $h \to 0^+$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(c) - \big(\frac{a+b}{2} - c\big)\frac{f(b)-f(a)}{b-a}\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma). \tag{2.2}$$

We note that the inequality (2.2) *is the Ostrowski-Grüss type inequality obtained by Dragomir and Wang in* [17]*.*

Corollary 1. *Let f, f', γ and Γ be defined as in Theorem 2 and $a+b=c+d$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{d-c}\int_c^d f(u)du\right| \le \frac{1}{2}(c-a)(\Gamma-\gamma). \tag{2.3}$$

Proof. Since $a+b=c+d$, by Theorem 2, the Corollary 1 holds immediately. □

Remark 2. *For $\Gamma\gamma > 0$, the inequality* (2.3) *is an improvement of the inequality* (1.3) *provided that $a+b=c+d$.*

For any $x \in (a,b)$ and some $\delta > 0$, let the function $F(x,\cdot) : [-\delta, \delta] \to R$ be defined by

$$F(x,t) = \frac{1}{t}\int_{x-t/2}^{x+t/2} f(u)du.$$

We obtain the following corollary.

Corollary 2. *Assume that the function $f : [a,b] \to R$ is absolutely continuous on $[a,b]$ and $\gamma < f'(t) < \Gamma$, for $t \in [a,b]$. Then the function $F(x,\cdot)$ is locally Lipschitzian and the Lipschitzian constant is $\frac{1}{4}(\Gamma-\gamma)$ and is independent of x.*

Proof. Assume that $x \in (a,b), t_1, t_2 \in [-\delta,\delta]$, with $t_2 > t_1$. For $[x-t_1/2, x+t_1/2] \subset [x-t_2, x+t_2] \subset (a,b)$, by Corollary 1, we obtain

$$\left|\frac{1}{t_2}\int_{x-t_2/2}^{x+t_2/2} f(u)du - \frac{1}{t_1}\int_{x-t_1/2}^{x+t_1/2} f(u)du\right| \le \frac{1}{4}(t_2-t_1)(\Gamma-\gamma)$$

which shows that

$$\left|F(x,t_2) - F(x,t_1)\right| \le \frac{1}{4}(t_2-t_1)(\Gamma-\gamma),$$

Similarly, for $t_1 > t_2$, we get

$$\left|F(x,t_2) - F(x,t_1)\right| \le \frac{1}{4}(t_1-t_2)(\Gamma-\gamma),$$

and then, we have

$$\left|F(x,t_2) - F(x,t_1)\right| \le \frac{1}{4}|t_2-t_1|(\Gamma-\gamma)$$

which proves the corollary. □

Remark 3. *We note that, for $\Gamma\gamma > 0$, the Corollary 2 is an improvement of the Corollary 2.4 in* [21].

Theorem 3. *Let $f : [a,b] \to R$ be an absolutely continuous function. Then we have the inequality*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{d-c}\int_c^d f(u)du - \frac{b-c-d+a}{2(b-a)}[f(b)-f(a)]\right| \\ &\le \frac{(b-a+c-d)}{2(b-a)}\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|dx,\end{aligned} \tag{2.4}$$

where $a \le c < d \le b$.

Proof. Set $f(x) = f'(x)$ and $g(x) = K_{c,d}(x)$ in Lemma 2. Since $K_{c,d}(c) \le K_{c,d}(x) \le K_{c,d}(d)$ for all $x \in [a, b]$, by Lemma 2, we obtain

$$\left|\frac{1}{b-a}\int_a^b K_{c,d}(x)f'(x)dx - \frac{1}{b-a}\int_a^b K_{c,d}(t)dt\frac{1}{b-a}\int_a^b f'(x)dx\right|$$
$$\le \frac{1}{2(b-a)}\big(K_{c,d}(d) - Kc,d(c)\big)\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|dx.$$

Further, by Lemma 3, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{d-c}\int_c^d f(u)du - \frac{1}{b-a}\int_a^b K_{c,d}(t)dt\big[f(b)-f(a)\big]\right|$$
$$\le \frac{1}{2}\big(K_{c,d}(d) - K_{c,d}(c)\big)\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|dx.$$

Now, using the fact that

$$\int_a^b K_{c,d}(t)dt = \frac{b-c-d+a}{2}, K_{c,d}(c) = \frac{a-c}{b-a}, K_{c,d}(d) = \frac{b-d}{b-a},$$

by the above inequality we get the desire inequality (2.4). This completes the proof of Theorem 3. □

Remark 4. *The inequality* (2.4) *is a generalization of the inequality* (1.3)*. If we set* $d = c + h$ *with* $c + h \in (a, b)$*, then by* (2.4)*, we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{h}\int_c^{c+h} f(u)du - \frac{b-2c-h+a}{2(b-a)}[f(b)-f(a)]\right|$$
$$\le \frac{1}{4}(b-a+h)\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|dx.$$

Now, letting $h \to 0^+$*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(c) - \big(\frac{a+b}{2} - c\big)\frac{f(b)-f(a)}{b-a}\right| \tag{2.5}$$
$$\le \frac{1}{4}(b-a)\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|dx.$$

We note that the condition imposed upon f' *in inequality* (2.5) *is weaker than the one in the inequality* (2.2) *given by Dragomir and Wang* [17].

