Received 20/09/12

A NOTE ON INVARIANT NORMS

LOREDANA CIURDARIU

ABSTRACT. The aim of this paper is to present several inequalities concerning unitarily invariant Hilbert-Schmidt norm $||.||_2$ using the recent generalizations of Young inequality. These inequalities contain Specht ratio and Kantorovich constant and are given for positive definite matrices from M_n.

1. INTRODUCTION

We will consider below $M_{m,n}$ as being the space of $m \times n$ complex matrices ($M_n = M_{n,n}$) and $||.||$ any unitarily invariant norm on M_n. We also consider for $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm of A which is defined by $||A||_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)}$, where $s_1(A) \geq ... \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = \sqrt{tr(AA^*)}$, arranged in decreasing order and repeated according to multiplicity, see [3]. Moreover, $||A||_2 = \sqrt{tr(AA^*)}$, tr being the usual trace functional.

For positive real numbers a, b and $\nu \in [0,1]$, S. Furuichi (see [1], Lemma 3.1) showed the following inequality:

$$S\left(\sqrt{\frac{a}{b}}\right) a^{1-\nu} b^{\nu} \geq (1-\nu)a + \nu b - r(\sqrt{a} - \sqrt{b})^2, \tag{1}$$

where $r \equiv \min\{\nu, 1-\nu\}$.

For positive real numbers, $a,\ b > 0$ and $\mu \in [0,1]$, Zuo H., Shi G. and Fujii M., see [7], showed the following result:

$$a \nabla_{\mu} b \geq K(h,2)^r a^{1-\mu} b^{\mu}, \tag{2}$$

where $r = \min\{\mu, 1-\mu\}$, $h = \frac{b}{a}$, $a\nabla_{\mu} b = (1-\mu)a + \mu b$ and $K(t,2) = \frac{(t+2)^2}{4t}$, $t > 0$ is the Kantorovich constant.

By Lemma 4.1, S. Furuichi showed in [1] a reverse difference inequality of the refined Young inequality given by F. Kittaneh and Y. Manasrah in [5] which proves that for positive real numbers $a,\ b$ and $\nu \in [0,1]$ we have:

$$\omega L(\sqrt{a}, \sqrt{b}) \log S(\frac{a}{b}) \geq (1-\nu)a + \nu b - a^{1-\nu} b^{\nu} - r(\sqrt{a} - \sqrt{b})^2, \tag{3}$$

Date: July 27, 2012.

2010 *Mathematics Subject Classification.* 15A45, 15A60.

Key words and phrases. Young inequality, Kantorovich constant, Specht ratio, unitarily invariant norms, positive definite matrices.

where $\omega = \max\{\sqrt{a}, \sqrt{b}\}$.

We need also to use the next inequality which was given in Theorem 2.1, see [6]. For all $a,\ b \geq 1$ and $\lambda \in (0,1)$ we have:

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)\log^2(\frac{a}{b}) \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)\log^2(\frac{a}{b}), \tag{4}$$

where $r = \min\{\lambda, 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

We recall also the Application 3.2 from [6].

For $0 < a, b \leq 1$ and $\lambda \in (0,1)$, the following inequality takes place, with $\lambda,\ A(\lambda),\ B(\lambda)$ given in Theorem 2.1:

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)ab\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)ab\log^2\left(\frac{a}{b}\right).$$

2. The Young's type inequalities for positive definite matrices

Using the refined scalar Young inequality we establish the following inequality for the Hilbert-Schmidt norm of two positive definite matrices using the technique given in [4], [3] and [7].

Theorem 1. *Let $A,\ B,\ X \in M_n$ and positive real numbers $m,\ m',\ M,\ M'$ such that A and B are positive definite and satisfy either of the following conditions:*

(i) $0 < m' \leq \frac{1}{||A^{-1}||_2},\ ||A||_2 \leq m < M \leq \frac{1}{||B^{-1}||_2},\ ||B||_2 \leq M'$

(ii) $0 < m' \leq \frac{1}{||B^{-1}||_2},\ ||B||_2 \leq m < M \leq \frac{1}{||A^{-1}||_2},\ ||A||_2 \leq M'$.

