

**SOME NUMERICAL RADIUS INEQUALITIES FOR POWER SERIES OF OPERATORS IN HILBERT SPACES**

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ABSTRACT. By the help of power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  we can naturally construct another power series that has as coefficients the absolute values of the coefficients of  $f$ , namely,  $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$ . Utilising these functions we show among others that

$$w[f(T)] \leq f_a[w(T)]$$

and

$$w[f(T)] \leq \frac{1}{2} \left[ f_a(\|T\|) + f_a(\|T^2\|^{1/2}) \right]$$

where  $w(T)$  denotes the numerical radius of the bounded linear operator  $T$  on a complex Hilbert space while  $\|T\|$  is its norm.

1. INTRODUCTION

The *numerical radius*  $w(T)$  of an operator  $T$  on  $H$  is given by [7, p. 8]:

$$(1.1) \quad w(T) = \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}.$$

Obviously, by (1.1), for any  $x \in H$  one has

$$(1.2) \quad |\langle Tx, x \rangle| \leq w(T) \|x\|^2.$$

It is well known that  $w(\cdot)$  is a norm on the Banach algebra  $B(H)$  of all bounded linear operators  $T : H \rightarrow H$ , i.e.,

- (i)  $w(T) \geq 0$  for any  $T \in B(H)$  and  $w(T) = 0$  if and only if  $T = 0$ ;
- (ii)  $w(\lambda T) = |\lambda| w(T)$  for any  $\lambda \in \mathbb{C}$  and  $T \in B(H)$ ;
- (iii)  $w(T + V) \leq w(T) + w(V)$  for any  $T, V \in B(H)$ .

This norm is equivalent with the operator norm. In fact, the following more precise result holds [7, p. 9]:

**Theorem 1** (Equivalent norm). *For any  $T \in B(H)$  one has*

$$(1.3) \quad w(T) \leq \|T\| \leq 2w(T).$$

Some improvements of (1.3) are as follows:

**Theorem 2** (Kittaneh, 2003 [11]). *For any operator  $T \in B(H)$  we have the following refinement of the first inequality in (1.3)*

$$(1.4) \quad w(T) \leq \frac{1}{2} \left( \|T\| + \|T^2\|^{1/2} \right).$$

Utilizing the *Cartesian decomposition* for operators, F. Kittaneh improved the inequality (1.3) as follows:

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1991 *Mathematics Subject Classification.* 47A63; 47A99.

*Key words and phrases.* Bounded linear operators, Functions of operators, Numerical radius, Power series.

**Theorem 3** (Kittaneh, 2005 [12]). *For any operator  $T \in B(H)$  we have*

$$(1.5) \quad \frac{1}{4} \|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2} \|T^*T + TT^*\|.$$

From a different perspective, we have the following result as well:

**Theorem 4** (Dragomir, 2007 [4]). *For any operator  $T \in B(H)$  we have*

$$(1.6) \quad w^2(T) \leq \frac{1}{2} \left[ w(T^2) + \|T\|^2 \right].$$

The following general result for the product of two operators holds [7, p. 37]:

**Theorem 5** (Holbrook, 1969 [9]). *If  $A, B$  are two bounded linear operators on the Hilbert space  $(H, \langle \cdot, \cdot \rangle)$ , then  $w(AB) \leq 4w(A)w(B)$ . In the case that  $AB = BA$ , then  $w(AB) \leq 2w(A)w(B)$ . The constant 2 is best possible here.*

The following results are also well known [7, p. 38].

**Theorem 6** (Holbrook, 1969 [9]). *If  $A$  is a unitary operator that commutes with another operator  $B$ , then*

$$(1.7) \quad w(AB) \leq w(B).$$

*If  $A$  is an isometry and  $AB = BA$ , then (1.7) also holds true.*

We say that  $A$  and  $B$  *double commute* if  $AB = BA$  and  $AB^* = B^*A$ . The following result holds [7, p. 38].

**Theorem 7** (Holbrook, 1969 [9]). *If the operators  $A$  and  $B$  double commute, then*

$$(1.8) \quad w(AB) \leq w(B) \|A\|.$$

As a consequence of the above, we have [7, p. 39]:

**Corollary 1.** *Let  $A$  be a normal operator commuting with  $B$ . Then*

$$(1.9) \quad w(AB) \leq w(A)w(B).$$

A related problem with the inequality (1.8) is to find the best constant  $c$  for which the inequality

$$w(AB) \leq cw(A) \|B\|$$

holds for any two commuting operators  $A, B \in B(H)$ . It is known that  $1.064 < c < 1.169$ , see [2], [13] and [14].