Corollary 3. *Let* f *and* f' *be defined as in Theorem 3 and* $a + b = c + d$*. Then*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{d-c}\int_c^d f(u)du\right| \tag{2.6}$$
$$\le \frac{c-a}{b-a}\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|dx.$$

Proof. Since $b - d = c - a$, using the inequality (2.4), the inequality (2.6) holds immediately. This completes the proofs of the corollary. □

Remark 5. *We note that the condition imposed upon* f' *in Corollary 3 is weaker than the one in Corollary 1.*

3. Applications to Special Means

In the following, we shall consider logarithmic, identric and generalized logarithmic means from two positive real numbers. We take

$$L(\alpha,\beta)=\frac{\beta-\alpha}{\ln\beta-\ln\alpha},\qquad \alpha,\beta\in R^{+},\alpha\neq\beta,$$

$$I(\alpha,\beta)=\frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right)^{\frac{1}{\beta-\alpha}}\qquad \alpha,\beta\in R^{+},\alpha\neq\beta,$$

$$L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}},\qquad p\in R\setminus\{-1,0\},\alpha,\beta\in R^{+},\alpha\neq\beta,$$

where R is the field of real numbers.

Proposition 1. *Let $a,b,x,y\in R, 0<a\le c<d\le b$, $a+b=c+d$ and $p\in R\setminus\{-1,0\}$. Then we have*

$$\left|L_p^p(a,b)-L_p^p(c,d)\right|\le\frac{1}{2}(c-a)\left|p(b^{p-1}-a^{p-1})\right|. \tag{3.1}$$

Proof. The proof is immediate from Corollary 1 with $f(x)=x^p, x\in R^{+}, p\in R\setminus\{-1,0\}$. □

Proposition 2. *Suppose $a,b,x,y\in R$, and $0<a\le c<d\le b$ with $a+b=c+d$. Then we have*

$$\left|L^{-1}(a,b)-L^{-1}(c,d)\right|\le\frac{(c-a)(b^2-a^2)}{2a^2b^2}. \tag{3.2}$$

Proof. The result follows from Corollary 1 with $f(x)=\frac{1}{x}$. □

Proposition 3. *Suppose $a,b,c,d\in R$, and $0<a\le c<d\le b$ with $a+b=c+d$. Then we have*

$$\left|\ln\left[\frac{I(a,b)}{I(c,d)}\right]\right|\le\frac{(c-a)(b-a)}{2ab}. \tag{3.3}$$

Proof. The result follows from Corollary 1 with $f(x)=\ln x$. □

Remark 6. *We note that the upper bounds in the inequalities* (3.1), (3.2) *and* (3.3) *are smaller than the ones of inequalities* (4.1), (4.2) *and* (4.3) *in* [21], *respectively.*

4. Applications for PDFs

In the following, assume that $f:[a,b]\to R^{+}$ is a probability density function of a certain random variable X and $F:[a,b]\to R^{+}$, $F(t)=\int_a^t f(x)dx$ is its cumulative distribution function. The we have the following propositions.

Proposition 4. *Let f and F be as above. Then we have*

$$\left|F(t)-\frac{t-a}{b-a}+\frac{(b-t)(t-a)}{2(b-a)}\left[f(b)-f(a)\right]\right|\le\frac{(b-t)(t-a)}{4}(\Gamma-\gamma),$$

provided that $\gamma<f'(t)<\Gamma, t\in[a,b]$.

Proof. Taking $c=a$ and $d=t$ in (2.1), we have the desired inequality immediately. □

Similarly, taking $c=a$ and $d=t$ in (2.4), we have the following proposition.

Proposition 5. *Let f and F be as above. Then we have*

$$\Big|F(t)-\frac{t-a}{b-a}+\frac{(b-t)(t-a)}{2(b-a)}\big[f(b)-f(a)\big]\Big|$$
$$\leq\frac{(b-t)(t-a)}{2(b-a)}\int_a^b\Big|f'(x)-\frac{f(b)-f(a)}{b-a}\Big|dx.$$

Remark 7. *The conditions imposed upon f' in the Proposition 4 and Proposition 5 are both weaker than the one in Proposition 3.1 in* [21].

Some other inequalities for the function $F(\cdot)$ are embodied in the following propositions.

Proposition 6. *Let f and F be as above and let*

$$E_t(X)=\int_a^t uf(u)du, u\in[a,b].$$

Then we have

$$\Big|\frac{(b-E(X))(t-a)}{b-a}+E_t(X)-tF(t)-\frac{(b-t)(t-a)}{2(b-a)}\Big|\leq\frac{(b-t)(t-a)}{4}(\Gamma-\gamma)$$

provided that $\gamma<f(t)<\Gamma, t\in[a,b]$.

Proof. Taking $f=F, c=a$ and $d=t$ in (2.1), we get

$$\Big|\frac{1}{b-a}\int_a^b F(x)dx-\frac{1}{t-a}\int_a^t F(u)du-\frac{b-t}{2(b-a)}\Big|\leq\frac{(b-t)}{4}(\Gamma-\gamma). \tag{4.1}$$

Since

$$\int_a^b F(x)dx=b-E(X),$$
$$\int_a^t F(x)dx=tF(t)-\int_a^t uF(u)du=tF(t)-E_t(X),$$

thus, by (4.1), we have the desired inequality. □

Similarly, taking $f=F, c=a$ and $d=t$ in (2.4), we have the following result.

Proposition 7. *Let f, F and $E_t(x)$ be as above. Then, for $t\in[a,b]$, we have*

$$\Big|\frac{(b-E(X))(t-a)}{b-a}+E_t(X)-tF(t)-\frac{(b-t)(t-a)}{2(b-a)}\Big|$$
$$\leq\frac{(b-t)(t-a)}{2(b-a)^2}\int_a^b\big|(b-a)f(x)-1\big|dx.$$

Remark 8. *We note that the conditions imposed upon f in the Proposition 6 and Proposition 7 are both weaker than the one of the Proposition 3.2 in* [21].

Let us consider the *Beta function*

$$B(p,q):=\int_a^b t^{p-1}(1-t)^{q-1}dt,\ \ p,q>-1$$

and the *incomplete Beta function*

$$B(x;p,q):=\int_a^x t^{p-1}(1-t)^{q-1}dt, p,q>-1.$$

If we define

$$f(t) = t^{p-1}(1-t)^{q-1},$$

we get

$$f'^{p-2}(1-t)^{q-2}[p-1-(p+q-2)t].$$

It is obvious that in the case : $p > 1$ and $q > 1$, we obtain $f(x)$ is increasing on $\left[0, \frac{p-1}{p+g-2}\right]$ and decreasing on $\left[\frac{p-1}{p+g-2}, 1\right]$, and then

$$0 \le f(t) \le \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}$$

for $t \in [0,1]$; in the case : $p > 1$ and $q < 1$, $f(t)$ is increasing on $[0,1]$, and in the case : $p < 1$ and $q > 1$, $f(t)$ is decreasing on $[0,1]$.

Now, consider the random variable X having the pdf $g(t) = \frac{f(t)}{B(p,q)}, t \in (0,1)$, and for $p \neq 1, q \neq 1, p+q \neq 0$ and $p+q \neq 2$, then we have

$$\begin{aligned}
&\int_0^1 f(t)dt = B(p,q),\\
&E(X) = \frac{1}{B(p,q)} \int_0^1 t^p (1-t)^{q-1} dt = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q},\\
&E_x(X) = \frac{1}{B(p,q)} \int_0^x t^p (1-t)^{q-1} dt = \frac{B(x;p+1,q)}{B(p,q)},\\
&F(x) = \frac{1}{B(p,q)} \int_0^x t^{p-1} (1-t)^{q-1} dt = \frac{B(x;p,q)}{B(p,q)},
\end{aligned}$$

and further, for $p > 1, q < 1$, we have

$$\begin{aligned}
\int_0^1 \left|f(t) - B(p,q)\right| dt &= \int_0^{B(p,q)} \left[B(p,q) - f(t)\right] dt + \int_{B(p,q)}^1 \left[f(t) - B(p,q)\right] dt\\
&= 2B^2(p,q) - 2B(B(p,q);p,q),
\end{aligned}$$

and, for $p < 1, q > 1$, we have

$$\begin{aligned}
\int_0^1 \left|f(t) - B(p,q)\right| dt &= \int_0^{B(p,q)} \left[f(t) - B(p,q)\right] dt + \int_{B(p,q)}^1 \left[B(p,q) - f(t)\right] dt\\
&= 2B(B(p,q);p,q) - 2B^2(p,q).
\end{aligned}$$

Using the Proposition 6 and Proposition 7, we may state the following results.

Proposition 8. *Let X be a Beta random variable with $p > 1$ and $q > 1$. Then we have the inequality*

$$\begin{aligned}
&\left| \left[\frac{qx}{p+q} - \frac{(1-x)x}{2} \right] B(p,q) + B(x;p+1,q) - xB(x;p,q) \right|\\
&\le \frac{(1-x)x}{4} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}
\end{aligned}$$

for all $x \in [0,1]$.

Proposition 9. *Let X be a Beta random variable. Then, for $x \in [a,b]$, we have the inequality*

$$\left|\left[\frac{qx}{p+q} - \frac{(1-x)x}{2}\right]B(p,q) + B(x;p+1,q) - xB(x;p,q)\right|$$
$$\leq \begin{cases} (1-x)x\big[B^2(p,q) - B(B(p,q);p,q)\big], & \text{if } p>1, q<1 \\ (1-x)x\big[B(B(p,q);p,q) - B^2(p,q)\big], & \text{if } p<1, q>1. \end{cases}$$

Remark 9. *The Proposition 8 provides a new type of inequality than the Proposition 3.3 in* [21]*. We also note that the result from Proposition 9 is an extension of the result from the Proposition 3.3 in* [21] *for the case: $p > 1$ and $q < 1$, and the case: $p < 1$ and $q > 1$.*

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