Then

$$||(1-\nu)AX + \nu XB||_2^2 \geq K(h,2)^{2r}||A^{1-\nu}XB^\nu||_2^2$$

for all $\nu \in [0,1]$, where $r = \min\{\nu, 1-\nu\}$ and $h = \frac{M}{m},\ h' = \frac{M'}{m'}$.

Proof. As in the proof of Theorem 3.1, see [3], A and B are positive definite matrices so are unitarily diagonalizable and then there are unitary matrices $U,\ V \in M_n$ which satisfy:

$$A = U\Lambda_1 U^*, \qquad B = V\Lambda_2 V^*,$$

where $\Lambda_1 = diag(\lambda_1, ..., \lambda_n)$, $\Lambda_2 = diag(\mu_1, ..., \mu_n)$, $\lambda_i,\ \mu_i > 0,\ i = 1, ..., n$.

Taking $Y = U^*XV = [y_{ij}]$ we have

$$(1-\nu)AX + \nu XB = U[(1-\nu)\Lambda_1 Y + \nu Y\Lambda_2]V^* = U[((1-\nu)\lambda_i + \nu\mu_j)y_{ij}]V^*$$

and

$$A^{1-\nu}XB^\nu = U[\lambda_i^{1-\nu}\mu_j^\nu y_{ij}]V^*.$$

Using inequality (1), we have

$$||(1-\nu)AX + \nu XB||_2^2 = \sum_{i,j=1}^n ((1-\nu)\lambda_i + \nu\mu_j)^2|y_{ij}|^2 \geq$$

$$\geq \sum_{i,j=1}^{n} K\left(\frac{\mu_j}{\lambda_i},2\right)^{2r} (\lambda_i^{1-\nu}\mu_j^{\nu})^2|y_{ij}|^2 \geq K(h,2)^{2r} \sum_{i,j=1}^{n} (\lambda_i^{1-\nu}\mu_j^{\nu})^2|y_{ij}|^2 =$$

$$= K(h,2)^{2r}||A^{1-\nu}XB^{\nu}||_2^2.$$

From hypothesis the matrices A and B are invertible and then so are the matrices Λ_1 and Λ_2. By construction of Λ_1 and Λ_2 we can observe that $||A||_2 = ||\Lambda_1||_2$ and $||B||_2 = ||\Lambda_2||_2$.

Thus $||\Lambda_1||_2 = \sqrt{\lambda_1^2 + ... + \lambda_n^2} = ||A||_2$, $||\Lambda_2||_2 = \sqrt{\mu_1^2 + ... + \mu_n^2} = ||B||_2$ and $||\Lambda_1^{-1}||_2 = \sqrt{\frac{1}{\lambda_1^2} + ... + \frac{1}{\lambda_n^2}} = ||A^{-1}||_2$, $||\Lambda_2^{-1}||_2 = \sqrt{\frac{1}{\mu_1^2} + ... + \frac{1}{\mu_n^2}} = ||B^{-1}||_2$.

Then in the first case, we have

$$\frac{1}{m} \leq \frac{1}{||A||_2} \leq \frac{1}{\lambda_i} \leq ||A^{-1}||_2 \leq \frac{1}{m'} \text{ and } M \leq \frac{1}{||B^{-1}||_2} \leq \mu_j \leq ||B||_2 \leq M'$$

and in the second case,

$$\frac{1}{M'} \leq \frac{1}{||A||_2} \leq \frac{1}{\lambda_i} \leq ||A^{-1}||_2 \leq \frac{1}{M} \text{ and } m' \leq \frac{1}{||B^{-1}||_2} \leq \mu_j \leq ||B||_2 \leq m.$$

Therefore in the first case,

$$h = \frac{M}{m} \leq \frac{1}{||A||_2||B^{-1}||_2} \leq \frac{\mu_j}{\lambda_i} \leq ||A^{-1}||_2||B||_2 \leq \frac{M'}{m'} = h',$$

and we will use the fact that function $K(x,2)$ is increasing for $x > 1$.

