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NEW INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL CALCULUS

MARCELA V. MIHAI

Abstract. We provide some new Hermite-Hadamard type inequalities for co-ordinated convex functions, via Riemann-Liouville fractional integration.

1. Introduction

The Hermite-Hadamard inequality states that if a function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is convex, then one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. Both inequalities hold in reversed direction if f is concave.

In recent years many researchers have returned to Hermite-Hadamard inequality and found many variations and generalizations of it for various types of convexity. Some of this research are related to functions convex on the co-ordinates (see, for instance, [1], [3], [4], [5], [6], [7], and the references therein).

Definition 1. [3] *Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b, c < d$. A function $f : \Delta \longrightarrow \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a, b] \longrightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \longrightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $y \in [c, d]$ and $x \in [a, b]$. Recall mapping $f : \Delta \longrightarrow \mathbb{R}$ is convex on the co-ordinates on Δ if the following inequality holds,*

$$\begin{aligned} f\left(tx + (1-t)\,y, su + (1-s)\,w\right) &\leq tsf(x,u) + t\,(1-s)\,f(x,w) \\ &\quad + s\,(1-t)\,f(y,u) + (1-t)\,(1-s)\,f(y,w) \end{aligned}$$

for all $(x, u), (y, w) \in \Delta$ and $t, s \in [0, 1]$.

In [6], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. *Suppose that $f : \Delta = [a,b] \times [c,d] \longrightarrow \mathbb{R}$ is a convex on the co-ordinates on Δ. Then one has the inequalities:*

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right] \\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,\mathrm{d}x + \frac{1}{b-a}\int_a^b f(x,d)\,\mathrm{d}x\right. \\
&\left. + \frac{1}{d-c}\int_c^d f(a,y)\,\mathrm{d}y + \frac{1}{d-c}\int_c^d f(b,y)\,\mathrm{d}y\right] \\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

The purpose of our paper is to establish, via the Riemann-Liouville fractional calculus, some Hermite-Hadamard type inequalities for co-ordinated convex functions, via Riemann-Liouville fractional integration.

Let $f \in L^1[a,b]$, where $a \ge 0$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$, of order $\alpha > 0$, are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \text{ for } x > a,$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t, \text{ for } x < b,$$

respectively. Here, $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\mathrm{d}t$ is the Gamma function. We also make the convention

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

For details about the Riemann-Liouville fractional integrals see [2].

2. Main results

We assume throughout the present paper that $\Delta = [a,b] \times [c,d]$ in $[0,\infty)^2$ and $f : \Delta \longrightarrow \mathbb{R}$ is a differentiable mapping on Δ and $\dfrac{\partial^2 f}{\partial s \partial t} \in L^1(\Delta)$, where $\alpha, \beta > 0$. Before stating the results we establish the notation.

We define the *cumulative to the left* $(\alpha,\beta)-$*gap* by

$$\begin{aligned}\mathcal{L}_{\Delta}(\alpha,\beta) &= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\ &-\frac{2^{\beta-1}\Gamma(\beta+1)}{(d-c)^{\beta}}\left[J^{\beta}_{\frac{c+d}{2}-}f\left(\frac{a+b}{2},c\right)+J^{\beta}_{d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\ &+\frac{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}-}f(a,c)+J^{\alpha,\beta}_{b-,d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\ &\left.+J^{\alpha,\beta}_{\frac{a+b}{2}-,d-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b-,\frac{c+d}{2}-}f\left(\frac{a+b}{2},c\right)\right],\end{aligned}$$

where

$$J^{\alpha,\beta}_{b-,d-}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d(u-x)^{\alpha-1}(v-y)^{\beta-1}f(u,v)\,\mathrm{d}v\mathrm{d}u,\ x<b,y<d,$$

is Riemann-Liouville integral and Γ is the Euler Gamma function.

Remark 1. *The particular case $\alpha=1$ and $\beta=1$ gives*

$$\begin{aligned}\mathcal{L}_{\Delta}(1,1) &= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &-\frac{1}{b-a}\int_a^b f\left(u,\frac{c+d}{2}\right)\mathrm{d}u-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},v\right)\mathrm{d}v\\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u.\end{aligned}$$

The right hand side term has its origins in the inequalities of the Theorem 1.

In order to prove our main results we need the following lemma.

