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SOME OPERATOR ORDER INEQUALITIES FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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This paper was started when the first author visited the University of Adelaide in October 2011 and is dedicated to Professor C.E.M. Pearce✠ who passed away in 08th of June 2012 in a tragic accident.

Abstract. Various bounds in the operator order for the following operator transform

$$\tilde{f}(A) := \frac{1}{2}\left[f(A) - f((m+M)1_H - A)\right],$$

where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ and $f : [m,M] \to \mathbb{C}$ is a continuous function on $[m,M]$ are given. Applications for the power and logaritmic functions are provided as well.

1. Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [12, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $B(H)$. We recall that $A \geq B$ in the operator order of $B(H)$ if $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ for any $x \in H$.

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For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [12] and the references therein.

For other recent results see the research papers [2], [3], [4], [13], [14], [15], [16] and the survey papers [1], [9] and [10].

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle), \tag{1.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

2. Trapezoidal and Ostrowski Type Inequalities in the Operator Order

Utilising scalar trapezoidal type inequalities, Dragomir obtained in [8] the following results:

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$.*

1. If $f : [m, M] \to \mathbb{R}$ is continuous on $[m, M]$, then

$$\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right| \le \left[\max_{t\in[m,M]} f(t) - \min_{t\in[m,M]} f(t)\right] 1_H. \tag{2.1}$$

2. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then

$$\begin{aligned} &\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right| \\ &\le \frac{M1_H - A}{M-m}\bigvee_m^A(f) + \frac{A - m1_H}{M-m}\bigvee_A^M(f) \\ &\le \left[\frac{1}{2} + \frac{\left|A - \frac{m+M}{2}1_H\right|}{M-m}\right]\bigvee_m^M(f), \end{aligned} \tag{2.2}$$

where $\bigvee_m^A(f)$ denotes the operator generated by the scalar function $[m, M] \ni t \longmapsto \bigvee_m^t(f) \in \mathbb{R}$. The same notation applies for $\bigvee_A^M(f)$.

3. If $f:[m,M]\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$ *on* $[m,M]$ *, then*

$$\left|\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\right| \le \frac{M1_H-A}{M-m}\left|f(A)-f(m)1_H\right|+\frac{A-m1_H}{M-m}\left|f(M)1_H-f(A)\right| \le \frac{1}{2}(M-m)L1_H. \tag{2.3}$$

4. If $f:[m,M]\to\mathbb{R}$ *is continuous convex on* $[m,M]$ *with finite lateral derivatives* $f'_-(M)$ *and* $f'_+(m)$ *, then we have the inequalities:*

$$0\le\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A) \le \frac{(M1_H-A)(A-m1_H)}{M-m}\left[f'_-(M)-f'_+(m)\right] \le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H. \tag{2.4}$$

When more information is available on the derivative of the function, the following inequalities may be stated as well, see [8]:

Theorem 2. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$*. Assume that the function* $f:I\to\mathbb{C}$ *with* $[m,M]\subset\mathring{I}$ *(the interior of* I*) is differentiable on* $\mathring{I}$*.*

1. If the derivative f' *is continuous and of bounded variation on* $[m,M]$ *, then we have the inequality*

$$\left|\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\right| \le \frac{(A-m1_H)(M1_H-A)}{M-m}\bigvee_m^M(f') \le \frac{1}{4}(M-m)\bigvee_m^M(f')1_H. \tag{2.5}$$

2. If the derivative f' *is Lipschitzian with the constant* $K>0$ *on* $[m,M]$ *, then we have the inequality*

$$\left|\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\right| \le \frac{1}{2}(M-m)(A-m1_H)(M1_H-A)K \le \frac{1}{8}(M-m)^2K1_H. \tag{2.6}$$

The dual case that provides Ostrowski type inequalities in the operator order have been obtained in [7]:

Theorem 3. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$*.*

(1) *If* $f:[m,M]\to\mathbb{C}$ *is a continuous function of bounded variation on* $[m,M]$, *then we have the inequality*

$$\left|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H\right|\le\left[\frac{1}{2}1_H+\left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\bigvee_m^M(f) \tag{2.7}$$

where $\bigvee_m^M(f)$ *denotes the total variation of* f *on* $[m,M]$. *The constant* $\frac{1}{2}$ *is best possible in (2.7).*