In the second case,

$$0 < \frac{1}{h'} = \frac{m'}{M'} \leq \frac{1}{||A||_2||B^{-1}||_2} \leq \frac{\mu_j}{\lambda_i} \leq ||A^{-1}||_2||B||_2 \leq \frac{m}{M} = \frac{1}{h} < 1$$

and we will use the fact that $K(x,2)$ is decreasing for $0 < x < 1$.

By the method from the proof of Theorem 7, see [7], we have

$$\min_{i,j=\overline{1,n}} K\left(\frac{\mu_j}{\lambda_i},2\right) = K(h,2) = \min_{h\leq x\leq h'} K(x,2),$$

using that $K(\frac{1}{h},2) = K(h,2),\ h > 0$.

∎

Theorem 2. *Let $A,\ B,\ X \in M_n$, A and B are positive definite and $||A||_2||B^{-1}||_2 < 1$. Then we have,*

$$||(1-\nu)AX + \nu XB||_2^2 \geq K\left(\frac{1}{||A||_2||B^{-1}||_2},2\right)^{2r} ||A^{1-\nu}XB^{\nu}||_2^2$$

for all $\nu \in [0,1]$, where $r = \min\{\nu, 1-\nu\}$.

Using now the reverse ratio inequality of the refined Young inequality for scalars, see [1], Lemma 3.1 we find the following result:

Theorem 3. *Let $A,\ B,\ X \in M_n$ with A and B positive definite with $m = \min\{\frac{1}{||A^{-1}||_2}, \frac{1}{||B^{-1}||_2}\}$ and $M = \max\{||A||_2, ||B||_2\}$. If $0 \leq \nu \leq 1$, then*

$$(1-\nu)||AX||_2^2 + \nu||XB||_2^2 \leq S(h)||A^{1-\nu}XB^{\nu}||_2^2 + r||AX - XB||_2^2,$$

where $h = \frac{M}{m}$, $r = \min\{\nu, 1-\nu\}$ and $S(t) = \frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}}$, $t > 0,\ t \neq 1$ is the Specht ratio.

Proof. As in the proof of Theorem 3.1, see [3], A and B are positive definite matrices so are unitarily diagonalizable and then there are unitary matrices $U,\ V \in M_n$ which satisfy:

$$A = U\Lambda_1 U^*, \qquad B = V\Lambda_2 V^*,$$

where $\Lambda_1 = diag(\lambda_1, ..., \lambda_n)$, $\Lambda_2 = diag(\mu_1, ..., \mu_n)$, $\lambda_i,\ \mu_i > 0,\ i = 1, ..., n$. Taking $Y = U^*XV = [y_{ij}]$ we have

$$(1-\nu)AX = U[(1-\nu)\Lambda_1 Y]V^* = U[((1-\nu)\lambda_i)y_{ij}]V^*,$$

$$\nu XB = U[(\nu\mu_j)y_{ij}]V^*,$$

$$AX - XB = U[(\lambda_i - \mu_j)y_{ij}]V^*$$

and

$$A^{1-\nu}XB^{\nu} = U[\lambda_i^{1-\nu}\mu_j^{\nu}y_{ij}]V^*.$$

Therefore

$$(1-\nu)||AX||_2^2 + \nu||XB||_2^2 = (1-\nu)\sum_{i,j=1}^{n}(\lambda_i)^2|y_{ij}|^2 + \nu\sum_{i,j=1}^{n}(\mu_j)^2|y_{ij}|^2 =$$

$$= \sum_{i,j=1}^{n}((1-\nu)\lambda_i^2 + \nu\mu_j^2)|y_{ij}|^2 \le \sum_{i,j=1}^{n}[r(\lambda_i - \mu_j)^2 + S(\frac{\lambda_i}{\mu_j})(\lambda_i^{1-\nu}\mu_j^{\nu})^2]|y_{ij}|^2 \le$$

$$\le r\sum_{i,j=1}^{n}(\lambda_i - \mu_j)^2|y_{ij}|^2 + S(h)\sum_{i,j=1}^{n}(\lambda_i^{1-\nu}\mu_j^{\nu})^2|y_{ij}|^2 =$$