Lemma 1. *It holds*

$$\begin{aligned}\mathcal{L}_{\Delta}(\alpha,\beta) &= \frac{(b-a)(d-c)}{16}\\ &\times\left[\int_0^1\int_0^1 t^{\alpha}s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\mathrm{d}s\mathrm{d}t\right.\\ &+\int_0^1\int_0^1(t^{\alpha}-1)\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\\ &+\int_0^1\int_0^1 t^{\alpha}\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\\ &\left.+\int_0^1\int_0^1(t^{\alpha}-1)s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\mathrm{d}s\mathrm{d}t\right],\end{aligned}$$

for all $t,s\in[0,1]$.

Proof. Calculate the four integrals by parts and change of variables $u = t\frac{a+b}{2} + (1-t)\,a$, $v = s\frac{c+d}{2} + (1-s)\,c$ and similar such

$$I_1 = \int_0^1 \int_0^1 t^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)\,a, s\frac{c+d}{2} + (1-s)\,c\right) \mathrm{d}s\mathrm{d}t$$

$$= \int_0^1 t^\alpha \left[\int_0^1 s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)\,a, s\frac{c+d}{2} + (1-s)\,c\right) \mathrm{d}s\right] \mathrm{d}t$$

$$= \frac{2}{d-c}\int_0^1 t^\alpha \frac{\partial f}{\partial t}\left(t\frac{a+b}{2} + (1-t)\,a, \frac{c+d}{2}\right) \mathrm{d}t$$

$$-\frac{2\beta}{d-c}\int_0^1 s^{\beta-1}\left[\int_0^1 t^\alpha \frac{\partial f}{\partial t}\left(t\frac{a+b}{2} + (1-t)\,a, s\frac{c+d}{2} + (1-s)\,c\right) \mathrm{d}t\right] \mathrm{d}s$$

$$= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$-\frac{4\alpha}{(b-a)(d-c)}\int_0^1 t^{\alpha-1} f\left(t\frac{a+b}{2} + (1-t)\,a, \frac{c+d}{2}\right) \mathrm{d}t$$

$$-\frac{4\beta}{(b-a)(d-c)}\int_0^1 s^{\beta-1} f\left(\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)\,c\right) \mathrm{d}s$$

$$-\frac{4\alpha\beta}{(b-a)(d-c)}\int_0^1 t^{\alpha-1} s^{\beta-1} f\left(t\frac{a+b}{2} + (1-t)\,a, s\frac{c+d}{2} + (1-s)\,c\right) \mathrm{d}s\mathrm{d}t$$

$$= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$-\frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} \cdot \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}} (u-a)^{\alpha-1} f\left(u, \frac{c+d}{2}\right) \mathrm{d}u$$

$$-\frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} \cdot \frac{1}{\Gamma(\beta)}\int_c^{\frac{c+d}{2}} (v-c)^{\beta-1} f\left(\frac{a+b}{2}, v\right) \mathrm{d}v$$

$$+\frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} \cdot \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (u-a)^{\alpha-1}(v-c)^{\beta-1} f(u,v)\mathrm{d}u\mathrm{d}v$$

$$= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$-\frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} J^\alpha_{\frac{a+b}{2}-} f\left(a, \frac{c+d}{2}\right) - \frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} J^\beta_{\frac{c+d}{2}-} f\left(\frac{a+b}{2}, c\right)$$

$$+\frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}-} f\,(a,c)\,.$$

Similarly,

$$I_2 = \int_0^1 \int_0^1 (t^\alpha - 1)\left(s^\beta - 1\right) \frac{\partial^2 f}{\partial t \partial s}\left(tb + (1-t)\,\frac{a+b}{2}, sd + (1-s)\,\frac{c+d}{2}\right) \mathrm{d}s\mathrm{d}t$$

$$= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$-\frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} J^\alpha_{b-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} J^\beta_{d-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$+\frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}J^{\alpha,\beta}_{b-,d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right),$$

$$I_3=\int_0^1\int_0^1 t^{\alpha}\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, sd+(1-s)\,\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t$$
$$=\frac{4}{(b-a)(d-c)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$-\frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)}J^{\alpha}_{\frac{a+b}{2}-}f\left(a,\frac{c+d}{2}\right)-\frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}}J^{\beta}_{d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$+\frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}J^{\alpha,\beta}_{\frac{a+b}{2}-,d-}f\left(a,\frac{c+d}{2}\right)$$

and

$$I_4=\int_0^1\int_0^1 (t^{\alpha}-1)\,s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right)\mathrm{d}s\mathrm{d}t$$
$$=\frac{4}{(b-a)(d-c)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$-\frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)}J^{\alpha}_{b-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}}J^{\beta}_{\frac{c+d}{2}-}f\left(\frac{a+b}{2},c\right)$$
$$+\frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}J^{\alpha,\beta}_{b-,\frac{c+d}{2}-}f\left(\frac{a+b}{2},c\right).$$