(2) *If* $f:[m,M]\to\mathbb{R}$ *is an absolutely continuous function such that there exists the real constants* γ *and* $\Gamma,\gamma<\Gamma$ *with the property that* $\gamma\le f'(s)\le\Gamma$ *for almost every* $s\in[m,M]$, *then we have the following double inequality in the operator order of* $B(H)$:

$$\begin{aligned}&-\frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m}\left[\left(A-\frac{M\Gamma-m\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2-\Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right]\\&\le f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H\\&\le\frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m}\left[\left(A-\frac{m\Gamma-M\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2-\Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right].\end{aligned} \tag{2.8}$$

(3) *If* $f:[m,M]\to\mathbb{C}$ *is an absolutely continuous function, then we have in the operator order the following inequalities*

$$\left|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H\right| \tag{2.9}$$

$$\le\begin{cases}\left[\frac{1}{4}1_H+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right](M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M];\\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{A-m1_H}{M-m}\right)^{p+1}+\left(\frac{M1_H-A}{M-m}\right)^{p+1}\right](M-m)^{\frac{1}{q}}\|f'\|_q \\ \qquad\qquad \text{if } f'\in L_p[m,M],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\[2ex] \left[\frac{1}{2}1_H+\left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\|f'\|_1.\end{cases}$$

Motivated by the above results we investigate in this paper the problem of bounding in the operator order the following operator transform

$$\tilde{f}(A):=\frac{1}{2}\left[f(A)-f((m+M)1_H-A)\right]$$

where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ and $f:[m,M]\to\mathbb{C}$ is a continuous function on $[m,M]$. Some applications for power and logarithmic functions are provided as well.

The same notation

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(m+M-t)\right]$$

can be used for the scalar function $f:[m,M]\to\mathbb{C}$ and could be seen as a "*measure of asymmetry*" for f.

3. Some Immediate Bounds for $\tilde{f}(A)$

The following result is a natural consequence of Theorem 1:

Theorem 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$.*

1. If $f : [m, M] \to \mathbb{R}$ is continuous on $[m, M]$, then

$$\left|\frac{f(M) - f(m)}{M - m}\left(A - \frac{m + M}{2}1_H\right) - \tilde{f}(A)\right| \le \left[\max_{t\in[m,M]} \tilde{f}(t) - \min_{t\in[m,M]} \tilde{f}(t)\right] 1_H \le \left[\max_{t\in[m,M]} f(t) - \min_{t\in[m,M]} f(t)\right] 1_H. \tag{3.1}$$

2. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then

$$\left|\frac{f(M) - f(m)}{M - m}\left(A - \frac{m + M}{2}1_H\right) - \tilde{f}(A)\right| \le \left[\frac{1}{2} + \frac{\left|A - \frac{m+M}{2}1_H\right|}{M - m}\right] \bigvee_m^M \left(\tilde{f}\right) \le \left[\frac{1}{2} + \frac{\left|A - \frac{m+M}{2}1_H\right|}{M - m}\right] \bigvee_m^M (f). \tag{3.2}$$

3. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\left|\frac{f(M) - f(m)}{M - m}\left(A - \frac{m + M}{2}1_H\right) - \tilde{f}(A)\right| \le \frac{1}{2}(M - m) L 1_H. \tag{3.3}$$

Proof. If we write the inequality (2.1) for $\tilde{f}$ we have

$$\left|\frac{\tilde{f}(m)(M1_H - A) + \tilde{f}(M)(A - m1_H)}{M - m} - \tilde{f}(A)\right| \le \left[\max_{t\in[m,M]} \tilde{f}(t) - \min_{t\in[m,M]} \tilde{f}(t)\right] 1_H. \tag{3.4}$$

Since

$$\tilde{f}(M) = -\tilde{f}(m) = \frac{f(M) - f(m)}{2},$$

then

$$\frac{\tilde{f}(m)(M1_H - A) + \tilde{f}(M)(A - m1_H)}{M - m} = \frac{f(M) - f(m)}{M - m}\left(A - \frac{m + M}{2}1_H\right)$$

and by (3.4) we deduce the first inequality in (3.1).