$$= r||AX - XB||_2^2 + S(h)||A^{1-\nu}XB^{\nu}||_2^2.$$

As in the proof of previous theorem,

$$\frac{1}{h} = \frac{m}{M} \le \frac{1}{||A^{-1}||_2||B||_2} \le \frac{\lambda_i}{\mu_j} \le ||A||_2||B^{-1}||_2 \le \frac{M}{m} = h,\ (\forall)i, j = \overline{1, n},$$

so

$$(1-\nu)\lambda_i^2 + \nu\mu_j^2 \le r(\lambda_i - \mu_j)^2 + S(\frac{\lambda_i}{\mu_j})(\lambda_i^{1-\nu}\mu_j^{\nu})^2 \le r(\lambda_i - \mu_j)^2 + S(h)(\lambda_i^{1-\nu}\mu_j^{\nu})^2,$$

where $S(h) = \max_{\frac{1}{h} \le t \le h} S(t)$.

In our case $h > 1$, and we used that $S(x)$ is monotone decreasing for $0 < x < 1$ and monotone increasing for $x > 1$.

∎

By the reverse difference inequality of the refined Young inequality for scalars given in [1] we obtain the next result:

Theorem 4. *Let $A,\ B,\ X \in M_n$ A and B being positive definite matrices with $m = \min\{\frac{1}{||A^{-1}||_2}, \frac{1}{||B^{-1}||_2}\}$ and $M = \max\{||A||_2, ||B||_2\}$. If $0 \le \nu \le 1$, then*

$$(1-\nu)||AX||_2^2 + \nu||XB||_2^2 \le$$

$$\le r||AX - XB||_2^2 + ||A^{1-\nu}XB^{\nu}||_2^2 + \frac{M}{m^2}L(M, m)\log S(h)||AX||_2^2$$

where $h = \frac{M}{m}$, $r = \min\{\nu, 1-\nu\}$, $S(t) = \frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}}$, $t > 0,\ t \ne 1$ is the Specht ratio and $L(x, y)$ is the logarithmic mean, i. e. $L(x, y) = \frac{y-x}{\log y - \log x}$, $x \ne y$.

Proof. If we replace in Lemma 4.1, see [1], a with a^2 and b with b^2 we have

$$\omega L(a,b)\log S(\frac{a}{b})+(a^{1-\nu}b^{\nu})^2+r(a-b)^2\geq(1-\nu)a^2+\nu b^2.$$

In our case this inequality becomes:

$$(1-\nu)\lambda_i^2+\nu\mu_j^2\leq r(\lambda_i-\mu_j)^2+(\lambda_i^{1-\nu}\mu_j^{\nu})^2+\max_{i,j=\overline{1,n}}\{\lambda_i,\mu_j\}L(\lambda_i,\mu_j)\log S\left(\frac{\lambda_i}{\mu_j}\right)=$$

$$=r(\lambda_i-\mu_j)^2+(\lambda_i^{1-\nu}\mu_j^{\nu})^2+\mu_j\max_{i,j=\overline{1,n}}\{\lambda_i,\mu_j\}L\left(\frac{\lambda_i}{\mu_j},1\right)\log S\left(\frac{\lambda_i}{\mu_j}\right)\leq$$

$$\leq r(\lambda_i-\mu_j)^2+(\lambda_i^{1-\nu}\mu_j^{\nu})^2+\mu_j\max_{i,j=\overline{1,n}}\{\lambda_i,\mu_j\}L(h,1)\log S(h)=$$

$$=r(\lambda_i-\mu_j)^2+(\lambda_i^{1-\nu}\mu_j^{\nu})^2+\mu_j^2\max_{i,j=\overline{1,n}}\{\frac{\lambda_i}{\mu_j},1\}\frac{L(M,m)}{m}\log S(h)\leq$$

$$\leq r(\lambda_i-\mu_j)^2+(\lambda_i^{1-\nu}\mu_j^{\nu})^2+\mu_j^2\frac{M}{m^2}L(M,n)\log S(h).$$

But analogously, we can obtain:

$$(1-\nu)\lambda_i^2+\nu\mu_j^2\leq r(\lambda_i-\mu_j)^2+(\lambda_i^{1-\nu}\mu_j^{\nu})^2+\lambda_i^2\frac{M}{m^2}L(M,n)\log S(h).$$