Multiplying the sum of I_1, I_2, I_3, I_4 with $\frac{(b-a)(d-c)}{16}$ we get $\mathcal{L}_{\Delta}(\alpha,\beta)$ and the proof is complete. □

We are now in a position to state and prove the following:

Theorem 2. *Assume* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is co-ordinated convex function on* Δ*. Then the following inequality holds:*

$$|\mathcal{L}_{\Delta}(\alpha,\beta)|\le\frac{(b-a)(d-c)}{16}\left[A\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+B\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|\right.$$
$$+C\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+D\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|+E\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|$$
$$\left.+F\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+G\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|+H\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+I\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\right]$$

where $A=\frac{\alpha^2\beta^2+5\alpha^2\beta+5\alpha\beta^2+2\alpha^2+2\beta^2+25\alpha\beta+10\alpha+10\beta+4}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}$,
$B=\frac{\alpha^2+5\alpha+2}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}$, $C=\frac{\beta^2+5\beta+2}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}$,
$D=\frac{\alpha(\beta^2+5\beta+2)}{4(\alpha+2)(\beta+1)(\beta+2)}$, $E=\frac{\beta(\alpha^2+5\alpha+2)}{4(\alpha+1)(\alpha+2)(\beta+2)}$,

$$F=\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)},\ \ G=\frac{\alpha\beta}{(4(\alpha+2))(\beta+2)},$$
$$H=\frac{\beta}{2(\alpha+1)(\alpha+2)(\beta+2)}\ \text{and}\ I=\frac{\alpha}{2(\alpha+2)(\beta+1)(\beta+2)}.$$

Proof. From Lemma 1 and by using the property of modulus, we can write

$$
\begin{aligned}
(2.1)\quad \left|\mathcal{L}_{\Delta}(\alpha,\beta)\right| &\le \frac{(b-a)(d-c)}{16}\\
&\times\left[\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|\mathrm{d}s\mathrm{d}t\right.\\
&+\int_0^1\int_0^1(1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|\mathrm{d}s\mathrm{d}t\\
&+\int_0^1\int_0^1 t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\right|\mathrm{d}s\mathrm{d}t\\
&\left.+\int_0^1\int_0^1(1-t^{\alpha})s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|\mathrm{d}s\mathrm{d}t\right]\\
&=\frac{(b-a)(d-c)}{16}\left(J_1+J_2+J_3+J_4\right).
\end{aligned}
$$

By co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$, we have

$$
\begin{aligned}
J_1&=\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|\mathrm{d}s\mathrm{d}t\\
&\le\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left[ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|\right.\\
&\left.+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right]\mathrm{d}s\mathrm{d}t\\
&=\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|\\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
J_2&=\int_0^1\int_0^1(1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|\mathrm{d}s\mathrm{d}t\\
&\le\frac{\alpha\beta}{4(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|+\frac{\alpha\beta(\beta+3)}{4(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\\
&+\frac{\alpha\beta(\alpha+3)}{4(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|\\
&+\frac{\alpha\beta(\alpha+3)(\beta+3)}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|,
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= \int_0^1 \int_0^1 t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2}+(1-t)\,a, sd+(1-s)\,\frac{c+d}{2}\right)\right| \mathrm{d}s\mathrm{d}t \\
&\leq \frac{\beta}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right| + \frac{\beta(\beta+3)}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \\
&+ \frac{\beta}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right| + \frac{\beta(\beta+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= \int_0^1 \int_0^1 \left(1-t^{\alpha}\right) s^{\beta}\left|\frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right)\right| \mathrm{d}s\mathrm{d}t \\
&\leq \frac{\alpha}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right| + \frac{\alpha}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right| \\
&\quad + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \\
&\quad\quad + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|.
\end{aligned}
$$

Considering the results J_1, J_2, J_3, J_4 in (2.1) and making appropriate calculations, we get the conclusion of the Theorem 2. □

