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NEW INTEGRAL INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper, we prove some new integral inequalities for co-ordinated convex functions by using a new lemma and fairly elementary analysis.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hadamard's inequality. Several papers have been written on convexity and inequalities (see [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein).

In [4], Dragomir defined convex functions on the co-ordinates as following;

Definition 1. *Let us consider the bidimensional interval* $\Delta = [a, b] \times [c, d]$ *in* $\mathbb{R}^2$ *with* $a < b$, $c < d$. *A function* $f : \Delta \to \mathbb{R}$ *will be called convex on the co-ordinates if the partial mappings* $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ *and* $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ *are convex where defined for all* $y \in [c, d]$ *and* $x \in [a, b]$. *Recall that the mapping* $f : \Delta \to \mathbb{R}$ *is convex on* Δ *if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w)$$

for all $(x, y), (z, w) \in \Delta$ *and* $\lambda \in [0, 1]$.

In [4], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$
\begin{aligned}
(1.1)\qquad & f(\frac{a+b}{2},\frac{c+d}{2}) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,\frac{c+d}{2})dx+\frac{1}{d-c}\int_c^d f(\frac{a+b}{2},y)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)dx\right. \\
& \left.+\frac{1}{(d-c)}\int_c^d f(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

In [1], Bakula and Pečarić established several Jensen type inequalities for co-ordinated convex functions and in [5], Hwang *et al.* gave a mapping F, discussed some properties of this mapping and proved some Hadamard-type inequalities for Lipschizian mapping in two variables. In [2], Özdemir *et al.* established new Hadamard-type inequalities for co-ordinated $m-$convex and $(\alpha,m)-$convex functions. In [7], Özdemir *et al.* proved an inequality of Simpson's type. On all of these, in [3], Sarıkaya *et al.* proved some Hadamard-type inequalities for co-ordinated convex functions as followings;

Theorem 2. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$
\begin{aligned}
(1.2)\qquad & \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
& \left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right| \\
\leq\ & \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|(b,d)}{4}\right)
\end{aligned}
$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx+\frac{1}{(d-c)}\int_c^d\left[f(a,y)dy+f(b,y)\right]dy\right].$$

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function on* *the co-ordinates on* Δ, *then one has the inequalities:*

$$(1.3)\quad \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx+\frac{1}{(d-c)}\int_c^d [f(a,y)dy+f(b,y)]\,dy\right]$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$(1.4)\quad \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx+\frac{1}{(d-c)}\int_c^d [f(a,y)dy+f(b,y)]\,dy\right].$$

In [6], Özdemir *et al.* proved following inequalities for co-ordinated convex functions.

Theorem 5. *Let* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$(1.5)\quad \left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|$$
$$\leq \frac{(b-a)(d-c)}{64}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].$$

Remark 1. *Suppose that all the assumptions of Theorem 5 are satisfied. If we choose* $\frac{\partial^2 f}{\partial t \partial s}$ *is bounded, i.e.,*

$$\left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty = \sup_{(t,s)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| < \infty,$$

we get

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\ &-\frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) dy - \frac{1}{(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \\ &\left. +\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx \right| \\ \leq\ & \frac{(b-a)(d-c)}{16} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty . \end{aligned} \tag{1.6}$$

Theorem 6. *Let* $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta = [a,b] \times [c,d]$. *If* $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$, $q > 1$, *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx \right. \\ &\left. -\frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) dy - \frac{1}{(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right| \\ \leq\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \\ &\times \left(\frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{4} \right)^{\frac{1}{q}} . \end{aligned} \tag{1.7}$$

Remark 2. *Suppose that all the assumptions of Theorem 6 are satisfied. If we choose* $\frac{\partial^2 f}{\partial t \partial s}$ *is bounded, i.e.,*

$$\left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty = \sup_{(t,s)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| < \infty,$$

we get

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\ &-\frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) dy - \frac{1}{(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \\ &\left. +\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx \right| \\ \leq\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty . \end{aligned} \tag{1.8}$$

Theorem 7. *Let $f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, is a convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right. \\
&\left.-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right| \\
\leq\ & \frac{(b-a)(d-c)}{16} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}\tag{1.9}$$

The main purpose of this paper is to prove some new inequalities of Hadamard-type for co-ordinated convex functions on the co-ordinates by using a new Lemma. These estimations give new upper bounds.

2. MAIN RESULTS

To prove our main results, we need following Lemma.

Lemma 1. *Let $f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then the following equality holds:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
=\ & \frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)dsdt\right. \\
&+\int_0^1\int_0^1(t-1)(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)dsdt \\
&+\int_0^1\int_0^1 t(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)dsdt \\
&\left.+\int_0^1\int_0^1 s(t-1)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)dsdt\right]
\end{aligned}$$

for all $t,s\in[0,1]$.