Applying now each of these inequalities we obtain:

$$(1-\nu)||AX||_2^2+\nu||XB||_2^2\leq$$

$$\leq r||AX-XB||_2^2+||A^{1-\nu}XB^{\nu}||_2^2+\frac{M}{m^2}L(M,n)\log S(h)||XB||_2^2,$$

or

$$(1-\nu)||AX||_2^2+\nu||XB||_2^2\leq$$

$$\leq r||AX-XB||_2^2+||A^{1-\nu}XB^{\nu}||_2^2+\frac{M}{m^2}L(M,n)\log S(h)||AX||_2^2.$$

∎

Using the refinement of the N. Minculete for the Kittaneh-Manasrah inequality which improves the inequality of Young , see [6], we will give two similar refinements of two inequalities for the Hilbert-Schmidt norm of two invertible positive definite matrices as below.

Theorem 5. *Let A, B, $X\in M_n$ with A and B two positive definite matrices satisfying $||B||_2<\frac{1}{||A^{-1}||_2}$ and $||B^{-1}||_2<1$. If $0\leq\nu\leq1$, then*

$$\nu||AX||_2^2+(1-\nu)||XB||_2^2-||A^{\nu}XB^{1-\nu}||_2^2\leq$$

$$\leq(1-r)||AX-XB||_2^2+4b(\nu)\log^2(h)||X||_2^2$$

where $h=||A||_2||B^{-1}||_2$, $r=\min\{\nu,1-\nu\}$, and $b(\nu)=\frac{\nu(1-\nu)}{2}-\frac{1-r}{4}$.

Proof. We will use again the proof of Theorem 3.1, see [3], where A and B are positive definite matrices so are unitarily diagonalizable and then there are unitary matrices $U,\ V \in M_n$ which satisfy:

$$A = U\Lambda_1 U^*, \qquad B = V\Lambda_2 V^*,$$

where $\Lambda_1 = diag(\lambda_1, ..., \lambda_n)$, $\Lambda_2 = diag(\mu_1, ..., \mu_n)$, $\lambda_i,\ \mu_i > 0,\ i = 1, ..., n$. If we put $Y = U^* X V = [y_{ij}]$ we have as in previous two theorems,

$$\nu||AX||_2^2 + (1-\nu)||XB||_2^2 - ||A^\nu X B^{1-\nu}||_2^2 =$$

$$= \nu \sum_{i,j=1}^{n} \lambda_i^2 |y_{ij}|^2 + (1-\nu) \sum_{i,j=1}^{n} \mu_j^2 |y_{ij}|^2 - \sum_{i,j=1}^{n} (\lambda_i^\nu \mu_j^{1-\nu})^2 |y_{ij}|^2 =$$

$$= \sum_{i,j=1}^{n} [\nu\lambda_i^2 + (1-\nu)\mu_j^2 - (\lambda_i^\nu \mu_j^{1-\nu})^2] |y_{ij}|^2.$$

If we take in inequality (4) λ_i instead of $\sqrt{\lambda_i}$ and μ_j instead of $\sqrt{\mu_j}$ we will obtain:

$$r(\lambda_i - \mu_j)^2 + a(\nu)\log^2(\frac{\lambda_i^2}{\mu_j^2}) \leq \nu\lambda_i^2 + (1-\nu)\mu_j^2 - (\lambda_i^\nu \mu_j^{1-\nu})^2 \leq$$

$$\leq (1-r)(\lambda_i - \mu_j)^2 + b(\nu)\log^2(\frac{\lambda_i^2}{\mu_j^2})$$

and then

$$\nu\lambda_i^2 + (1-\nu)\mu_j^2 - (\lambda_i^\nu \mu_j^{1-\nu})^2 \leq (1-r)(\lambda_i - \mu_j)^2 + 4b(\nu)\log^2(h),$$

for all $1 \leq \lambda_i \leq ||A||_2$ and $1 \leq \frac{1}{||B^{-1}||_2} \leq \mu_j \leq ||B||_2,\ i, j = \overline{1, n}$.