We recall that the Beta function (the Euler integral of the first kind), is defined by

$$
\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t
$$

for $x, y > 0$.
Our next result reads as:

Theorem 3. *Assume* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q, q > 1$ *is co-ordinated convex function on* Δ*. Then the following inequality holds:*

$$
\begin{aligned}
&\left|\mathcal{L}_{\Delta}(\alpha,\beta)\right| \leq \frac{(b-a)(d-c)}{16\cdot 4^{1/q}}\left\{\left[\frac{1}{(\alpha p+1)(\beta p+1)}\right]^{1/p}\times\right. \\
&\left[\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q\right]^{1/q} \\
&\quad + \left[\frac{1}{\alpha\beta}\mathrm{B}\left(p+1, \frac{1}{\alpha}\right)\mathrm{B}\left(p+1, \frac{1}{\beta}\right)\right]^{1/p}\times \\
&\left[\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|^q\right]^{1/q} \\
&\quad + \left[\frac{1}{\alpha(\beta p+1)}\mathrm{B}\left(p+1, \frac{1}{\alpha}\right)\right]^{1/p}\times \\
&\left[\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q\right]^{1/q}
\end{aligned}
$$

$$+\left[\frac{1}{(\alpha p+1)\beta}\mathrm{B}\left(p+1,\frac{1}{\beta}\right)\right]^{1/p}\times$$

$$\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\right]^{1/q}$$

for $\frac{1}{p}+\frac{1}{q}=1.$

Proof. According to Lemma 1 and Hölder's inequality, we have

$$|\mathcal{L}_\Delta(\alpha,\beta)|\le\frac{(b-a)(d-c)}{16}\times$$
$$\left\{\left[\int_0^1\int_0^1\left(t^\alpha s^\beta\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_1^{1/q}+\left[\int_0^1\int_0^1\left(\left(1-t^\alpha\right)\left(1-s^\beta\right)\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_2^{1/q}\right.$$
$$\left.+\left[\int_0^1\int_0^1\left(t^\alpha\left(1-s^\beta\right)\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_3^{1/q}+\left[\int_0^1\int_0^1\left(\left(1-t^\alpha\right)s^\beta\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_4^{1/q}\right\}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is co-ordinated convex function, we have:

$$K_1=\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,s\frac{c+d}{2}+(1-s)c\right)\right|^q\mathrm{d}s\mathrm{d}t$$
$$\le\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\int_0^1\int_0^1 ts\mathrm{d}s\mathrm{d}t+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\int_0^1\int_0^1 t(1-s)\mathrm{d}s\mathrm{d}t$$
$$+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\int_0^1\int_0^1(1-t)s\mathrm{d}s\mathrm{d}t+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\int_0^1\int_0^1(1-t)(1-s)\mathrm{d}s\mathrm{d}t=$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right]$$

and similarly

$$K_2=\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\le$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right],$$

$$K_3=\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a,sd+(1-s)\frac{c+d}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\le$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\right],$$

$$K_4=\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)c\right)\right|^q\mathrm{d}s\mathrm{d}t\le$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right].$$

A simple computation shows that

$$\int_0^1\int_0^1 (t^\alpha s^\beta)^p \mathrm{d}s\mathrm{d}t = \frac{1}{(\alpha p+1)(\beta p+1)},$$
$$\int_0^1\int_0^1 (1-t^\alpha)^p\left(1-s^\beta\right)^p \mathrm{d}s\mathrm{d}t = \frac{1}{\alpha\beta}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\mathrm{B}\left(p+1,\frac{1}{\beta}\right),$$
$$\int_0^1\int_0^1 (t^\alpha)^p(1-s^\beta)^p \mathrm{d}s\mathrm{d}t = \frac{1}{(\alpha p+1)\beta}\mathrm{B}\left(p+1,\frac{1}{\beta}\right),$$
$$\int_0^1\int_0^1 (1-t^\alpha)^p(s^\beta)^p \mathrm{d}s\mathrm{d}t = \frac{1}{\alpha(\beta p+1)}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)$$

and the proof is complete. □

Theorem 4. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q, q>1$ *is co-ordinated convex function on* Δ*, then the following inequality holds:*