If we denote $\delta = \min_{t\in[m,M]} f(t)$ and $\Delta = \max_{t\in[m,M]} f(t)$ then $\delta \le f(t) \le \Delta$ and $-\Delta \le -f(m + M - t) \le -\delta$ which gives that

$$-\frac{1}{2}(\Delta - \delta) \le \tilde{f}(t) \le \frac{1}{2}(\Delta - \delta)$$

therefore

$$\max_{t\in[m,M]} \tilde{f}(t) \le \frac{1}{2}(\Delta-\delta) \text{ and } -\frac{1}{2}(\Delta-\delta) \le \min_{t\in[m,M]} \tilde{f}(t)$$

which implies that

$$\max_{t\in[m,M]} \tilde{f}(t) - \min_{t\in[m,M]} \tilde{f}(t) \le \Delta-\delta$$

and the second inequality in (3.1) is proved.

The first inequality in (3.2) follows from (2.2).

If f is of bounded variation, then obviously $\tilde{f}$ is of bounded variation and

$$\bigvee_m^M \left(\tilde{f}\right) \le \frac{1}{2}\left[\bigvee_m^M (f) + \bigvee_m^M (f(m+M-\cdot))\right] = \bigvee_m^M (f).$$

This proves the last part of (3.2).

Now, if f is Lipschitzian with the constant $L>0$ then $\tilde{f}$ is also Lipschitzian with at least the same constant L and by (2.3) we deduce the desired result (3.3). □

We need the following notation

$$\widehat{g}(s) := \frac{1}{2}[g(s)+g(m+M-s)], s\in[m,M]$$

where $g:[m,M]\to\mathbb{C}$.

Theorem 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$.*

(1) *If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality*

$$\left|\tilde{f}(A)\right| \le \left[\frac{1}{2}1_H + \left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\bigvee_m^M\left(\tilde{f}\right) \le \left[\frac{1}{2}1_H + \left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\bigvee_m^M(f). \tag{3.5}$$

(2) *If $f:[m,M]\to\mathbb{R}$ is an absolutely continuous function such that there exists the real constants γ and $\Gamma, \gamma<\Gamma$ with the property that $\gamma\le f'(s)\le\Gamma$ for almost every $s\in[m,M]$, then we have the following double inequality in the operator order of $B(H)$:*

$$\begin{aligned} &-\frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m}\left[\left(A-\frac{M\Gamma-m\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2-\Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right] \\ &\le \tilde{f}(A) \\ &\le \frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m}\left[\left(A-\frac{m\Gamma-M\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2-\Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right]. \end{aligned} \tag{3.6}$$

(3) *If* $f:[m,M]\to\mathbb{C}$ *is an absolutely continuous function, then we have in the operator order the following inequalities*

$$\left|\tilde{f}(A)\right| \tag{3.7}$$
$$\leq\begin{cases}\left[\frac{1}{4}1_H+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right](M-m)\left\|\widehat{(f')}\right\|_\infty & \text{if } f'\in L_\infty[m,M];\\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{A-m1_H}{M-m}\right)^{p+1}+\left(\frac{M1_H-A}{M-m}\right)^{p+1}\right](M-m)^{\frac{1}{q}}\left\|\widehat{(f')}\right\|_q & \\ \qquad\text{if } f'\in L_p[m,M],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; & \\ \left[\frac{1}{2}1_H+\left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\left\|\widehat{(f')}\right\|_1 & \end{cases}$$
$$\leq\begin{cases}\left[\frac{1}{4}1_H+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right](M-m)\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[m,M];\\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{A-m1_H}{M-m}\right)^{p+1}+\left(\frac{M1_H-A}{M-m}\right)^{p+1}\right](M-m)^{\frac{1}{q}}\left\|f'\right\|_q & \\ \qquad\text{if } f'\in L_p[m,M],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; & \\ \left[\frac{1}{2}1_H+\left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\left\|f'\right\|_1. & \end{cases}$$

Proof. Follows by Theorem 3 applied for $\tilde{f}$ and observing that

$$\frac{1}{M-m}\int_m^M\tilde{f}(t)\,dt=0$$

and the fact that if $\gamma\leq f'(s)\leq\Gamma$ for almost every $s\in[m,M]$, then

$$\begin{aligned}\left(\tilde{f}(s)\right)' &= \frac{1}{2}\left[f(s)-f(m+M-s)\right]'\\ &= \frac{1}{2}\left[f'(s)+f'(m+M-s)\right]\\ &= \widehat{(f')}(s)\in[\gamma,\Gamma]\end{aligned}$$

for almost every $s\in[m,M]$, where we have used the notation

$$\widehat{g}(s):=\frac{1}{2}\left[g(s)+g(m+M-s)\right],s\in[m,M].$$

The last part from (3.7) follows from the fact that

$$\begin{aligned}\|\widehat{g}\|_q &\leq \frac{1}{2}\left[\|g\|_q+\|g(m+M-\cdot)\|_q\right]\\ &= \|g\|_q\end{aligned}$$

for any $q\in[1,\infty]$. □

Finally, we can state the following result as well:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. Assume that the function $f:I\to\mathbb{C}$ with $[m,M]\subset\mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

1. If the derivative f' is continuous and of bounded variation on $[m,M]$, then we have the inequality

$$\left|\frac{f(M)-f(m)}{M-m}\left(A-\frac{m+M}{2}1_H\right)-\tilde{f}(A)\right| \le \frac{(A-m1_H)(M1_H-A)}{M-m}\bigvee_m^M\left(\widehat{(f')}\right) \le \frac{1}{4}(M-m)\bigvee_m^M\left(\widehat{(f')}\right)1_H. \tag{3.8}$$

2. If the derivative f' is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$\left|\frac{f(M)-f(m)}{M-m}\left(A-\frac{m+M}{2}1_H\right)-\tilde{f}(A)\right| \le \frac{1}{2}(M-m)(A-m1_H)(M1_H-A)K \le \frac{1}{8}(M-m)^2K1_H. \tag{3.9}$$

This is a direct consequence of Theorem 2 and the details are omitted.

4. Other Bounds

The following simple bounds for the operator $\left|\tilde{f}(A)\right|$ hold:

Theorem 7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$.*

(1) *If the function $f:[m,M]\to\mathbb{C}$ is continuous, then*

$$\left|\tilde{f}(A)\right| \le \frac{1}{2}\left[\max_{t\in[m,M]}f(t)-\min_{t\in[m,M]}f(t)\right]1_H. \tag{4.1}$$

(2) *If the function $f:[m,M]\to\mathbb{C}$ is continuous and of bounded variation, then*

$$\left|\tilde{f}(A)\right| \le \frac{1}{2}\bigvee_m^M\left(\tilde{f}\right)1_H \le \frac{1}{2}\bigvee_m^M(f)\,1_H. \tag{4.2}$$

(3) *If the function $f:[m,M]\to\mathbb{C}$ is $r-H$-Hölder continuous, i.e. for fixed $r\in(0,1]$ and $H>0$ we have*

$$|f(t)-f(s)|\le|t-s| \text{ for any } t,s\in[m,M],$$

then

$$\left|\tilde{f}(A)\right| \le \frac{1}{2^{1-r}}H\left|A-\frac{m+M}{2}1_H\right|^r. \tag{4.3}$$

(4) *If the function $f:[m,M]\to\mathbb{C}$ is absolutely continuous on $[m,M]$, then*

$$\left|\tilde{f}(A)\right| \le \begin{cases} \left|A-\frac{m+M}{2}1_H\right|\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \frac{1}{2^{1-1/q}}\left|A-\frac{m+M}{2}1_H\right|^{1/q}\left\|f'\right\|_p & \text{if } f'\in L_\infty[m,M], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1. \end{cases} \tag{4.4}$$

Proof. 1. As above, if we denote $\delta=\min_{t\in[m,M]}f(t)$ and $\Delta=\max_{t\in[m,M]}f(t)$ then $\delta\le f(t)\le\Delta$ and $-\Delta\le -f(m+M-t)\le-\delta$ which gives that

$$\left|\tilde{f}(t)\right|\le\frac{1}{2}(\Delta-\delta)$$

for any $t\in[m,M]$.

Applying the property (P) we deduce the desired result.

2. Since $\tilde{f}(M)=-\tilde{f}(m)$, then we have

$$\begin{aligned}\left|\tilde{f}(t)\right| &= \left|\tilde{f}(t)-\frac{\tilde{f}(M)+\tilde{f}(m)}{2}\right| \\ &\le \frac{\left|\tilde{f}(t)-\tilde{f}(m)\right|+\left|\tilde{f}(M)-\tilde{f}(t)\right|}{2}\le\frac{1}{2}\bigvee_m^M\left(\tilde{f}\right),\end{aligned}$$

for any $t\in[m,M]$.

Applying the property (P) we deduce the first inequality in (4.2). The second part was proven before.

