Received 11/09/12

NEW INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. We provide some new Hermite-Hadamard type inequalities for co-ordinated convex functions.

1. Introduction

The Hermite-Hadamard inequality states that if a function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is convex, then one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. Both inequalities hold in reversed direction if f is concave.

Since 1893 when Hadamard proved his famous inequality, many mathematicians have been working about and around it, in many different directions and with a lot of applications (see, for instance, [1], [3], [4], [5], [6], [7], and the references therein).

Definition 1. [3] *Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b, c < d$. A function $f : \Delta \longrightarrow \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a, b] \longrightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \longrightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $y \in [c, d]$ and $x \in [a, b]$. Recall mapping $f : \Delta \longrightarrow \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$\begin{aligned} f(tx + (1-t)y, su + (1-s)w) &\leq tsf(x,u) + t(1-s)f(x,w) \\ &\quad + s(1-t)f(y,u) + (1-t)(1-s)f(y,w) \end{aligned}$$

for all $(x, u), (y, w) \in \Delta$ and $t, s \in [0, 1]$.

In [6], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

Date: 16.08.2012.
2000 *Mathematics Subject Classification.* 26A51.
Key words and phrases. co-ordinated convex function, Hermite-Hadamard inequality.

Theorem 1. *Suppose that $f:\Delta=[a,b]\times[c,d]\longrightarrow\mathbb{R}$ is a convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\mathrm{d}y\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,\mathrm{d}x+\frac{1}{b-a}\int_a^b f(x,d)\,\mathrm{d}x\right.\\
&\left.+\frac{1}{d-c}\int_c^d f(a,y)\,\mathrm{d}y+\frac{1}{d-c}\int_c^d f(b,y)\,\mathrm{d}y\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

The purpose of our paper is to establish some Hermite-Hadamard type inequalities for co-ordinated convex functions.

2. Main results

We assume throughout the present paper that $\Delta=[a,b]\times[c,d]$ in $[0,\infty)^2$ and $f:\Delta\longrightarrow\mathbb{R}$ is a differentiable mapping on Δ and $\frac{\partial^2 f}{\partial s\partial t}\in L^1(\Delta)$, where $\alpha,\beta>0$.

In order to prove our main results we need the following lemma.

Lemma 1. *It holds*

$$\begin{aligned}
&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\int_a^b\int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u\\
&-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(u,c)+f(u,d))\,\mathrm{d}u+\frac{1}{d-c}\int_c^d (f(a,v)+f(b,v))\,\mathrm{d}v\right]\\
&=\frac{(b-a)(d-c)}{16}\\
&\times\left[\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\right.\\
&+\int_0^1\int_0^1 (t-1)(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)b,s\frac{c+d}{2}+(1-s)d\right)\mathrm{d}s\mathrm{d}t\\
&+\int_0^1\int_0^1 t(s-1)\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)d\right)\mathrm{d}s\mathrm{d}t\\
&\left.+\int_0^1\int_0^1 (t-1)s\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)b,sc+(1-s)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\right],
\end{aligned}$$

for all $t,s\in[0,1]$.

Proof. Calculate the four integrals by parts and change of variables $u = t\frac{a+b}{2} + (1-t)\,a$, $v = s\frac{c+d}{2} + (1-s)\,c$ and similar such

$$\begin{aligned}
I_1 &= \int_0^1 \int_0^1 ts\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\,\frac{a+b}{2}, sc + (1-s)\,\frac{c+d}{2}\right) \mathrm{d}s\mathrm{d}t \\
&= \int_0^1 t\left[\int_0^1 s\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\,\frac{a+b}{2}, sc + (1-s)\,\frac{c+d}{2}\right) \mathrm{d}s\right] \mathrm{d}t \\
&= -\frac{2}{d-c}\int_0^1 t\frac{\partial f}{\partial t}\left(ta + (1-t)\,\frac{a+b}{2}, c\right) \mathrm{d}t \\
&+ \frac{2}{d-c}\int_0^1 \left[\int_0^1 t\frac{\partial f}{\partial t}\left(ta + (1-t)\,\frac{a+b}{2}, sc + (1-s)\,\frac{c+d}{2}\right) \mathrm{d}t\right] \mathrm{d}s \\
&= \frac{4}{(b-a)(d-c)}f\,(a,c) - \frac{4}{(b-a)(d-c)}\int_0^1 f\left(ta + (1-t)\,\frac{a+b}{2}, c\right) \mathrm{d}t \\
&- \frac{4}{(b-a)(d-c)}\int_0^1 f\left(a, sc + (1-s)\,\frac{c+d}{2}\right) \mathrm{d}s \\
&+ \frac{4}{(b-a)(d-c)}\int_0^1 f\left(ta + (1-t)\,\frac{a+b}{2}, sc + (1-s)\,\frac{c+d}{2}\right) \mathrm{d}s\mathrm{d}t \\
&= \frac{4}{(b-a)(d-c)}f\,(a,c) + \frac{8}{(b-a)^2(d-c)}\int_{\frac{a+b}{2}}^{a} f\,(u,c)\,\mathrm{d}u \\
&+ \frac{8}{(b-a)(d-c)^2}\int_{\frac{c+d}{2}}^{c} f\,(a,v)\,\mathrm{d}v + \frac{16}{(b-a)^2(d-c)^2}\int_{\frac{a+b}{2}}^{a}\int_{\frac{c+d}{2}}^{c} f(u,v)\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= \int_0^1 \int_0^1 (t-1)\,(s-1)\,\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)\,b, s\frac{c+d}{2} + (1-s)\,d\right) \mathrm{d}s\mathrm{d}t \\
&= \frac{4}{(b-a)(d-c)}f\,(b,d) - \frac{8}{(b-a)^2(d-c)}\int_{\frac{a+b}{2}}^{b} f\,(u,d)\,\mathrm{d}u \\
&- \frac{8}{(b-a)(d-c)^2}\int_{\frac{c+d}{2}}^{d} f\,(b,v)\,\mathrm{d}v + \frac{16}{(b-a)^2(d-c)^2}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f\,(u,v)\,\mathrm{d}v\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_0^1 \int_0^1 t\,(s-1)\,\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\,\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)\,d\right) \mathrm{d}s\mathrm{d}t \\
&= \frac{4}{(b-a)(d-c)}f\,(a,d) - \frac{8}{(b-a)^2(d-c)}\int_a^{\frac{a+b}{2}} f\,(u,d)\,\mathrm{d}u \\
&- \frac{8}{(b-a)(d-c)^2}\int_{\frac{c+d}{2}}^{d} f\,(a,v)\,\mathrm{d}v + \frac{16}{(b-a)^2(d-c)^2}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f\,(u,v)\,\mathrm{d}v\mathrm{d}u
\end{aligned}$$

and

$$I_4 = \int_0^1 \int_0^1 (t-1)\, s \frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)\, b, sc + (1-s)\frac{c+d}{2}\right) \mathrm{d}s\mathrm{d}t$$
$$= \frac{4}{(b-a)(d-c)} f(b,c) - \frac{8}{(b-a)^2(d-c)} \int_{\frac{a+b}{2}}^{b} f(u,c)\,\mathrm{d}u$$
$$- \frac{8}{(b-a)(d-c)^2} \int_c^{\frac{c+d}{2}} f(b,v)\,\mathrm{d}v + \frac{16}{(b-a)^2(d-c)^2} \int_{\frac{a+b}{2}}^{b} \int_c^{\frac{c+d}{2}} f(u,v)\,\mathrm{d}v\mathrm{d}u.$$

This completes the proof. □

We are now in a position to state and prove the following:

Theorem 2. *Assume* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ *is co-ordinated convex function on* Δ*. Then the following inequality holds:*

$$\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \int_a^b \int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u \right.$$
$$\left. -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(u,c) + f(u,d))\,\mathrm{d}u + \frac{1}{d-c}\int_c^d (f(a,v) + f(b,v))\,\mathrm{d}v\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{144}\left[\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|\right.$$
$$+ \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|$$
$$\left. + \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|\right].$$

Proof. From Lemma 1 and by using the property of modulus, we can write

$$(2.1) \quad \left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \int_a^b \int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u \right.$$
$$\left. -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(u,c) + f(u,d))\,\mathrm{d}u + \frac{1}{d-c}\int_c^d (f(a,v) + f(b,v))\,\mathrm{d}v\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{16}(J_1 + J_2 + J_3 + J_4),$$

where, using convexity on the co-ordinated of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$,

$$\begin{aligned}
J_1 &= \int_0^1 \int_0^1 ts \left|\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right)\right| \mathrm{d}s\mathrm{d}t \\
&\leq \int_0^1 \int_0^1 ts \left[ts \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right| + t(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|\right. \\
&\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right| + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|\right] \mathrm{d}s\mathrm{d}t \\
&= \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right| + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right| + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right| \\
&\quad + \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|.
\end{aligned}$$

Similarly

$$\begin{aligned}
J_2 &= \int_0^1 \int_0^1 (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)b, s\frac{c+d}{2} + (1-s)d\right)\right| \mathrm{d}s\mathrm{d}t \\
&\leq \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right| + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right| \\
&\quad + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|,
\end{aligned}$$

$$\begin{aligned}
J_3 &= \int_0^1 \int_0^1 t(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)d\right)\right| \mathrm{d}s\mathrm{d}t \\
&\leq \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right| + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right| + \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \\
&\quad + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|
\end{aligned}$$

and

$$\begin{aligned}
J_4 &= \int_0^1 \int_0^1 (1-t)s\left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)b, sc + (1-s)\frac{c+d}{2}\right)\right| \mathrm{d}s\mathrm{d}t \\
&\leq \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right| + \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right| \\
&\quad + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|.
\end{aligned}$$

Considering the results J_1, J_2, J_3, J_4 in (2.1) and making appropriate calculations, we get the conclusion of the Theorem 2. □

Our next result reads as:

Theorem 3. *Assume* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q, q>1$ *is co-ordinated convex function on* Δ. *Then the following inequality holds:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\int_a^b\int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u\right.$$
$$\left.-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(u,c)+f(u,d))\,\mathrm{d}u+\frac{1}{d-c}\int_c^d (f(a,v)+f(b,v))\,\mathrm{d}v\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{16\cdot 4^{1/q}(p+1)^{2/p}}\times$$
$$\left\{\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right]^{1/q}\right.$$
$$+\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\right]^{1/q}$$
$$+\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right]^{1/q}$$
$$\left.+\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\right]^{1/q}\right\},$$

for $\frac{1}{p}+\frac{1}{q}=1$.

Proof. According to Lemma 1 and Hölder's inequality, we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\int_a^b\int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u\right.$$
$$\left.-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(u,c)+f(u,d))\,\mathrm{d}u+\frac{1}{d-c}\int_c^d (f(a,v)+f(b,v))\,\mathrm{d}v\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\times$$
$$\left\{\left[\int_0^1\int_0^1 (ts)^p\,\mathrm{d}s\mathrm{d}t\right]^{1/p}K_1^{1/q}+\left[\int_0^1\int_0^1 ((1-t)(1-s))^p\,\mathrm{d}s\mathrm{d}t\right]^{1/p}K_2^{1/q}\right.$$
$$\left.+\left[\int_0^1\int_0^1 (t(1-s))^p\,\mathrm{d}s\mathrm{d}t\right]^{1/p}K_3^{1/q}+\left[\int_0^1\int_0^1 ((1-t)s)^p\,\mathrm{d}s\mathrm{d}t\right]^{1/p}K_4^{1/q}\right\}.$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is co-ordinated convex function, we have:

$$K_1 = \int_0^1 \int_0^1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t$$
$$\leq \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q \int_0^1 \int_0^1 ts\mathrm{d}s\mathrm{d}t + \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q \int_0^1 \int_0^1 t(1-s)\mathrm{d}s\mathrm{d}t$$
$$+\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q \int_0^1 \int_0^1 (1-t)s\mathrm{d}s\mathrm{d}t + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q \int_0^1 \int_0^1 (1-t)(1-s)\mathrm{d}s\mathrm{d}t =$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q\right]$$

and similarly

$$K_2 = \int_0^1 \int_0^1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)b, s\frac{c+d}{2} + (1-s)d\right)\right|^q \mathrm{d}s\mathrm{d}t \leq$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q\right],$$

$$K_3 = \int_0^1 \int_0^1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)d\right)\right|^q \mathrm{d}s\mathrm{d}t \leq$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q\right],$$

$$K_4 = \int_0^1 \int_0^1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)b, sc + (1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t \leq$$
$$\frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q\right].$$

A simple computation shows that

$$\int_0^1 \int_0^1 t^p s^p \mathrm{d}s\mathrm{d}t = \int_0^1 \int_0^1 (1-t)^p (1-s)^p \mathrm{d}s\mathrm{d}t$$
$$= \int_0^1 \int_0^1 t^p (1-s)^p \mathrm{d}s\mathrm{d}t = \int_0^1 \int_0^1 (1-t)^p s^p \mathrm{d}s\mathrm{d}t = \frac{1}{(p+1)^2}$$

and the proof is complete. □

Theorem 4. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q, q > 1$ *is co-ordinated convex function on* Δ*, then the following inequality holds:*

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \int_a^b \int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u \right.$$
$$\left. -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(u,c)+f(u,d))\,\mathrm{d}u + \frac{1}{d-c}\int_c^d (f(a,v)+f(b,v))\,\mathrm{d}v\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{64\cdot 3^{2/q}}\times$$
$$\left\{\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + 2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right.\right.$$
$$\left. +2\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q + 4\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right]^{1/q}$$
$$+\left[4\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q + 2\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q\right.$$
$$\left. +2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right]^{1/q}$$
$$+\left[2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right.$$
$$\left. +4\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + 2\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\right]^{1/q}$$
$$+\left[2\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q + 4\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right.$$
$$\left.\left. +\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + 2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right]^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and the power mean inequality, we have

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \int_a^b \int_c^d f(u,v)\,\mathrm{d}v\mathrm{d}u \right.$$
$$\left. -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(u,c)+f(u,d))\,\mathrm{d}u + \frac{1}{d-c}\int_c^d (f(a,v)+f(b,v))\,\mathrm{d}v\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\times$$
$$\left\{\left[\int_0^1\int_0^1 ts\,\mathrm{d}s\mathrm{d}t\right]^{1/p} M_1^{1/q} + \left[\int_0^1\int_0^1 (1-t)(1-s)\,\mathrm{d}s\mathrm{d}t\right]^{1/p} M_2^{1/q}\right.$$
$$\left. +\left[\int_0^1\int_0^1 t(1-s)\,\mathrm{d}s\mathrm{d}t\right]^{1/p} M_3^{1/q} + \left[\int_0^1\int_0^1 (1-t)s\,\mathrm{d}s\mathrm{d}t\right]^{1/p} M_4^{1/q}\right\}.$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is co-ordinated convex function, we have:

$$\begin{aligned}
M_1 &= \int_0^1 \int_0^1 ts \left|\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q \int_0^1 \int_0^1 t^2 s^2 \mathrm{d}s\mathrm{d}t \\
&\quad + \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q \int_0^1 \int_0^1 t^2 (s-s^2) \mathrm{d}s\mathrm{d}t \\
&\quad + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q \int_0^1 \int_0^1 (t-t^2) s^2 \mathrm{d}s\mathrm{d}t \\
&\quad + \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q \int_0^1 \int_0^1 (t-t^2)(s-s^2) \mathrm{d}s\mathrm{d}t \\
&= \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q \\
&\quad + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q
\end{aligned}$$

and similarly

$$\begin{aligned}
M_2 &= \int_0^1 \int_0^1 (1-t)(1-s) \left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)b, s\frac{c+d}{2} + (1-s)d\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|^q \\
&\quad + \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q,
\end{aligned}$$

$$\begin{aligned}
M_3 &= \int_0^1 \int_0^1 t(1-s) \left|\frac{\partial^2 f}{\partial t \partial s}\left(ta + (1-t)\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)d\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|^q + \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\
&\quad + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right|^q,
\end{aligned}$$

$$\begin{aligned}
M_4 &= \int_0^1 \int_0^1 (1-t)s \left|\frac{\partial^2 f}{\partial t \partial s}\left(t\frac{a+b}{2} + (1-t)b, sc + (1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\leq \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right|^q + \frac{1}{36}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \frac{1}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q \\
&\quad + \frac{1}{18}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right|^q.
\end{aligned}$$

Hence the proof of the theorem is complete. □

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