$$\begin{aligned}
|\mathcal{L}_\Delta(\alpha,\beta)| \leq{}& \frac{(b-a)(d-c)}{16\left[(\alpha+1)(\beta+1)\right]^{1/p}\left[(\alpha+2)(\beta+2)\right]^{1/q}}\times \\
&\left\{\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{1}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right.\right. \\
&\left.+\frac{1}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\frac{1}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right]^{1/q} \\
&+\frac{\alpha\beta}{4^{1/q}}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\frac{\beta+3}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q\right. \\
&\left.+\frac{\alpha+3}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\frac{(\alpha+3)(\beta+3)}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right]^{1/q} \\
&+\frac{\beta}{2^{1/q}}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\frac{\beta+3}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right. \\
&\left.+\frac{1}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\frac{\beta+3}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\right]^{1/q} \\
&+\frac{\alpha}{2^{1/q}}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\frac{1}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right. \\
&\left.\left.+\frac{\alpha+3}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{\alpha+3}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right]^{1/q}\right\}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and the power mean inequality, we have

$$
\begin{aligned}
&|\mathcal{L}_{\Delta}(\alpha,\beta)| \leq \frac{(b-a)(d-c)}{16}\times \\
&\left\{\left[\int_0^1\int_0^1 t^{\alpha}s^{\beta}\mathrm{d}s\mathrm{d}t\right]^{1/p} M_1^{1/q} + \left[\int_0^1\int_0^1 (1-t^{\alpha})\left(1-s^{\beta}\right)\mathrm{d}s\mathrm{d}t\right]^{1/p} M_2^{1/q}\right. \\
&\left.+\left[\int_0^1\int_0^1 t^{\alpha}\left(1-s^{\beta}\right)\mathrm{d}s\mathrm{d}t\right]^{1/p} M_3^{1/q} + \left[\int_0^1\int_0^1 (1-t^{\alpha})\, s^{\beta}\mathrm{d}s\mathrm{d}t\right]^{1/p} M_4^{1/q}\right\}
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is co-ordinated convex function, we have:

$$
\begin{aligned}
M_1 &= \int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \int_0^1\int_0^1 t^{\alpha+1}s^{\beta+1}\mathrm{d}s\mathrm{d}t \\
&+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q \int_0^1\int_0^1 t^{\alpha+1}(s^{\beta}-s^{\beta+1})\mathrm{d}s\mathrm{d}t \\
&+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q \int_0^1\int_0^1 (t^{\alpha}-t^{\alpha+1})s\mathrm{d}s\mathrm{d}t \\
&+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \int_0^1\int_0^1 (t^{\alpha}-t^{\alpha+1})(s^{\beta}-s^{\beta+1})\mathrm{d}s\mathrm{d}t =
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + \frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q \\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q + \frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q
\end{aligned}
$$

and similarly

$$
\begin{aligned}
M_2 &= \int_0^1\int_0^1 (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \frac{\alpha\beta}{4(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \\
&+\frac{\alpha\beta(\beta+3)}{4(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q \\
&+\frac{\alpha\beta(\alpha+3)}{4(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q \\
&+\frac{\alpha\beta(\alpha+3)(\beta+3)}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q,
\end{aligned}
$$

$$\begin{aligned}M_3 &= \int_0^1 \int_0^1 t^\alpha \left(1-s^\beta\right) \left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \frac{\beta}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|^q \\
&+ \frac{\beta(\beta+3)}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q \\
&+ \frac{\beta}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\
&+ \frac{\beta(\beta+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q,\end{aligned}$$

$$\begin{aligned}M_4 &= \int_0^1 \int_0^1 \left(1-t^\alpha\right) s^\beta \left|\frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \frac{\alpha}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q \\
&+ \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q \\
&+ \frac{\alpha}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q \\
&+ \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q.\end{aligned}$$

Hence the proof of the theorem is complete. □

Remark 2. *For $\alpha = 1$ and $\beta = 1$ in the Theorems 2, 3, respectively 4, we recover the results stated in (*[4, Theorems 8-10]*). Also for $\alpha = 1$ and $\beta = 1$ in Lemma 1, we get (*[4, Lemma 1]*).*

We end our paper by considering the *cumulative to the right* $(\alpha,\beta)-$*gap* defined as

$$\begin{aligned}\mathcal{R}_\Delta(\alpha,\beta) &= \frac{f(a,c)+f(b,d)+f(a,d)+f(b,c)}{4} \\
&- \frac{2^{\alpha-2}\Gamma(\alpha+1)}{(b-a)^\alpha} \\
&\times \left[J_{a+}^\alpha f\left(\frac{a+b}{2}, c\right) + J_{\frac{a+b}{2}+}^\alpha f(b,d) + J_{a+}^\alpha f\left(\frac{a+b}{2}, d\right) + J_{\frac{a+b}{2}+}^\alpha f(b,c)\right] \\
&- \frac{2^{\beta-2}\Gamma(\beta+1)}{(d-c)^\beta} \\
&\times \left[J_{c+}^\beta f\left(a, \frac{c+d}{2}\right) + J_{\frac{c+d}{2}+}^\beta f(b,d) + J_{\frac{c+d}{2}+}^\beta f(a,d) + J_{c+}^\beta f\left(b, \frac{c+d}{2}\right)\right] \\
&+ \frac{2^{\beta-2}\Gamma(\beta+1)}{(d-c)^\beta}\left[J_{a+,c+}^{\alpha,\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + J_{\frac{a+b}{2}+,\frac{c+d}{2}+}^{\alpha,\beta} f(b,d)\right. \\
&\left. + J_{a+,\frac{c+d}{2}+}^{\alpha,\beta} f\left(\frac{a+b}{2}, d\right) + J_{\frac{a+b}{2}+,c+}^{\alpha,\beta} f\left(b, \frac{c+d}{2}\right)\right],\end{aligned}$$

where $f:\Delta\longrightarrow\mathbb{R}$ be a differentiable function on Δ and

$$J_{a+,c+}^{\alpha,\beta}f\left(x,y\right)=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int_{a}^{x}\int_{c}^{y}\left(x-u\right)^{\alpha-1}\left(y-v\right)^{\beta-1}f\left(u,v\right)\mathrm{d}v\mathrm{d}u,x>a,y>c,$$

is Riemann-Liouville integral and Γ is the Euler Gamma function.

Remark 3. *The particular case $\alpha=1$ and $\beta=1$ gives*

$$\mathcal{R}_{\Delta}\left(1,1\right)=\frac{f(a,c)+f(b,d)+f(a,d)+f(b,c)}{4}$$
$$-\frac{1}{2(b-a)}\left(\int_{a}^{b}f(x,c)\mathrm{d}x+\int_{a}^{b}f(x,d)\mathrm{d}x\right)-\frac{1}{2(d-c)}\left(\int_{c}^{d}f(a,y)\mathrm{d}y+\int_{c}^{d}f(b,y)\mathrm{d}y\right)$$
$$+\frac{1}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d}f(x,y)\mathrm{d}y\mathrm{d}x$$

The right hand side term can be recognized from the Hermite-Hadamard's inequality concerning the co-cordinated convex functions. See [5].

Using the above technique, the reader can find companions of the results we proved for the cumulative to the right $(\alpha,\beta)-$gap. We give below one of the results, but omit the proof.

Lemma 2. *It holds*

$$\mathcal{R}_{\Delta}\left(\alpha,\beta\right)=\frac{(b-a)(d-c)}{16}$$
$$\times\left[\int_{0}^{1}\int_{0}^{1}t^{\alpha}s^{\beta}\frac{\partial^{2}f}{\partial t\partial s}\left(ta+\left(1-t\right)\frac{a+b}{2},sc+\left(1-s\right)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\right.$$
$$+\int_{0}^{1}\int_{0}^{1}\left(t^{\alpha}-1\right)\left(s^{\beta}-1\right)\frac{\partial^{2}f}{\partial t\partial s}\left(\frac{a+b}{2}+\left(1-t\right)b,\frac{c+d}{2}+\left(1-s\right)d\right)\mathrm{d}s\mathrm{d}t$$
$$+\int_{0}^{1}\int_{0}^{1}t^{\alpha}\left(s^{\beta}-1\right)\frac{\partial^{2}f}{\partial t\partial s}\left(ta+\left(1-t\right)\frac{a+b}{2},\frac{c+d}{2}+\left(1-s\right)d\right)\mathrm{d}s\mathrm{d}t$$
$$\left.+\int_{0}^{1}\int_{0}^{1}\left(t^{\alpha}-1\right)s^{\beta}\frac{\partial^{2}f}{\partial t\partial s}\left(t\frac{a+b}{2}+\left(1-t\right)b,sc+\left(1-s\right)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\right],$$

for all $t,s\in[0,1]$.

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(Marcela V. Mihai) University of Craiova, Department of Mathematics, Street A. I. Cuza 13, Craiova, RO-200585, Romania

E-mail address: marcelamihai58@yahoo.com