3. Utilising the definition, we have

$$\begin{aligned}\left|\tilde{f}(t)\right| &= \frac{1}{2}\left|f(t)-f(m+M-t)\right| \\ &\le \frac{1}{2}H\left|2t-(m+M)\right|^r=\frac{1}{2^{1-r}}H\left|t-\frac{m+M}{2}\right|^r\end{aligned}$$

for any $t\in[m,M]$.

Applying the property (P) we deduce the desired inequality in (4.3).

4. Since f is absolutely continuous on $[m,M]$, then

$$\left|\tilde{f}(t)\right|=\frac{1}{2}\left|f(t)-f(m+M-t)\right|=\frac{1}{2}\left|\int_t^{m+M-t}f'(s)\,ds\right|$$

for any $t\in[m,M]$.

Utilising the integral Hölder's inequality we have

$$\begin{aligned}\left|\tilde{f}(t)\right| &\le \frac{1}{2}\left|\int_t^{m+M-t}\left|f'(s)\right|ds\right| \\ &\le \frac{1}{2}\times\begin{cases}\left|2t-(m+M)\right|\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \\ \left|2t-(m+M)\right|^{1/q}\left\|f'\right\|_p & \begin{array}{c}\text{if } f'\in L_\infty[m,M], \\ p>1,\frac{1}{p}+\frac{1}{q}=1.\end{array}\end{cases} \\ &= \begin{cases}\left|t-\frac{m+M}{2}\right|\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \\ \frac{1}{2^{1-1/q}}\left|t-\frac{m+M}{2}\right|^{1/q}\left\|f'\right\|_p & \begin{array}{c}\text{if } f'\in L_\infty[m,M], \\ p>1,\frac{1}{p}+\frac{1}{q}=1.\end{array}\end{cases}\end{aligned}$$

for any $t\in[m,M]$.

Applying the property (P) we deduce the desired inequality in (4.4). □

The following result the provided upper and lower bounds for $\tilde{f}(A)$ in the operator order of $B(H)$ also holds:

Theorem 8. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$. If the derivative f' is continuous and convex on $[m, M]$ then*

$$\begin{aligned} &\frac{1}{2}[f(M) - f(m)]1_H - \frac{f'(m) + f'(M)}{2}(M1_H - A) \\ &\leq \tilde{f}(A) \\ &\leq \frac{1}{2}[f(M) - f(m)]1_H - f'\left(\frac{m+M}{2}\right)(M1_H - A) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &f'\left(\frac{m+M}{2}\right)(A - m1_H) - \frac{1}{2}[f(M) - f(m)]1_H \\ &\leq \tilde{f}(A) \\ &\leq \frac{f'(m) + f'(M)}{2}(A - m1_H) - \frac{1}{2}[f(M) - f(m)]1_H. \end{aligned} \tag{4.6}$$

We also have the inequality

$$\begin{aligned} &\frac{1}{2}\left[f'\left(\frac{m+M}{2}\right)(A - m1_H) - \frac{f'(m) + f'(M)}{2}(M1_H - A)\right] \\ &\leq \tilde{f}(A) \\ &\leq \frac{1}{2}\left[\frac{f'(m) + f'(M)}{2}(A - m1_H) - f'\left(\frac{m+M}{2}\right)(M1_H - A)\right]. \end{aligned} \tag{4.7}$$

Proof. Let $\{E_\lambda\}_\lambda$ be the *spectral family* of the operator A. For $x \in H, \|x\| = 1$, consider the function $g : [m, M] \to \mathbb{R}$,

$$g(\lambda) := \left\langle \frac{1}{2}(E_\lambda + E_{m+M-\lambda})x, x \right\rangle.$$

Then $g(\lambda) = g(m + M - \lambda)$ for any $\lambda \in [m, M]$, i.e., g is symmetrical on $[m, M]$ and $g(\lambda) \geq 0$ for any $\lambda \in [m, M]$.

By the spectral representation (1.1) we also have that

$$\begin{aligned} \int_{m-0}^{M} g(\lambda)\,d\lambda &= \int_{m-0}^{M} \left\langle \frac{1}{2}(E_\lambda + E_{m+M-\lambda})x, x \right\rangle d\lambda \\ &= \int_{m-0}^{M} \langle E_\lambda x, x\rangle\,d\lambda \\ &= \langle E_\lambda x, x\rangle \lambda\Big|_{m-0}^{M} - \int_{m-0}^{M} \lambda d\langle E_\lambda x, x\rangle \\ &= \langle (M1_H - A)x, x\rangle \end{aligned}$$

for any $x \in H, \|x\| = 1$.