Proof. We note that

$$I_1=\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)dsdt.$$

Integration by parts, we can write

$$\begin{aligned}
&\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)dsdt\\
=&\int_0^1 t\left[\frac{2s}{d-c}\frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)\Big|_0^1\right.\\
&\left.\frac{2}{d-c}\int_0^1\frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)ds\right]dt\\
=&\frac{2}{d-c}\int_0^1 t\frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)dt\\
&-\frac{2}{d-c}\int_0^1\int_0^1 t\frac{\partial f}{\partial t}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)dsdt.
\end{aligned}$$

By integrating again, we get

$$\begin{aligned}
&I_1\\
=&\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)dsdt\\
=&\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\,a,\frac{c+d}{2}\right)dt\right.\\
&\left.-\int_0^1 f\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right)ds+\int_0^1\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)dtds\right].
\end{aligned}$$

Analogously, we can easily obtain

$$\begin{aligned}
&I_2\\
=&\int_0^1\int_0^1 (t-1)(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)dsdt\\
=&\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(tb+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dt\right.\\
&\left.-\int_0^1 f\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)ds+\int_0^1\int_0^1 f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)dtds\right],
\end{aligned}$$

$$\begin{aligned}
&I_3\\
=&\int_0^1\int_0^1 t(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, sd+(1-s)\frac{c+d}{2}\right)dsdt\\
=&\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\,a,\frac{c+d}{2}\right)dt\right.\\
&\left.-\int_0^1 f\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)ds+\int_0^1\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\,a, sd+(1-s)\frac{c+d}{2}\right)dtds\right]
\end{aligned}$$

and

$$
\begin{aligned}
& I_4 \\
= & \int_0^1\int_0^1 (t-1)\, s\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right) dsdt \\
= & \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(tb+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dt\right. \\
& \left.-\int_0^1 f\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right)ds+\int_0^1\int_0^1 f\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right)dtds\right].
\end{aligned}
$$

By adding I_1, I_2, I_3, I_4 and by changing of the variables, we get the desired result. □

Theorem 8. *Let $f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is co-ordinated convex function on the co-ordinates on Δ, then the following inequality holds;*

$$
\begin{aligned}
(2.1)\qquad & I=\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
& -\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
& \left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right| \\
\leq\ & \frac{(b-a)(d-c)}{144}\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right. \\
& +4\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right| \\
& \left.+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\right].
\end{aligned}
$$

Proof. From Lemma 1 and by using the property of modulus, we can write

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \right.\\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|\\
=\;&\frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right.\\
&+\int_0^1\int_0^1 |(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\\
&+\int_0^1\int_0^1 |t(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\\
&\left.+\int_0^1\int_0^1 |s(t-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right].
\end{aligned}$$

By co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$, we have

$$\begin{aligned}
&I\\
=\;&\frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|\right.\right.\\
&\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,s\frac{c+d}{2}+(1-s)c\right)\right|\right\}dsdt\\
&+\int_0^1\int_0^1 |(t-1)(s-1)|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sd+(1-s)\frac{c+d}{2}\right)\right|\right.\\
&\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\right\}\\
&+\int_0^1\int_0^1 |t(s-1)|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\right.\\
&\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\right\}\\
&+\int_0^1\int_0^1 |s(t-1)|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,s\frac{c+d}{2}+(1-s)c\right)\right|\right.\\
&\left.\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right\}\right].
\end{aligned}$$

By computing these integrals, we obtain

$$
\begin{aligned}
& I \\
= \; & \frac{(b-a)(d-c)}{16}\left[\frac{1}{3}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|ds\right. \\
& +\frac{1}{6}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, s\frac{c+d}{2}+(1-s)c\right)\right|ds \\
& +\frac{1}{6}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
& +\frac{1}{3}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
& +\frac{1}{3}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
& +\frac{1}{6}\int_0^1 |s-1|\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, sd+(1-s)\frac{c+d}{2}\right)\right|ds \\
& +\frac{1}{6}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, s\frac{c+d}{2}+(1-s)c\right)\right|ds \\
& \left.+\frac{1}{3}\int_0^1 s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|ds\right].
\end{aligned}
$$

By using co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ again and computing the integrals, we get desired result. □

Remark 3. *Suppose that all the assumptions of Theorem 8 are satisfied. If we choose* $\frac{\partial^2 f}{\partial t\partial s}$ *is bounded, i.e.,*

$$
\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty = \sup_{(t,s)\in(0,1)\times(0,1)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| < \infty,
$$

we get

$$
\begin{aligned}
& \left|f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
& -\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2}, y\right)dy - \frac{1}{(b-a)}\int_a^b f\left(x, \frac{c+d}{2}\right)dx \\
& \left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right| \\
\leq \; & \frac{(b-a)(d-c)}{16}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty
\end{aligned}
$$

which is the inequality (1.6).

