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# GENERALIZATION OF SOME INEQUALITIES VIA RIEMANN-LIOUVILLE FRACTIONAL CALCULUS

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ABSTRACT. Some Hermite-Hadamard type inequalities are provided. We deal with functions whose derivatives in absolute value are convex or concave. By defining two cumulative gaps which enable us to generalize known rezults in the framework of Riemann-Liouville fractional calculus, we open a new perspective on the classic statement of the inequality.

## 1. INTRODUCTION

The Hermite-Hadamard inequality states that if a function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is convex, then one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. Both inequalities hold in reversed direction if $f$ is concave.

A large variety of generalizations of Hermite-Hadamard inequality involving convex functions have been found; see, for instance, [1], [2], [4], [6], [8], [10] and the references therein.

The purpose of our paper is to establish, via the Riemann-Liouville fractional calculus, some generalized Hermite-Hadamard type inequalities, for functions whose derivatives in absolute value are convex or concave.

Let $f \in L^1[a, b]$, where $a \geq 0$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$, of order $\alpha > 0$, are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \text{ for } x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t, \text{ for } x < b,$$

respectively. Here, $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\mathrm{d}t$ is the Gamma function. We also make the convention

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

For details about the Riemann-Liouville fractional integrals see [3].

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## 2. Main results

We assume throughout the present paper that $[a,b]$ is a subinterval of $[0,\infty)$ and $f:[a,b]\longrightarrow\mathbb{R}$ is a differentiable function on $(a,b)$ such that $f'\in L^1[a,b]$ and $n$ is an odd number. Before stating the results we establish the notation.

We define the *cumulative to the left* $(\alpha,n)-$*gap* by

$$\mathcal{L}_{\alpha,n}(a,b)=\sum_{k=0}^{(n-1)/2}2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)$$
$$-\Gamma(\alpha+1)\left(\frac{n+1}{b-a}\right)^{\alpha}\sum_{k=0}^{(n-1)/2}\left[J^{\alpha}_{\frac{a(n-2k)+b(2k+1)}{n+1}-}f\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right.$$
$$\left.+J^{\alpha}_{\frac{a(n-2k-1)+b(2k+2)}{n+1}-}f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right].$$

**Remark 1.** *The particular case $\alpha=1$ and $n=3$ gives*

$$\frac{\mathcal{L}_{1,3}(a,b)}{4}=\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t.$$

*The right hand side term has its origins in the inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\geq\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right].$$

*See* [7], *p. 52.*

In order to prove our main results we need the following lemma.

**Lemma 1.** *It holds*

$$\mathcal{L}_{\alpha,n}(a,b)=\frac{b-a}{n+1}\times$$
$$\sum_{k=0}^{(n-1)/2}\left[\int_0^1 t^{\alpha}f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\mathrm{d}t\right.$$
$$\left.+\int_0^1(t^{\alpha}-1)f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\mathrm{d}t\right].$$

*Proof.* Firstly, we denote

$$I_{1k}=\int_0^1 t^{\alpha}f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\mathrm{d}t,$$

and

$$I_{2k}=\int_0^1(t^{\alpha}-1)f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\mathrm{d}t.$$

We use the integration by parts and the substitutions $u = t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}$, $v = t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}$ to show that

$$\begin{aligned} I_{1k} + I_{2k} = \frac{2(n+1)}{b-a} f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \Gamma(\alpha+1)\cdot\left(\frac{n+1}{b-a}\right)^{\alpha+1} \\ \times \left[ J^{\alpha}_{\frac{a(n-2k)+b(2k+1)}{n+1}-} f\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right. \\ \left. + J^{\alpha}_{\frac{a(n-2k-1)+b(2k+2)}{n+1}-} f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right] \end{aligned}$$

The proof is completed. □

We are now in a position to state and prove the following:

**Theorem 1.** *Assume* $|f'|$ *is convex on* $[a, b]$*. Then*

$$\begin{aligned} |\mathcal{L}_{\alpha,n}(a,b)| \le \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left[ \frac{\alpha^2+5\alpha+2}{2(\alpha+1)(\alpha+2)} \left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right| \right. \\ + \frac{1}{(\alpha+1)(\alpha+2)} \left| f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right| \\ \left. + \frac{\alpha}{2(\alpha+2)} \left| f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right| \right]. \end{aligned}$$

*Proof.* Using Lemma 1 and taking modulus, we infer from the convexity of $|f'|$ that

$$\begin{aligned} |\mathcal{L}_{\alpha,n}(a,b)| \le \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left[ \left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right| \int_0^1 t^{\alpha+1}\mathrm{d}t \right. \\ + \left| f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right| \int_0^1 t^{\alpha}(1-t)\mathrm{d}t \\ + \left| f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right| \int_0^1 t(1-t^{\alpha})\mathrm{d}t \\ \left. + \left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right| \int_0^1 (1-t^{\alpha})(1-t)\mathrm{d}t \right]. \end{aligned}$$