We use Fejér's inequality, see for instance [11, pp. 1-2], which says that if $h : [a, b] \to \mathbb{R}$ is convex and g is symmetrical on $[a, b]$ and nonnegative, then

$$h\left(\frac{a+b}{2}\right)\int_a^b g(\lambda)\,d\lambda \leq \int_a^b h(\lambda)g(\lambda)\,d\lambda \leq \frac{h(a) + h(b)}{2}\int_a^b g(\lambda)\,d\lambda.$$

By writing this inequality for $h = f'$, we can state that

$$f'\left(\frac{m+M}{2}\right)\int_{m-0}^{M} g(\lambda)\,d\lambda \leq \int_{m-0}^{M} f'(\lambda)\,g(\lambda)\,d\lambda \leq \frac{f'(m)+f'(M)}{2}\int_{m-0}^{M} g(\lambda)\,d\lambda, \tag{4.8}$$

for any $x \in H, \|x\| = 1$.

Integrating by parts, we observe that

$$\begin{aligned} I &:= \int_{m-0}^{M} f'(\lambda)\,g(\lambda)\,d\lambda = f(\lambda)\,g(\lambda)\big|_{m-0}^{M} - \int_{m-0}^{M} f(\lambda)\,dg(\lambda) \\ &= \frac{1}{2}\left[f(M) - f(m)\right] \\ &- \frac{1}{2}\left[\int_{m-0}^{M} f(\lambda)\,d(\langle E_\lambda x, x\rangle) + \int_{m-0}^{M} f(\lambda)\,d(\langle E_{m+M-\lambda} x, x\rangle)\right]. \end{aligned} \tag{4.9}$$

Utilising the change of variable $t = m + M - \lambda$ and the spectral representation (1.1), we get that

$$\int_{m-0}^{M} f(\lambda)\,d(\langle E_{m+M-\lambda} x, x\rangle) = -\langle f((m+M)1_H - A)\,x, x\rangle$$

for any $x \in H, \|x\| = 1$ and since

$$\int_{m-0}^{M} f(\lambda)\,d(\langle E_\lambda x, x\rangle) = \langle f(A)\,x, x\rangle$$

for any $x \in H, \|x\| = 1$, then by (4.9) we obtain

$$I = \frac{1}{2}\left[f(M) - f(m)\right] - \frac{1}{2}\langle\left[f(A) - f((m+M)1_H - A)\right]x, x\rangle,$$

for any $x \in H, \|x\| = 1$.

On making use of the inequality (4.8) we can state that

$$\begin{aligned} &f'\left(\frac{m+M}{2}\right)\langle(M1_H - A)\,x, x\rangle \\ &\leq \frac{1}{2}\left[f(M) - f(m)\right] - \frac{1}{2}\langle\left[f(A) - f((m+M)1_H - A)\right]x, x\rangle \\ &\leq \frac{f'(m)+f'(M)}{2}\langle(M1_H - A)\,x, x\rangle, \end{aligned}$$

for any $x \in H, \|x\| = 1$, which is equivalent with (4.5).

Now, if we replace in the inequality (4.5) the operator A with the operator $(m+M)1_H - A$, then we get the inequality

$$\begin{aligned} &f'\left(\frac{m+M}{2}\right)(A - m1_H) \\ &\leq \frac{1}{2}\left[f(M) - f(m)\right]1_H + \frac{1}{2}\left[f(A) - f((m+M)1_H - A)\right] \\ &\leq \frac{f'(m)+f'(M)}{2}(A - m1_H), \end{aligned}$$

which is equivalent with (4.6).

Finally, we observe that the inequality (4.7) is obtained by adding the inequalities (4.5) with (4.6). □

The following result may be stated as well:

Theorem 9. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

(1) *If $|f'|$ is convex on $[m, M]$, then*

$$\left|\tilde{f}(A)\right| \leq (M-m)\left[\frac{|f'(m)| + |f'(M)|}{2}\right]\left[\frac{1}{4} + \left(\frac{A - \frac{m+M}{2}1_H}{M-m}\right)^2\right]. \tag{4.10}$$

(2) *If $|f'|$ is concave on $[m, M]$, then*

$$\left|\tilde{f}(A)\right| \leq (M-m)\left|f'\left(\frac{m+M}{2}\right)\right|\left[\frac{1}{4} + \left(\frac{A - \frac{m+M}{2}1_H}{M-m}\right)^2\right]. \tag{4.11}$$

(3) *If $|f'|$ is quasiconvex on $[m, M]$, then*

$$\left|\tilde{f}(A)\right| \leq (M-m)\max\{|f'(m)|, |f'(M)|\}\left[\frac{1}{4} + \left(\frac{A - \frac{m+M}{2}1_H}{M-m}\right)^2\right]. \tag{4.12}$$

Proof. Integrating by parts in the Riemann integral, we get the following representation:

$$\begin{aligned} \tilde{f}(t) &= \tilde{f}(t) - \frac{1}{M-m}\int_m^M \tilde{f}(s)\,ds \\ &= \frac{1}{M-m}\int_m^t (s-m)\left(\tilde{f}(s)\right)' ds + \frac{1}{M-m}\int_t^M (s-M)\left(\tilde{f}(s)\right)' ds \\ &= \frac{1}{M-m}\int_m^t (s-m)\widehat{(f')}(s)\,ds + \frac{1}{M-m}\int_t^M (s-M)\widehat{(f')}(s)\,ds \end{aligned} \tag{4.13}$$

for any $t \in [m, M]$.

Taking the modulus in (4.14) we get

$$\begin{aligned} &\left|\tilde{f}(t)\right| \\ &\leq \frac{1}{M-m}\int_m^t (s-m)\left|\widehat{(f')}(s)\right| ds + \frac{1}{M-m}\int_t^M (M-s)\left|\widehat{(f')}(s)\right| ds \\ &\leq \frac{1}{M-m}\int_m^t (s-m)\widehat{|f'|}(s)\,ds + \frac{1}{M-m}\int_t^M (M-s)\widehat{|f'|}(s)\,ds \end{aligned} \tag{4.14}$$

for any $t \in [m, M]$.

1. If $|f'|$ is convex on $[m, M]$, then

$$|f'(s)| \leq \frac{(s-m)|f'(M)| + (M-s)|f'(m)|}{M-m}$$

and

$$|f'(m+M-s)| \leq \frac{(s-m)|f'(m)| + (M-s)|f'(M)|}{M-m}$$

for any $s \in [m, M]$.

If we add the above two inequalities and divide by 2, then we get

$$\widehat{|f'|}(s) \le \frac{1}{2}\left[|f'(m)| + |f'(M)|\right] \tag{4.15}$$

for any $s \in [m, M]$.

On making use of (4.14) and (4.15) we deduce

$$\begin{aligned} \left|\tilde{f}(t)\right| &\le \frac{1}{2}\frac{1}{M-m}\left[|f'(m)| + |f'(M)|\right] \\ &\quad\times \left[\int_m^t (s-m)\,ds + \int_t^M (M-s)\,ds\right] \\ &= \frac{1}{2}\frac{1}{M-m}\left[|f'(m)| + |f'(M)|\right] \\ &\quad\times \left[\frac{(t-m)^2 + (M-t)^2}{2}\right] \\ &= \frac{1}{2}\left[|f'(m)| + |f'(M)|\right]\left[\frac{1}{4} + \left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m) \end{aligned} \tag{4.16}$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the desired inequality in (4.10).

2. If $|f'|$ is concave on $[m, M]$, then

$$\widehat{|f'|}(s) = \frac{1}{2}\left[|f'(s)| + |f'(m+M-s)|\right] \le \left|f'\left(\frac{m+M}{2}\right)\right|$$

for any $s \in [m, M]$ and by (4.14) we deduce

$$\begin{aligned} \left|\tilde{f}(t)\right| &\le \frac{1}{M-m}\left|f'\left(\frac{m+M}{2}\right)\right| \\ &\quad\times \left[\int_m^t (s-m)\,ds + \int_t^M (M-s)\,ds\right] \\ &= \left|f'\left(\frac{m+M}{2}\right)\right|\left[\frac{1}{4} + \left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m) \end{aligned} \tag{4.17}$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the desired inequality in (4.11).

3. If $|f'|$ is quasiconvex on $[m, M]$, then

$$\widehat{|f'|}(s) = \frac{1}{2}\left[|f'(s)| + |f'(m+M-s)|\right] \le \max\left\{|f'(m)|, |f'(M)|\right\}$$

for any $s \in [m, M]$ from where we similarly get the desired result (4.12). □

5. Applications

Consider the function $f : [m, M] \to \mathbb{R}$ with $[m, M] \subset (0, \infty)$ given by $f(t) = \ln t$. Then $f'(t) = \frac{1}{t}$ is convex and on making use of Theorem 8 we get for any A a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ that

$$\ln\sqrt{\frac{M}{m}}1_H-\frac{m+M}{2mM}\left(M1_H-A\right)\tag{5.1}$$
$$\le\frac{1}{2}\left\{\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right\}$$
$$\le\ln\sqrt{\frac{M}{m}}1_H-\frac{2}{m+M}\left(M1_H-A\right)$$

and

$$\frac{2}{m+M}\left(A-m1_H\right)-\ln\sqrt{\frac{M}{m}}1_H\tag{5.2}$$
$$\le\frac{1}{2}\left\{\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right\}$$
$$\le\frac{m+M}{2mM}\left(A-m1_H\right)-\ln\sqrt{\frac{M}{m}}1_H.$$

We also have the inequality

$$\frac{1}{2}\left[\frac{2}{m+M}\left(A-m1_H\right)-\frac{m+M}{2mM}\left(M1_H-A\right)\right]\tag{5.3}$$
$$\le\frac{1}{2}\left\{\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right\}$$
$$\le\frac{1}{2}\left[\frac{m+M}{2mM}\left(A-m1_H\right)-\frac{2}{m+M}\left(M1_H-A\right)\right].$$

Now, if we use the first statement in Theorem 9, then we get

$$\frac{1}{2}\left|\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right|\tag{5.4}$$
$$\le\left(M-m\right)\frac{m+M}{2mM}\left[\frac{1}{4}+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right].$$

Further, if we consider the power function $f:[m,M]\subset(0,\infty)\to\mathbb{R}$, $f\left(t\right)=t^p, p>0$ then $f'\left(t\right)=pt^{p-1}$and for $p\ge 2$ we have that f' is convex and by Theorem 8 we have for any A a selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right)\subseteq[m,M]$ that

$$\frac{1}{2}\left(M^p-m^p\right)1_H-p\frac{m^{p-1}+M^{p-1}}{2}\left(M1_H-A\right)\tag{5.5}$$
$$\le\frac{1}{2}\left[A^p-\left(\left(m+M\right)1_H-A\right)^p\right]$$
$$\le\frac{1}{2}\left(M^p-m^p\right)1_H-p\left(\frac{m+M}{2}\right)^{p-1}\left(M1_H-A\right)$$

and

$$p\left(\frac{m+M}{2}\right)^{p-1}(A-m1_H)-\frac{1}{2}(M^p-m^p)1_H \tag{5.6}$$
$$\leq \frac{1}{2}\left[A^p-((m+M)1_H-A)^p\right]$$
$$\leq p\frac{m^{p-1}+M^{p-1}}{2}(A-m1_H)-\frac{1}{2}(M^p-m^p)1_H.$$

We also have the inequality

$$\frac{1}{2}p\left[\left(\frac{m+M}{2}\right)^{p-1}(A-m1_H)-\frac{m^{p-1}+M^{p-1}}{2}(M1_H-A)\right] \tag{5.7}$$
$$\leq \frac{1}{2}\left[A^p-((m+M)1_H-A)^p\right]$$
$$\leq \frac{1}{2}p\left[\frac{m^{p-1}+M^{p-1}}{2}(A-m1_H)-\left(\frac{m+M}{2}\right)^{p-1}(M1_H-A)\right].$$

Now, if we apply the first statement from Theorem 9, then we get for $p\geq 2$ that

$$\frac{1}{2}\left|A^p-((m+M)1_H-A)^p\right| \tag{5.8}$$
$$\leq p(M-m)\frac{m^{p-1}+M^{p-1}}{2}\left[\frac{1}{4}+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right].$$

By the second statement of the same theorem we also have for $1\leq p<2$ that

$$\frac{1}{2}\left|A^p-((m+M)1_H-A)^p\right| \tag{5.9}$$
$$\leq p(M-m)\left(\frac{m+M}{2}\right)^{p-1}\left[\frac{1}{4}+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right].$$

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