Motivated by the above results we establish in this paper some inequalities for the numerical radius of functions of operators defined by power series, which incorporate many fundamental functions of interest such as the exponential function, some trigonometric functions, the functions  $f(z) = (1 \pm z)^{-1}$ ,  $g(z) = \log(1 \pm z)^{-1}$  and others. Some examples of interest are also provided.

## 2. SOME INEQUALITIES FOR ONE OPERATOR

Now, by the help of power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely,  $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$ . It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients  $a_n \geq 0$ , then  $f_a = f$ .

The following simple result provides some nice inequalities for operator functions defined by power series.

**Theorem 8.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with complex coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . For any  $T \in B(H)$  we have the inequality

$$(2.1) \quad w[f(T)] \leq f_a[w(T)]$$

provided  $w(T) < R$ , and the inequality

$$(2.2) \quad w[f(T)] \leq \frac{1}{2} \left[ f_a(\|T\|) + f_a\left(\|T^2\|^{1/2}\right) \right]$$

provided  $\|T\| < R$ .

*Proof.* Let  $m \in \mathbb{N}$  with  $m \geq 1$ . Utilising the properties of the numerical radius we have

$$(2.3) \quad w \left[ \sum_{n=0}^m a_n T^n \right] \leq \sum_{n=0}^m |a_n| w(T^n) \leq \sum_{n=0}^m |a_n| w^n(T).$$

Since the series  $\sum_{n=0}^{\infty} |a_n| w^n(T)$  is convergent on  $\mathbb{R}$ , the series  $\sum_{n=0}^{\infty} a_n T^n$  is convergent in  $B(H)$  and by the continuity of the numerical radius we have

$$\lim_{m \rightarrow \infty} w \left[ \sum_{n=0}^m a_n T^n \right] = w \left[ \sum_{n=0}^{\infty} a_n T^n \right],$$

then by letting  $m \rightarrow \infty$  in the inequality (2.3) we deduce the desired result (2.1).

Utilising the properties of the numerical radius and the Kittaneh inequality (1.4) we also have

$$(2.4) \quad \begin{aligned} w \left[ \sum_{n=0}^m a_n T^n \right] &\leq \sum_{n=0}^m |a_n| w(T^n) \leq \frac{1}{2} \sum_{n=0}^m |a_n| \left( \|T^n\| + \|T^{2n}\|^{1/2} \right) \\ &\leq \frac{1}{2} \sum_{n=0}^m |a_n| \left( \|T\|^n + \|T^2\|^{n/2} \right) \\ &= \frac{1}{2} \left[ \sum_{n=0}^m |a_n| \|T\|^n + \sum_{n=0}^m |a_n| \left( \|T^2\|^{1/2} \right)^n \right]. \end{aligned}$$

Since the series  $\sum_{n=0}^{\infty} |a_n| \|T\|^n$ ,  $\sum_{n=0}^m |a_n| \left( \|T^2\|^{1/2} \right)^n$  are convergent on  $\mathbb{R}$ , the series  $\sum_{n=0}^{\infty} a_n T^n$  is convergent in  $B(H)$ , then by letting  $m \rightarrow \infty$  in the inequality (2.4) we deduce the desired result (2.2).  $\square$

**Corollary 2.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with nonnegative coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . For any  $T \in B(H)$  we have the inequality

$$(2.5) \quad w[f(T)] \leq f[w(T)]$$

if  $w(T) < R$ , and the inequality

$$(2.6) \quad w[f(T)] \leq \frac{1}{2} \left[ f(\|T\|) + f\left(\|T^2\|^{1/2}\right) \right]$$

if  $\|T\| < R$ .

From a different perspective, we have:

**Theorem 9.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with complex coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . For any  $T \in B(H)$  with  $\|T\|^2 < R$  and  $z \in \mathbb{C}$  with  $|z|^2 < r$  we have the inequality

$$(2.7) \quad w^2 [f(zT)] \leq \frac{1}{2} f_a(|z|^2) \left[ f_a(w(T^2)) + f_a(\|T\|^2) \right]$$

and the inequality

$$(2.8) \quad w^2 [f(zT)] \leq f_a(|z|^2) f_a \left( \left\| \frac{T^*T + TT^*}{2} \right\| \right).$$

*Proof.* Let  $m \in \mathbb{N}$  with  $m \geq 1$ . Utilising the properties of the numerical radius we have

$$(2.9) \quad w^2 \left[ \sum_{n=0}^m a_n z^n T^n \right] \leq \left( \sum_{n=0}^m |a_n| |z|^n w(T^n) \right)^2.$$

By the weighted Cauchy-Bunyakovsky-Schwarz discrete inequality we have

$$(2.10) \quad \left( \sum_{n=0}^m |a_n| |z|^n w(T^n) \right)^2 \leq \sum_{n=0}^m |a_n| |z|^{2n} \sum_{n=0}^m |a_n| w^2(T^n).$$

Now, on writing the inequality (1.6) for the operators  $T^n$  we have

$$w^2(T^n) \leq \frac{1}{2} [w(T^{2n}) + \|T^n\|^2] \leq \frac{1}{2} [w^n(T^2) + \|T\|^{2n}]$$

for any  $n \in \mathbb{N}$ , which implies that

$$(2.11) \quad \sum_{n=0}^m |a_n| w^2(T^n) \leq \frac{1}{2} \left[ \sum_{n=0}^m |a_n| w^n(T^2) + \sum_{n=0}^m |a_n| \|T\|^{2n} \right].$$

On making use of the inequalities (2.9) - (2.11) we get

$$(2.12) \quad w^2 \left[ \sum_{n=0}^m a_n z^n T^n \right] \leq \frac{1}{2} \sum_{n=0}^m |a_n| |z|^{2n} \left[ \sum_{n=0}^m |a_n| w^n(T^2) + \sum_{n=0}^m |a_n| \|T\|^{2n} \right].$$

Since the series  $\sum_{n=0}^m |a_n| |z|^{2n}$ ,  $\sum_{n=0}^m |a_n| w^n(T^2)$  and  $\sum_{n=0}^m |a_n| \|T\|^{2n}$  are convergent on  $\mathbb{R}$  and  $\sum_{n=0}^m a_n z^n T^n$  is convergent on  $B(H)$ , then by letting  $m \rightarrow \infty$  in the inequality (2.12) we deduce the desired result (2.7).

Now, on making use of Kittaneh's inequality (1.5) we also have

$$w^2(T^n) \leq [w^2(T)]^n \leq \left\| \frac{T^*T + TT^*}{2} \right\|^n$$

for any  $n \in \mathbb{N}$ , which implies

$$\sum_{n=0}^m |a_n| w^2(T^n) \leq \sum_{n=0}^m |a_n| \left\| \frac{T^*T + TT^*}{2} \right\|^n.$$

By the inequalities (2.9) and (2.10) we then get

$$w^2 \left[ \sum_{n=0}^m a_n z^n T^n \right] \leq \sum_{n=0}^m |a_n| |z|^{2n} \sum_{n=0}^m |a_n| \left\| \frac{T^*T + TT^*}{2} \right\|^n$$

for any  $m \in \mathbb{N}$  with  $m \geq 1$ .

The proof follows now as above and we get the desired inequality (2.8).  $\square$

**Corollary 3.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with nonnegative coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . For any  $T \in B(H)$  with  $\|T\|^2 < R$  and  $z \in \mathbb{C}$  with  $|z|^2 < r$  we have the inequality*

$$(2.13) \quad w^2 [f(zT)] \leq \frac{1}{2} f(|z|^2) \left[ f(w(T^2)) + f(\|T\|^2) \right]$$

and the inequality

$$(2.14) \quad w^2 [f(zT)] \leq f(|z|^2) f \left( \left\| \frac{T^*T + TT^*}{2} \right\| \right).$$

### 3. SOME INEQUALITIES FOR TWO OPERATORS

We start with the following result:

**Theorem 10.** *Let  $A, B \in B(H)$  and  $k > 0$  such that*

$$(3.1) \quad w(AB) \leq k^2 w(A) w(B).$$

*If  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is a function defined by power series with complex coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$  and  $k^p w^p(A), k^q w^q(B) < R$ , for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ , then we have the inequalities*

$$(3.2) \quad w(f(AB)) \leq \min \{M_1, M_2\}$$

where

$$M_1 := f_a^{1/p}(k^p w^p(A)) f_a^{1/q}(k^q w^q(B))$$

and

$$M_2 := \frac{f_a(k^p w^p(A)) f_a(k^q w^q(B))}{f_a(k^{(p+q-2)} w^{p-1}(A) w^{q-1}(B))}.$$

*Proof.* By the properties of the numerical radius and by (3.3) we have

$$(3.3) \quad w[(AB)^n] \leq w^n(AB) \leq k^{2n} w^n(A) w^n(B)$$

for any  $n \in \mathbb{N}$ .

Let  $m \in \mathbb{N}$  with  $m \geq 1$ . We have by the above inequality

$$(3.4) \quad w \left[ \sum_{n=0}^m a_n (AB)^n \right] \leq \sum_{n=0}^m |a_n| w[(AB)^n] \leq \sum_{n=0}^m |a_n| k^{2n} w^n(A) w^n(B).$$

By Hölder's weighted inequality we have

$$(3.5) \quad \sum_{n=0}^m |a_n| k^{2n} w^n(A) w^n(B) \leq \left( \sum_{n=0}^m |a_n| k^{pn} w^{pn}(A) \right)^{1/p} \left( \sum_{n=0}^m |a_n| k^{qn} w^{qn}(B) \right)^{1/q}.$$

Then by (3.4) and by (3.5) we get

$$(3.6) \quad w \left[ \sum_{n=0}^m a_n (AB)^n \right] \leq \left( \sum_{n=0}^m |a_n| k^{pn} w^{pn}(A) \right)^{1/p} \left( \sum_{n=0}^m |a_n| k^{qn} w^{qn}(B) \right)^{1/q}$$

for any  $m \in \mathbb{N}$  with  $m \geq 1$ .

Since the series whose partial sums are involved in (3.6) are convergent, then by taking  $m \rightarrow \infty$  in (3.6) we deduce the first inequality in (3.2).

Further, by utilizing the following Hölder's type inequality obtained by Dragomir and Sándor in 1990 [5] (see also [3, Corollary 2.34]):

$$(3.7) \quad \sum_{k=0}^n m_k |x_k|^p \sum_{k=0}^n m_k |y_k|^q \geq \sum_{k=0}^n m_k |x_k y_k| \sum_{k=0}^n m_k |x_k|^{p-1} |y_k|^{q-1},$$

that holds for nonnegative numbers  $m_k$  and complex numbers  $x_k, y_k$  where  $k \in \{0, \dots, n\}$ , we observe that the convergence of the series  $\sum_{k=0}^{\infty} m_k |x_k|^p$  and  $\sum_{k=0}^{\infty} m_k |y_k|^q$  imply the convergence of the series  $\sum_{k=0}^{\infty} m_k |x_k|^{p-1} |y_k|^{q-1}$ .

Utilising (3.7) we can state that

$$\sum_{n=0}^m |a_n| k^{2n} w^n(A) w^n(B) \leq \frac{\sum_{n=0}^m |a_n| k^{pn} w^{pn}(A) \sum_{n=0}^m |a_n| k^{qn} w^{qn}(B)}{\sum_{n=0}^m |a_n| k^{(p+q-2)n} w^{(p-1)n}(A) w^{(q-1)n}(B)}$$

for any  $m \in \mathbb{N}$  with  $m \geq 1$ .

This together with (3.4) provides

$$(3.8) \quad w \left[ \sum_{n=0}^m a_n (AB)^n \right] \leq \frac{\sum_{n=0}^m |a_n| k^{pn} w^{pn}(A) \sum_{n=0}^m |a_n| k^{qn} w^{qn}(B)}{\sum_{n=0}^m |a_n| k^{(p+q-2)n} w^{(p-1)n}(A) w^{(q-1)n}(B)}$$

for any  $m \in \mathbb{N}$  with  $m \geq 1$ .

Since all the series whose partial sums are involved in (3.8) are convergent, then by taking  $n \rightarrow \infty$  in (3.8) we deduce the second inequality in (3.2).  $\square$

**Remark 1.** If we take  $p = q = 2$  in the first inequality in (3.2) we have

$$(3.9) \quad w^2(f(AB)) \leq f_a(k^2 w^2(A)) f_a(k^2 w^2(B))$$

provided  $k^2 w^2(A), k^2 w^2(B) < R$ .

**Corollary 4.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with complex coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . Then for any  $A, B \in B(H)$  with  $2^p w^p(A), 2^q w^q(B) < R$ , for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ , we have the inequalities

$$(3.10) \quad w(f(AB)) \leq \min\{N_1, N_2\}$$

where

$$N_1 := f_a^{1/p}(2^p w^p(A)) f_a^{1/q}(2^q w^q(B))$$

and

$$N_2 := \frac{f_a(2^p w^p(A)) f_a(2^q w^q(B))}{f_a(2^{(p+q-2)} w^{p-1}(A) w^{q-1}(B))}.$$

If  $A, B \in B(H)$  are commutative with  $2^{p/2} w^p(A), 2^{q/2} w^q(B) < R$ , for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ , then we have the inequalities

$$(3.11) \quad w(f(AB)) \leq \min\{P_1, P_2\}$$

where

$$P_1 := f_a^{1/p}(2^{p/2} w^p(A)) f_a^{1/q}(2^{q/2} w^q(B))$$

and

$$P_2 := \frac{f_a(2^{p/2} w^p(A)) f_a(2^{q/2} w^q(B))}{f_a(2^{(p+q-2)/2} w^{p-1}(A) w^{q-1}(B))}.$$

The proof of the inequality (3.10) follows by Theorem 5 since in this case we can take  $k = 2$  in (3.1), while the inequality (3.11) follows by the commutative case in which case we can take  $k = \sqrt{2}$  in (3.1).

The case of commuting operators can be treated in a different way as well:

**Proposition 1.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with complex coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . If  $A, B \in B(H)$  are commutative with  $w^p(A), w^q(B) < R$ , for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ , then we have the inequalities*

$$(3.12) \quad w(f(AB)) \leq 2 \min \{Q_1, Q_2\}$$

where

$$Q_1 := f_a^{1/p}(w^p(A)) f_a^{1/q}(w^q(B))$$

and

$$Q_2 := \frac{f_a(w^p(A)) f_a(w^q(B))}{f_a(w^{p-1}(A) w^{q-1}(B))}.$$

*Proof.* Since  $A, B \in B(H)$  are commutative, then for any  $n \in \mathbb{N}$  the operators  $A^n$  and  $B^n$  are commutative and  $A^n B^n = (AB)^n$ .

Applying Theorem 5 for the commutative case, we have

$$w((AB)^n) = w(A^n B^n) \leq 2w(A^n) w(B^n) \leq 2w^n(A) w^n(B)$$

for any  $n \in \mathbb{N}$ .

Let  $m \in \mathbb{N}$  with  $m \geq 1$ . We have by the above inequality

$$w \left[ \sum_{n=0}^m a_n (AB)^n \right] \leq \sum_{n=0}^m |a_n| w[(AB)^n] \leq 2 \sum_{n=0}^m |a_n| w^n(A) w^n(B).$$

Now, on making use of a similar approach to the one employed in the proof of Theorem 10 we deduce the desired result (3.12).  $\square$

**Remark 2.** *If we take  $p = q = 2$  in the first inequality in (3.12) we get*

$$(3.13) \quad w^2(f(AB)) \leq 2f_a(w^2(A)) f_a(w^2(B))$$

provided  $w^2(A), w^2(B) < R$ .

As pointed out in the introduction, the inequality

$$(3.14) \quad w(AB) \leq cw(A) \|B\|$$

holds for any two commuting operators  $A, B \in B(H)$  and for some  $c > 1$ . It is known that  $1.064 < c < 1.169$ , see [2], [13] and [14].

**Proposition 2.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with complex coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . If  $A, B \in B(H)$  are commutative with  $w^p(A), \|B\|^q < R$ , for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ , then we have the inequalities*

$$(3.15) \quad w(f(AB)) \leq c \min \{Q_1, Q_2\}$$

where

$$S_1 := f_a^{1/p}(w^p(A)) f_a^{1/q}(\|B\|^q)$$

and

$$S_2 := \frac{f_a(w^p(A)) f_a(\|B\|^q)}{f_a(w^{p-1}(A) \|B\|^{q-1})}.$$

Moreover, if the operators  $A$  and  $B$  double commute, then the constant  $c$  can be taken to be 1 in (3.15).