The result follows after a straightforward computation in the right hand side term. This ends the proof. □

We recall that the Beta function (the Euler integral of the first kind), is defined by

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

for $x, y > 0$.
Our next result reads as:

**Theorem 2.** *Assume $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$. Then*

$$|\mathcal{L}_{\alpha,n}(a,b)| \leq \frac{b-a}{2^{\frac{1}{q}}(n+1)} \sum_{k=0}^{(n-1)/2} \left\{ \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \times \right.$$
$$\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\right]^{\frac{1}{q}}$$
$$+\left[\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right]^{\frac{1}{p}} \times$$
$$\left.\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\right\}.$$

*for* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* According to Lemma 1 and Hölder's inequality, we have

$$|\mathcal{L}_{\alpha,n}(a,b)| \leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left[\left(\int_0^1 (t^\alpha)^p \,\mathrm{d}t\right)^{\frac{1}{p}} \times \right.$$
$$\left(\int_0^1 \left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t^\alpha)^p \,\mathrm{d}t\right)^{\frac{1}{p}} \times$$
$$\left.\left(\int_0^1 \left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\right].$$

Since $|f|^q$ is convex on $[a,b]$, we have:

$$\int_0^1 \left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t$$
$$\leq \frac{1}{2}\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\right]$$

and

$$\int_0^1 \left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t$$
$$\leq \frac{1}{2}\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right].$$

A simple computation shows that $\int_0^1 t^{\alpha p}\mathrm{d}t = \frac{1}{\alpha p+1}$, $\int_0^1 (1-t^\alpha)^p \,\mathrm{d}t = \frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)$ and the proof is complete. □

**Theorem 3.** *Assume $|f'|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$. Then the following inequality holds:*

$$\left|\mathcal{L}_{\alpha,n}(a,b)\right| \leq \frac{b-a}{(n+1)(\alpha+1)^{\frac{1}{p}}(\alpha+2)^{\frac{1}{q}}} \times$$
$$\sum_{k=0}^{(n-1)/2}\left\{\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q + \frac{1}{\alpha+1}\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \frac{\alpha}{2^{\frac{1}{q}}}\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \frac{\alpha+3}{\alpha+1}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\right\},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Using Lemma 1 and the power mean inequality, we have

$$\left|\mathcal{L}_{\alpha,n}(a,b)\right| \leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{\frac{1}{p}} \times\right.$$
$$\left(\int_0^1 t^\alpha\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t^\alpha)\,\mathrm{d}t\right)^{\frac{1}{p}} \times$$
$$\left.\left(\int_0^1 (1-t^\alpha)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\right].$$

Since $|f|^q$ is convex on $[a,b]$, we have:

$$\int_0^1 t^\alpha\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t$$
$$\leq \frac{1}{\alpha+2}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q$$
$$+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q$$

and

$$\int_0^1 (1-t^\alpha)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t$$
$$\leq \frac{\alpha}{2(\alpha+2)}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q$$
$$+\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q.$$

Hence the proof of the theorem is complete. □

**Theorem 4.** *Assume $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$. Then*

$$|\mathcal{L}_{\alpha,n}(a,b)| \leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\frac{1}{\alpha p+1}\right)^{1/p}\left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right| + \left(\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right)^{1/p}\left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|\right],$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* From Lemma 1 and Hölder's integral inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$|\mathcal{L}_{\alpha,n}(a,b)| \leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 (t^\alpha)^p\,\mathrm{d}t\right)^{\frac{1}{p}} \times \left(\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t^\alpha)^p\,\mathrm{d}t\right)^{p} \times \left(\int_0^1\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$ is concave on $[a,b]$ and by using the Jensen's inequality for concave functions, we have

$$\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\mathrm{d}t \leq \left|f'\left(\frac{\frac{a(n-2k)+b(2k+1)}{n+1}+\frac{a(n-2k+1)+b\cdot 2k}{n+1}}{2}\right)\right|^q = \left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q$$

and similarly

$$\int_0^1\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\mathrm{d}t \leq \left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|^q.$$

Using $\int_0^1 t^{\alpha p}\mathrm{d}t=\dfrac{1}{\alpha p+1}$, $\int_0^1(1-t^\alpha)^p\,\mathrm{d}t=\dfrac{1}{\alpha}\mathrm{B}\left(p+1,\dfrac{1}{\alpha}\right)$ we complete the proof. □

Our final result is the following:

**Theorem 5.** *Assume $|f'|$ is concave on $[a,b]$ for some fixed $q>1$. Then*

$$|\mathcal{L}_{\alpha,n}(a,b)| \le \frac{b-a}{(\alpha+1)(n+1)} \times$$
$$\sum_{k=0}^{(n-1)/2}\left[\left|f'\left(\frac{\alpha+1}{\alpha+2}\cdot\frac{a(n-2k)+b(2k+1)}{(n+1)}+\frac{1}{\alpha+2}\cdot\frac{a(n-2k+1)+b\cdot 2k}{(n+1)}\right)\right| +\right.$$
$$\left.\alpha\left|f'\left(\frac{\alpha+1}{2(\alpha+2)}\cdot\frac{a(n-2k-1)+b(2k+2)}{(n+1)}+\frac{\alpha+3}{2(\alpha+2)}\cdot\frac{a(n-2k)+b(2k+1)}{(n+1)}\right)\right|\right].$$

*Proof.* From Lemma 1 we have

$$|\mathcal{L}_{\alpha,n}(a,b)| \le \frac{b-a}{n+1} \times$$
$$\sum_{k=0}^{(n-1)/2}\left[\int_0^1 t^\alpha\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|\mathrm{d}t\right.$$
$$\left.+\int_0^1(1-t^\alpha)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\mathrm{d}t\right].$$

Since $|f'|$ is concave, by using Jensen's inequality, we obtain

$$|\mathcal{L}_{\alpha,n}(a,b)| \le \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 t^\alpha \mathrm{d}t\right)I_1+\left(\int_0^1(1-t^\alpha)\,\mathrm{d}t\right)I_2\right],$$

where

$$I_1=\left|f'\left(\frac{\int_0^1 t^\alpha\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\mathrm{d}t}{\int_0^1 t^\alpha \mathrm{d}t}\right)\right|$$
$$=\left|f'\left(\frac{\alpha+1}{\alpha+2}\cdot\frac{a(n-2k)+b(2k+1)}{n+1}+\frac{1}{\alpha+2}\cdot\frac{a(n-2k+1)+b\cdot 2k)}{n+1}\right)\right|$$

and

$$I_2=$$
$$\left|f'\left(\frac{\int_0^1(1-t^\alpha)\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\mathrm{d}t}{\int_0^1(1-t^\alpha)\,\mathrm{d}t}\right)\right|$$
$$=\left|f'\left(\frac{\alpha+1}{2(\alpha+2)}\cdot\frac{a(n-2k-1)+b(2k+2)}{n+1}+\frac{\alpha+3}{2(\alpha+2)}\cdot\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|$$

and the proof is complete. □

**Remark 2.** *For $\alpha=1$ in the Theorems 1, 2, 3, 4 respectively 5, we recover the results stated in (*[9, Theorems 6-10]*). Also for $\alpha=1$ in Lemma 1, we get (*[9, Lemma 1]*).*

**Remark 3.** *For $n=3$ in the Theorems 1, 2, 3, respectively 4 , we recover the results stated in (*[5, Theorems 1-4]*). Also for $\alpha=1$ in Lemma 1, we get (*[5, Lemma 1]*).*

We end our paper by considering the *cumulative to the right* $(\alpha,n)-$*gap* defined as

$$\mathcal{R}_{\alpha,n}(a,b)= -\sum_{k=0}^{(n-1)/2}\left[f\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)+f\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right] + \Gamma(\alpha+1)\left(\frac{n+1}{b-a}\right)^{\alpha}\sum_{k=0}^{(n-1)/2}\left[J^{\alpha}_{\frac{a(n-2k+1)+b\cdot 2k}{n+1}+}f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) + J^{\alpha}_{\frac{a(n-2k)+b(2k+1)}{n+1}+}f\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right],$$

where $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function on $[a,b]$.

**Remark 4.** *The particular case $\alpha=1$ and $n=3$ gives*

$$\frac{\mathcal{R}_{1,3}(a,b)}{4}=-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t.$$

*The right hand side term can be recognized from the well known inequality concerning the continuous convex functions defined on intervals:*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\le\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right].$$

*See* [7], *p. 52.*

Using the above technique, the reader can find companions of the results we proved for the cumulative to the left $(\alpha,n)-$gap. We succinctly state below such results but we are omitting their proofs.

**Lemma 2.** *It holds*

$$\mathcal{R}_{\alpha,n}(a,b)=\frac{b-a}{n+1}\times \sum_{k=0}^{(n-1)/2}\left[\int_0^1 t^{\alpha}f'\left(t\frac{a(n-2k+1)+b\cdot 2k}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\mathrm{d}t + \int_0^1(t^{\alpha}-1)f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\mathrm{d}t\right].$$

Our final result is the following:

**Theorem 6.** *It holds:*

$$|\mathcal{R}_{\alpha,n}(a,b)|\le\frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\frac{1}{\alpha+2}\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right| + \frac{\alpha^2+\alpha+2}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right| + \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right].$$

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