Thus we have:

$$\nu||AX||_2^2 + (1-\nu)||XB||_2^2 - ||A^\nu X B^{1-\nu}||_2^2 \leq$$

$$\leq (1-r) \sum_{i,j=1}^{n} (\lambda_i - \mu_j)^2 |y_{ij}|^2 + 4b(\nu)\log^2(h) \sum_{i,j=1}^{n} |y_{ij}|^2 =$$

$$= (1-r)||AX - XB||_2^2 + 4b(\nu)\log^2(h)||X||_2^2.$$

∎

Theorem 6. *Let $A,\ B,\ X \in M_n$ and A and B positive definite matrices satisfying $||A||_2 < 1$ and $||B||_2 < \frac{1}{||A^{-1}||_2}$. If $0 \leq \nu \leq 1$, then*

$$\nu||AX||_2^2 + (1-\nu)||XB||_2^2 - ||A^\nu X B^{1-\nu}||_2^2 \leq$$

$$\leq (1-r)||AX - XB||_2^2 + 4b(\nu)\log^2(h)||AXB||_2^2$$

where $h = ||A||_2||B^{-1}||_2$, $r = \min\{\nu, 1-\nu\}$, and $b(\nu) = \frac{\nu(1-\nu)}{2} - \frac{1-r}{4}$.

Proof. Like before we compute

$$\nu||AX||_2^2 + (1-\nu)||XB||_2^2 - ||A^\nu X B^{1-\nu}||_2^2 =$$

$$= \sum_{i,j=1}^{n} [\nu\lambda_i^2 + (1-\nu)\mu_j^2 - (\lambda_i^\nu \mu_j^{1-\nu})^2] |y_{ij}|^2 \leq$$

$$\leq \sum_{i,j=1}^{n} [(1-r)(\lambda_i - \mu_j)^2 + b(\nu)\lambda_i^2\mu_j^2 \log^2(\frac{\lambda_i^2}{\mu_j^2})] |y_{ij}|^2 \leq$$

$$\leq \sum_{i,j=1}^{n} [(1-r)(\lambda_i - \mu_j)^2 + 4b(\nu)\lambda_i^2\mu_j^2 \log^2(h)]|y_{ij}|^2 =$$

$$= (1-r)||AX - XB||_2^2 + 4b(\nu)\log^2(h)||AXB||_2^2,$$

where we used that $1 < \frac{\lambda_i}{\mu_j} < h$ and the fact that the logarithm is an increasing function and then so is $\log^2(x)$ when $x > 1$.

∎

Applying now the inequalities (i) and (ii) from Corollary 2.3 as in Theorem 2.3, see [2] we obtain also two similar inequalities for the Hilbert-Schmidt norm of two invertible positive definite matrices.

Theorem 7. *Let A, B, $X \in M_n$ with A and B positive definite matrices with $m = \min\{\frac{1}{||A^{-1}||_2}, \frac{1}{||B^{-1}||_2}\}$ and $||A||_2 \leq \frac{1}{||B^{-1}||_2} \leq ||B||_2 = M \leq 1$. If $0 \leq \nu \leq 1$, then*

$$(1-\nu)||AX||_2^2 + \nu||XB||_2^2 \leq \exp\{\nu(1-\nu)(1-\frac{1}{h})^2\}||A^{1-\nu}XB^{\nu}||_2^2,$$

where $h = \frac{M^2}{m^2}$, $r = \min\{\nu, 1-\nu\}$.

Proof. Using hypothesis we have $\frac{1}{\sqrt{h}} = \frac{m}{M} \leq \frac{\lambda_i}{\mu_j} \leq ||A||_2||B^{-1}||_2 \leq \frac{M}{m} = \sqrt{h}$ and $0 < \lambda_i \leq ||A||_2 \leq \frac{1}{||B^{-1}||_2} \leq \mu_j$, $(\forall)\ i,j = \overline{1,n}$. Therefore we can use the inequality form the proof of Theorem 2.3 (i), see [2],

$$(1-\nu)t + \nu \leq t^{1-\nu}e^{\nu(1-\nu)(1-\frac{1}{t})^2},\ (\forall)\ 0 < t \leq 1.$$