*Proof.* Applying the inequality (3.14) for the commuting operators  $A^n$  and  $B^n$  with  $n \in \mathbb{N}$  we have

$$w((AB)^n) = w(A^n B^n) \leq cw(A^n) \|B^n\| \leq cw^n(A) \|B\|^n$$

for any  $n \in \mathbb{N}$ .

On making use of a similar argument as in the proof of Theorem 10 we deduce the desired inequality (3.15).

If the operators  $A$  and  $B$  double commute, then the operators  $A^n$  and  $B^n$  also double commute and by Theorem 7 we deduce the second part of the proposition.  $\square$

From a different perspective, we have the following result as well:

**Proposition 3.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a function defined by power series with complex coefficients and convergent on the open disk  $D(0, R) \subset \mathbb{C}$ ,  $R > 0$ . If  $A, B \in B(H)$  such that  $\|A\|^2, \|B\|^2 < R$ , then*

$$(3.16) \quad w(f(AB)) \leq \begin{cases} f_a\left(\left\|\frac{AA^*+B^*B}{2}\right\|\right) \\ f_a\left(\left\|\frac{AA^*+BB^*}{2}\right\|\right) \end{cases} \leq \frac{f_a(\|A\|^2) + f_a(\|B\|^2)}{2}.$$

*Proof.* We use the following two inequalities obtained by Kittaneh in [12]

$$w(AB) \leq \begin{cases} \left\|\frac{AA^*+B^*B}{2}\right\| \\ \left\|\frac{AA^*+BB^*}{2}\right\| \end{cases}$$

for any  $A, B \in B(H)$ .

Let  $n \in \mathbb{N}$ . We have by the above inequalities

$$w[(AB)^n] \leq w^n(AB) \leq \begin{cases} \left\|\frac{AA^*+B^*B}{2}\right\|^n \\ \left\|\frac{AA^*+BB^*}{2}\right\|^n \end{cases} \leq \frac{1}{2} [\|A\|^{2n} + \|B\|^{2n}].$$

The proof follows now as above and the details are omitted.  $\square$



## 4. EXAMPLES

As some natural examples that are useful for applications, we can point out that, if

$$(4.1) \quad \begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z}, \quad z \in D(0, 1); \\ g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \quad z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \quad z \in \mathbb{C}; \\ l(z) &= \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \quad z \in D(0, 1); \end{aligned}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$(4.2) \quad \begin{aligned} f_A(z) &= \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \ln \frac{1}{1-z}, \quad z \in D(0, 1); \\ g_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbb{C}; \\ h_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \quad z \in \mathbb{C}; \\ l_A(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in D(0, 1). \end{aligned}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$(4.3) \quad \begin{aligned} \exp(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad z \in \mathbb{C}, \\ \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0, 1); \\ \sin^{-1}(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \quad z \in D(0, 1); \\ \tanh^{-1}(z) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0, 1) \\ {}_2F_1(\alpha, \beta, \gamma, z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \quad \alpha, \beta, \gamma > 0, \\ &z \in D(0, 1); \end{aligned}$$

where  $\Gamma$  is *Gamma function*.

For any operator  $T \in B(H)$  with  $w(T) < 1$ , by making use of the inequality (2.1) we have the simple inequalities

$$\begin{aligned} w \left[ (I \pm T)^{-1} \right] &\leq [1 - w(T)]^{-1}, \\ w \left[ \ln (I \pm T)^{-1} \right] &\leq \ln [1 - w(T)]^{-1}, \\ w \left[ \sin^{-1}(T) \right] &\leq \sin^{-1} [w(T)] \end{aligned}$$

and

$$w [ {}_2F_1(\alpha, \beta, \gamma, T) ] \leq {}_2F_1(\alpha, \beta, \gamma, w(T)).$$

For any operator  $T \in B(H)$  we also have

$$\begin{aligned} w [\exp(T)] &\leq \exp [w(T)], \\ w [\sin(T)], w [\sinh(T)] &\leq \sinh (w(T)) \end{aligned}$$

and

$$w [\cos(T)], w [\cosh(T)] \leq \cosh (w(T)).$$

Similar inequalities may be stated by employing the other results obtained for one or two operators. However, the details are left to the interested reader.

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