Theorem 9. *Let* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$*,* $q>1$*, is a convex function on the co-ordinates on* Δ*,* *then the following inequality holds;*

(2.2)

$$
\begin{aligned}
& \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \right. \\
& \left. -\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \right| \\
\leq\; & \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|}{2}\right. \\
& \left.\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)\right)^{\frac{1}{q}}
\end{aligned}
$$

Proof. From Lemma 1, we have

$$
\begin{aligned}
& \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \right. \\
& -\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
& \left. +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \right| \\
=\; & \frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right. \\
& +\int_0^1\int_0^1 |(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|dsdt \\
& +\int_0^1\int_0^1 |t(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\right|dsdt \\
& \left.+\int_0^1\int_0^1 |s(t-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right].
\end{aligned}
$$

By applying the well-known Hölder inequality for double integrals, then one has

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \right. \\
& -\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
& \left. +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \right| \\
= & \frac{(b-a)(d-c)}{16}\left[\left(\int_0^1\int_0^1 (ts)^p\,dsdt\right)^{\frac{1}{p}}\right. \\
& \times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 |(t-1)(s-1)|^p\,dsdt\right)^{\frac{1}{p}} \\
& \times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 |t(s-1)|^p\,dsdt\right)^{\frac{1}{p}} \\
& \times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 |s(t-1)|^p\,dsdt\right)^{\frac{1}{p}} \\
& \left.\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|^q dsdt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is a co-ordinated convex function on Δ, we can write for all $(t,s)\in[0,1]\times[0,1]$

$$\begin{aligned}
& \left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|^q \\
\leq\ & t\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|^q \\
& +(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,s\frac{c+d}{2}+(1-s)c\right)\right|^q
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)\right|^q \\
\leq\;& ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\
&+t\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q \\
&+(1-t)\,s\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q \\
&+(1-t)\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q .
\end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2}, sd+(1-s)\,\frac{c+d}{2}\right)\right|^q \\
\leq\;& ts\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \\
&+t\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q \\
&+(1-t)\,s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q \\
&+(1-t)\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q ,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, sd+(1-s)\,\frac{c+d}{2}\right)\right|^q \\
\leq\;& ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q \\
&+t\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\
&+(1-t)\,s\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\
&+(1-t)\,(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q ,
\end{aligned} \tag{2.5}$$

(2.6)
$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q \\
\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q \\
&+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \\
&+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\
&+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q
\end{aligned}$$

Using inequalities of (2.3)-(2.6) in (**??**), we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right. \\
&\left.-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right| \\
\leq\ & \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|}{2}\right. \\
&\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

This completes the proof. □

Remark 4. *Suppose that all the assumptions of Theorem 9 are satisfied. If we choose* $\frac{\partial^2 f}{\partial t\partial s}$ *is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty=\sup_{(t,s)\in(0,1)\times(0,1)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|<\infty,$$

we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right| \\
\leq\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty
\end{aligned}$$

which is the inequality (1.8).

Theorem 10. *Let* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

(2.7)
$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|\\
\leq\ &\frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{36}\right.\\
&\left.+\frac{4\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q}{9}\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|\\
=\ &\frac{(b-a)(d-c)}{16}\left[\int_0^1\int_0^1 ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right.\\
&+\int_0^1\int_0^1|(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\\
&+\int_0^1\int_0^1|t(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\right|dsdt\\
&\left.+\int_0^1\int_0^1|s(t-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|dsdt\right].
\end{aligned}$$

By applying the well-known Power mean inequality for double integrals, then one has

$$\begin{aligned}
& I \\
= & \frac{(b-a)(d-c)}{16}\left[\left(\int_0^1\int_0^1 (ts)\,dsdt\right)^{1-\frac{1}{q}}\right. \\
& \times\left(\int_0^1\int_0^1 (ts)\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 |(t-1)(s-1)|\,dsdt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1\int_0^1 |(t-1)(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 |t(s-1)|\,dsdt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1\int_0^1 |t(s-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 |s(t-1)|\,dsdt\right)^{1-\frac{1}{q}} \\
& \left.\times\left(\int_0^1\int_0^1 |s(t-1)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q dsdt\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.8}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is a co-ordinated convex function on Δ, we can write for all $(t,s)\in[0,1]\times[0,1]$

$$\begin{aligned}
& \left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|^q \\
\leq & \; ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\
& +t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q \\
& +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q \\
& +(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q.
\end{aligned} \tag{2.9}$$

$$\begin{aligned} &\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|^q \\ \leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \\ &+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q \\ &+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q \\ &+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q \\ \leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q \\ &+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\ &+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\ &+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q \\ \leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q \\ &+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \\ &+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \\ &+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q \end{aligned} \tag{2.12}$$

If we use (2.9)−(2.12) in (2.8), we get

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \right. \\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
&\left. +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \right| \\
\leq\ & \frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{36}\right. \\
&\left.+\frac{4\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q}{9}\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

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