So

$$(1-\nu)t + \nu \leq t^{1-\nu} \max_{\frac{1}{h} \leq t \leq 1} e^{\nu(1-\nu)(1-\frac{1}{t})^2} \leq$$

$$\leq t^{1-\nu} \max_{\frac{1}{h} \leq t \leq h} e^{\nu(1-\nu)(1-\frac{1}{t})^2} = t^{1-\nu}e^{\nu(1-\nu)(1-\frac{1}{h})^2},\ (\forall)\ \frac{1}{h} < t \leq 1$$

and putting $\frac{\lambda_i^2}{\mu_j^2}$ instead of t we have:

$$(1-\nu)||AX||_2^2 + \nu||XB||_2^2 = (1-\nu)\lambda_i^2 + \nu\mu_j^2 \leq (\lambda_i^{1-\nu}\mu_j^{\nu})^2 e^{\nu(1-\nu)(1-\frac{1}{h})^2} =$$

$$= \exp\{\nu(1-\nu)(1-\frac{1}{h})^2\}||A^{1-\nu}XB^{\nu}||_2^2.$$

∎

Theorem 8. *Let A, B, $X \in M_n$ with A and B positive definite matrices with $m = \min\{\frac{1}{||A^{-1}||_2}, \frac{1}{||B^{-1}||_2}\}$ and $||A||_2 \leq \frac{1}{||B^{-1}||_2} \leq ||B||_2 = M \leq 1$. If $0 \leq \nu \leq 1$, then*

$$(1-\nu)||AX||_2^2 + \nu||XB||_2^2 - ||A^{1-\nu}XB^{\nu}||_2^2 \leq \nu(1-\nu)(\log h)^2||XB||_2^2,$$

where $h = \frac{M^2}{m^2}$, $r = \min\{\nu, 1-\nu\}$.

Proof. Using the inequality from Theorem 2.3 (ii), see [2],

$$(1-\nu)t+\nu-t^{1-\nu} \leq \nu(1-\nu)(\log t)^2,\ (\forall)\ 0<t\leq 1$$

in fact,

$$(1-\nu)t+\nu-t^{1-\nu} \leq \nu(1-\nu) \max_{\frac{1}{h}\leq t\leq 1} (\log t)^2 \leq$$

$$\leq \nu(1-\nu) \max_{\frac{1}{h}\leq t\leq h} (\log t)^2 = \nu(1-\nu)(\log h)^2,\ (\forall)\ \frac{1}{h}\leq t\leq 1,$$

and putting as in previous theorem $\frac{\lambda_i^2}{\mu_j^2}$ instead of t we obtain:

$$(1-\nu)||AX||_2^2+\nu||XB||_2^2-||A^{1-\nu}XB^{\nu}||_2^2 = (1-\nu)\lambda_i^2+\nu\mu_j^2-(\lambda_i^{1-\nu}\mu_j^{\nu})^2 \leq$$

$$\leq \nu(1-\nu)(\log h)^2\mu_j^2 = \nu(1-\nu)(\log h)^2||XB||_2^2.$$

∎

References

[1] Furuichi S., On refined Young inequalities and reverse inequalities, *Journal of Mathematical Inequalities*, 5, 1 (2011), 21-31.

[2] Furuichi S., Minculete N., Alternative reverse inequalities for Young's inequality, *Journal of Mathematical inequalities*, 5, 4 (2011), 595-600.

[3] He Chuanjian and Zou L., Some inequalities involving unitarily invariant norms, *Mathematical Inequalities and Applications*, 12, 4 (2012), 767-776.

[4] Kittaneh F., Manasrah Y., Reverse Young and Heinz inequalities for matrices, *Linear and Multilinear Algebra*, 59, 9 (2011), 1031-1037.

[5] Kittaneh F., Manasrah Y., Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.*, 361 (2010), 262-269.

[6] Minculete N., A refinement of the Kittaneh-Manasrah inequality, *Creat. Math. Inform.*, 20 (2011), No. 2, 157-162.

[7] Zuo H., Shi G. and Fujii M., Reefined Young inequality with Kantorovich constant, *Journal of Mathematical Inequalities*, 5, 4 (2011), 551-556.

Department of Mathematics,"Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara