Received 11/09/12

SOME HERMITE-HADAMARD TYPE INEQUALITIES WITH APPLICATIONS

MARCELA V. MIHAI

ABSTRACT. We provide some new Hermite-Hadamard type inequalities for functions whose derivatives in absolute value are convex and we give some applications for special means.

1. INTRODUCTION

The Hermite-Hadamard inequality asserts that for every convex function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. Both inequalities hold in reversed direction if f is concave.

Since its descovery in 1883, Hermite-Hadamard's inequality [3] has been considered the most useful inequality. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the functions f.Over time, researchers have discovered many extensions, generalizations and applications of this inequality; see, for example, [1], [2], [4], [5], [6] and the references therein.

The main aim of this paper is to establish some Hermite-Hadamard inequalities which give an estimate between $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$ and $\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]$ for functions whose derivatives in absolute value are convex.

2. MAIN RESULTS

We assume throughout the present paper that $[a, b]$ is a subinterval of $\mathbb{R}$ and $f : [a, b] \longrightarrow \mathbb{R}$ is a function differentiable on (a, b) such that $f' \in L^1[a, b]$. In order to prove our main results we need the following lemma.

Lemma 1. *We have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] = \frac{b-a}{16}\times$$
$$\left[\int_0^1 tf'\left(ta+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t + \int_0^1 (t-1)\,f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right.$$
$$\left.+\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t + \int_0^1 (t-1)\,f'\left(t\frac{a+3b}{4}+(1-t)b\right)\mathrm{d}t\right],$$

2000 *Mathematics Subject Classification.* 26A51.

Key words and phrases. convex function, Hermite-Hadamard inequality, Hölder's inequality, Jensen's inequality.

for all $x \in [a,b]$.

Proof. We use the integration by parts and appropriate substitutions (such as $u = ta + (1-t)\frac{3a+b}{4}$, $v = t\frac{3a+b}{4} + (1-t)\frac{a+b}{2}$, ...) to show that

$$\begin{aligned}
&\frac{b-a}{16}\times \\
&\left[\int_0^1 tf'\left(ta+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t + \int_0^1 (t-1)\,f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right. \\
&\left.+\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t + \int_0^1 (t-1)\,f'\left(t\frac{a+3b}{4}+(1-t)b\right)\mathrm{d}t\right] \\
&\qquad = \frac{b-a}{16}\left[\frac{4}{a-b}f\left(a\right) - \frac{4}{a-b}\int_0^1 f\left(ta+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t\right. \\
&\qquad +\frac{4}{a-b}f\left(\frac{a+b}{2}\right) - \frac{4}{a-b}\int_0^1 f\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\mathrm{d}t \\
&\qquad +\frac{4}{a-b}f\left(\frac{a+b}{2}\right) - \frac{4}{a-b}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t \\
&\qquad \left.+\frac{4}{a-b}f\left(b\right) - \frac{4}{a-b}\int_0^1 f\left(t\frac{a+3b}{4}+(1-t)b\right)\mathrm{d}t\right] \\
&\qquad = \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x,
\end{aligned}$$

and the proof is completed. □

We are now in a position to state and prove the following:

Theorem 1. *Assume* $|f'|$ *is convex on* $[a,b]$. *Then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \\
&\le \frac{b-a}{48}\left[\left|f'\left(a\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right| + 2\left|f'\left(\frac{a+b}{2}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(b\right)\right|\right].
\end{aligned}$$

Proof. Using Lemma 1 and taking modulus, we infer from the convexity of $|f'|$ that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{16}\times \\
&\left[\int_0^1 t\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|\mathrm{d}t + \int_0^1 (1-t)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right. \\
&\left.+\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|\mathrm{d}t + \int_0^1 (1-t)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|\mathrm{d}t\right] \\
&\qquad = \frac{b-a}{16}\left(I_1+I_2+I_3+I_4\right),
\end{aligned}$$

where

$$I_1 = \int_0^1 t\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|\mathrm{d}t \le \int_0^1 \left[t^2\left|f'(a)\right| + (t-t^2)\left|f'\left(\frac{3a+b}{4}\right)\right|\right]\mathrm{d}t$$
$$= \frac{1}{3}\left|f'(a)\right| + \frac{1}{6}\left|f'\left(\frac{3a+b}{4}\right)\right|,$$

$$I_2 \le \frac{1}{6}\left|f'\left(\frac{3a+b}{4}\right)\right| + \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|, I_3 \le \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{1}{6}\left|f'\left(\frac{a+3b}{4}\right)\right|$$

and

$$I_4 \le \frac{1}{6}\left|f'\left(\frac{a+3b}{4}\right)\right| + \frac{1}{3}\left|f'(b)\right|.$$

This ends the proof. □

Corollary 1. *In the conditions of Theorem 1, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{16}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Moreover, if $|f'(x)| \le M$, for all $x \in [a,b]$, then we have the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{8}M.$$

Proof. It follows from Theorem 1 and convexity of $|f'|$. □

Our next results reads as:

Theorem 2. *Assume $|f'|^q$ is convex on $[a,b]$ for some fixed $q > 1$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{16(p+1)^{1/p}\cdot 2^{1/q}}\times$$
$$\left\{\left[\left|f'(a)\right|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]^{1/q} + \left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right]^{1/q} + \left[\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \left|f'(b)\right|^q\right]^{1/q}\right\},$$

for all $x \in [a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. According to Lemma 1, Hölder's inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{16}\times$$
$$\left[\left(\int_0^1 t^p\mathrm{d}t\right)^{1/p}\left((J_1)^{1/q} + (J_3)^{1/q}\right) + \left(\int_0^1 (1-t)^p\ dt\right)^{1/p}\left((J_2)^{1/q} + (J_4)^{1/q}\right)\right],$$

where

$$J_1=\int_0^1\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|^q \mathrm{d}t \le \int_0^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]\mathrm{d}t$$
$$=\frac{1}{2}\left[\left|f'(a)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right],$$

$$J_2\le\frac{1}{2}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right],\ J_3\le\frac{1}{2}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right]$$

and

$$J_4\le\frac{1}{2}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right].$$

The proof is complete. □

Corollary 2. *Suppose all the conditions of Theorem 2 are satisfied, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|$$
$$\le\frac{b-a}{16(p+1)^{1/p}\cdot 2^{3/q}}\left(1+3^{1/q}+5^{1/q}+7^{1/q}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

for all $x\in[a,b]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Theorem 2, convexity of $|f'|^q$ and the fact

$$\sum_{k=1}^n (u_k+v_k)^s\le\sum_{k=1}^n (u_k)^s+\sum_{k=1}^n (v_k)^s,\ u_k,v_k\ge 0,\ 1\le k\le n,\ 0\le s<1,$$

we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|$$
$$\le\frac{b-a}{16(p+1)^{1/p}}\left[(J_1)^{1/q}+(J_2)^{1/q}+(J_3)^{1/q}+(J_4)^{1/q}\right],$$

where

$$J_1^{1/q}\le\left\{\frac{1}{2}\left[\left|f'(a)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]\right\}^{1/q}\le\left\{\frac{1}{2}\left[\frac{7}{4}\left|f'(a)\right|^q+\frac{1}{4}\left|f'(b)\right|^q\right]\right\}^{1/q}$$
$$\le\frac{1}{2^{3/q}}\left[7^{1/q}\left|f'(a)\right|+\left|f'(b)\right|\right],$$

$$J_2^{1/q}\le\frac{1}{2^{3/q}}\left[5^{1/q}\left|f'(a)\right|+3^{1/q}\left|f'(b)\right|\right],\ J_3^{1/q}\le\frac{1}{2^{3/q}}\left[3^{1/q}\left|f'(a)\right|+5^{1/q}\left|f'(b)\right|\right],$$

and

$$J_4^{1/q}\le\frac{1}{2^{3/q}}\left[\left|f'(a)\right|+7^{1/q}\left|f'(b)\right|\right],$$

which completes the proof. □

Theorem 3. *If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{32\cdot 3^{1/q}}\times$$

$$\left\{\left[2\left|f'(a)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]^{1/q}+\left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$

$$\left.+\left[2\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right]^{1/q}+\left[\left|f'\left(\frac{a+3b}{4}\right)\right|^q+2\left|f'(b)\right|^q\right]^{1/q}\right\},$$

for all $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1, the convexity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{16}\times$$

$$\left[\left(\int_0^1 t\mathrm{d}t\right)^{1/p}\left((K_1)^{1/q}+(K_3)^{1/q}\right)+\left(\int_0^1(1-t)\ dt\right)^{1/p}\left((K_2)^{1/q}+(K_4)^{1/q}\right)\right],$$

where

$$K_1=\int_0^1 t\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|^q\mathrm{d}t\leq\int_0^1\left[t^2\left|f'(a)\right|^q+(t-t^2)\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]\mathrm{d}t$$

$$=\frac{1}{3}\left|f'(a)\right|^q+\frac{1}{6}\left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$

$$K_2=\int_0^1(1-t)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\leq\frac{1}{6}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q,$$

$$K_3=\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|^q\mathrm{d}t\leq\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'\left(\frac{a+3b}{4}\right)\right|^q$$

and

$$K_4=\int_0^1(1-t)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|^q\mathrm{d}t\leq\frac{1}{6}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{3}\left|f'(b)\right|^q$$

Hence the proof of the theorem is complete. □

Corollary 3. *In the Theorem 3 assumptions, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|$$

$$\leq\frac{b-a}{32\cdot 12^{1/q}}\left(1+5^{1/q}+7^{1/q}+11^{1/q}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

for all $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4. *Assume $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{16(p+1)^{1/p}}\times$$
$$\left[\left|f'\left(\frac{7a+b}{8}\right)\right|^q+\left|f'\left(\frac{5a+3b}{8}\right)\right|^q\right]^{1/q}+\left[\left|f'\left(\frac{3a+5b}{8}\right)\right|^q+\left|f'\left(\frac{a+7b}{8}\right)\right|^q\right],$$

for all $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1, Hölder's inequality for $q>1$ and Jensen's inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{16}\times$$
$$\left[\left(\int_0^1 t^p\mathrm{d}t\right)^{1/p}\left((J_1)^{1/q}+(J_3)^{1/q}\right)+\left(\int_0^1(1-t)^p\ dt\right)^{1/p}\left((J_2)^{1/q}+(J_4)^{1/q}\right)\right],$$

where

$$J_1=\int_0^1\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|^q\mathrm{d}t\leq\left|f'\left(\frac{a+\frac{3a+b}{4}}{2}\right)\right|^q=\left|f'\left(\frac{7a+b}{8}\right)\right|^q,$$

$$J_2=\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\leq\left|f'\left(\frac{\frac{3a+b}{4}+\frac{a+b}{2}}{2}\right)\right|^q=\left|f'\left(\frac{5a+3b}{8}\right)\right|^q,$$

$$J_3=\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|^q\mathrm{d}t\leq\left|f'\left(\frac{\frac{a+b}{2}+\frac{a+3b}{4}}{2}\right)\right|^q=\left|f'\left(\frac{3a+5b}{8}\right)\right|^q,$$

and

$$J_4=\int_0^1\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|^q\mathrm{d}t\leq\left|f'\left(\frac{\frac{a+3b}{4}+b}{2}\right)\right|^q=\left|f'\left(\frac{a+7b}{8}\right)\right|^q.$$

This completes the proof of the theorem. □

Corollary 4. *In the Theorem 4 assumptions and in addition that $|f'|^q$ is a linear map, we get the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{8(p+1)^{1/p}}\left|f'\left(a+b\right)\right|^q,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. It is a direct consequence of Theorem 4 and using the linearity of $|f'|^q$. □

Theorem 5. *Assume $|f'|$ is concave on $[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{32}\times$$
$$\left[\left|f'\left(\frac{11a+b}{12}\right)\right|+\left|f'\left(\frac{7a+5b}{12}\right)\right|+\left|f'\left(\frac{5a+7b}{12}\right)\right|+\left|f'\left(\frac{a+11b}{12}\right)\right|\right],$$

for all $x\in[a,b]$.

Proof. Using Lemma 1 and Jensen's inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{16}\times$$
$$\left[\int_0^1 t\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|\mathrm{d}t+\int_0^1(1-t)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right.$$
$$\left.+\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|\mathrm{d}t+\int_0^1(1-t)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|\mathrm{d}t\right]$$
$$\leq\frac{b-a}{16}\times\left[\left(\int_0^1 t\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1 t\left(ta+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t}{\int_0^1 t\mathrm{d}t}\right)\right|\right.$$
$$+\left(\int_0^1(1-t)\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1(1-t)\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\mathrm{d}t}{\int_0^1(1-t)\mathrm{d}t}\right)\right|$$
$$+\left(\int_0^1 t\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1 t\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t}{\int_0^1 t\mathrm{d}t}\right)\right|$$
$$\left.+\left(\int_0^1(1-t)\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1(1-t)\left(t\frac{a+3b}{4}+(1-t)b\right)\mathrm{d}t}{\int_0^1(1-t)\mathrm{d}t}\right)\right|\right]$$
$$=\frac{b-a}{32}\times$$
$$\left[\left|f'\left(\frac{11a+b}{12}\right)\right|+\left|f'\left(\frac{7a+5b}{12}\right)\right|+\left|f'\left(\frac{5a+7b}{12}\right)\right|+\left|f'\left(\frac{a+11b}{12}\right)\right|\right],$$

and the proof is complete. □

Corollary 5. *In the Theorem 5 assumptions and in addition that $|f'|^q$ is a linear map, we get the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\leq\frac{b-a}{16}\left|f'\left(a+b\right)\right|.$$

3. Applications to special means

We consider the means for arbitrary real numbers $a,b\in\mathbb{R}$. We take

- The arithmetic mean:

$$A(a,b)=\frac{a+b}{2}.$$

- The harmonic mean:

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}; a,b \in \mathbb{R}\setminus\{0\}.$$

- The logarithmic mean:

$$L(a,b) = \frac{ln\,|b| - ln\,|a|}{b-a};\; a,b \in \mathbb{R},\; |a| \neq |b|,\; a \neq b.$$

- Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}};\; a,b \in \mathbb{R},\; n \in \mathbb{Z}\setminus\{-1,0\},\; a \neq b.$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let* $a,b \in \mathbb{R},\; a < b,\; 0 \notin [a,b]$ *and* $n \in \mathbb{Z},\; |n| \geq 2$. *Then*

$$|A(a^n,b^n) + A^n(a,b) - 2L_n^n(a,b)| \leq \frac{b-a}{4}|n|\,A\left(|a|^{n-1},|b|^{n-1}\right).$$

Proof. The assertion follows from Corollary 1 when applied to the function $f(x) = x^n, x \in \mathbb{R},\; n \in \mathbb{Z},\; |n| \geq 2$. □

Proposition 2. *Let* $a,b \in \mathbb{R},\; a < b,\; 0 \notin [a,b]$ *and* $n \in \mathbb{Z},\; |n| \geq 2$. *Then for* $p,q > 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1$, *we have*

$$|A(a^n,b^n) + A^n(a,b) - 2L_n^n(a,b)| \leq \frac{(b-a)\,|n|}{4(p+1)\cdot 2^{3/q}}\left(1+3^{1/q}+5^{1/q}+7^{1/q}\right)A\left(|a|^{n-1},|b|^{n-1}\right).$$

Proof. The assertion follows from Corollary 2 when applied to the function $f(x) = x^n, x \in \mathbb{R},\; n \in \mathbb{Z},\; |n| \geq 2$. □

Proposition 3. *Let* $a,b \in \mathbb{R},\; a < b,\; 0 \notin [a,b]$ *and* $n \in \mathbb{Z},\; |n| \geq 2$. *Then for* $p,q > 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1$, *we have*

$$|A(a^n,b^n) + A^n(a,b) - 2L_n^n(a,b)| \leq \frac{(b-a)\,|n|}{8\cdot 12^{1/q}}\left(1+5^{1/q}+7^{1/q}+11^{1/q}\right)A\left(|a|^{n-1},|b|^{n-1}\right).$$

Proof. The assertion follows from Corollary 3 when applied to the function $f(x) = x^n, x \in \mathbb{R},\; n \in \mathbb{Z},\; |n| \geq 2$. □

Proposition 4. *Let* $a,b \in \mathbb{R},\; a < b,\; 0 \notin [a,b]$. *Then*

$$\left|A^{-1}(a,b) + A\left(a^{-1},b^{-1}\right) - 2L(a,b)\right| \leq \frac{b-a}{4}A\left(|a|^{-2},|b|^{-2}\right).$$

Proof. The assertion follows from Corollary 4 when applied to the function $f(x) = \frac{1}{x}, x \in [a,b]$. □

Remark 1. *Let* $a,b \in \mathbb{R},\; a < b$, *then* $a^{-1} > b^{-1}$ *and* $A^{-1}\left(a^{-1},b^{-1}\right) = H(a,b)$. *Hence one can get several inequalities containing harmonic mean and logarithmic means and we omit the details for the interested readers.*

References

[1] S. S. Dragomir, C. E. M. Pearce, *Selected Topic on Hermite-Hadamard Inequalities and Applications*, Melbourne and Adelaide, December, 2000.

[2] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applicationsto special means of real numbers and to trapeziodal formula*, Appl. Math. Lett., 11(**5**)(1998) 91-95.

[3] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann*, J. Math Pures Appl., 58(1893), 171-215.

[4] H. Kavurmaci, M. Avci, M. E. Özdemir, *New inequalities of Hermite-Hadamard type for convex functions with applications*, arXiv: 1006.1593v1[math. CA].

[5] M. A. Latif, *New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications*, RGMIA Research Report Collection, **15**(2012), Article 34, 13 pp.

[6] C. P. Niculescu, L.-E. Persson, *Convex Functions and their Applications. A Contemporary Approach.* CMS Books in Mathematics vol. **23**, Springer-Verlag, New York, 2006.

(Marcela V. Mihai) University of Craiova, Department of Mathematics, Street A. I. Cuza 13, Craiova, RO-200585, Romania

E-mail address: marcelamihai58@yahoo.com