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# SHARP INTEGRAL INEQUALITIES OF OSTROWSKI-GRÜSS INVOLVING SEVERAL INTERIOR POINTS AND APPLICATIONS

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Abstract. A way to solve the open problem for sharp versions of Ostrowski-Grüss's inequality in two variables is obtained, and a new application for numerical integral formula is given.

## 1. Introduction

In [1], Dragomir and Wang proved the following Ostrowski-Grüss type integral inequalities.

**Theorem 1.** *Let $f : I \subseteq R \to R$ be mapping differentiable in the interior $I^0$ of $I$, and let $a, b \in I^0$ with $a < b$. If $\gamma < f'(x) < \Gamma$, for some constants $\gamma$, $\Gamma \in R$, then the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\Big(x - \frac{a+b}{2}\Big)\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.1}$$

*holds.*

This inequality is a connection between Ostrowski's inequality ([2, p.468] or [3, p.468]) and Grüss's inequality [4]. It can be applied to some special mean and some numerical quadrature rules, see [1, 5, 6, 7, 8, 9, 10, 11].

In [5, Theorem 1.5], Cheng presented the sharp version for (1.1) in the following theorem.

**Theorem 2.** *Let the assumptions of Theorem 1 hold. Then for all $x \in [a, b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\Big(x - \frac{a+b}{2}\Big)\right| \le \frac{1}{8}(b-a)(\Gamma-\gamma). \tag{1.2}$$

*The inequality (1.2) is sharp in the sense that the constant $\frac{1}{8}$ cannot be replaced by a smaller one.*

Recently, Feng and Meng [12] established the following generalizations of Theorem 1 and 2 for one dimension case involving $(k-1)$ interior points, and gave application on the estimate of error bound for numerical integral formula.

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**Theorem 3.** *Under the conditions of Theorem 1, suppose that $x_i \in [a,b]$, $i = 0,1,\dots,k$, $I_k : a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a,b]$, and $m_i \in [x_{i-1}, x_i]$, $i = 1,2,\dots,k$, $m_0 = a$, $m_{k+1} = b$, then we have*

$$\begin{aligned}(1.3)\qquad &\Big|\frac{1}{b-a}\sum_{i=0}^{k}(m_{i+1}-m_i)f(x_i) - \frac{1}{b-a}\int_a^b f(t)dt \\ &- \frac{f(b)-f(a)}{(b-a)^2}\Big[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1}-x_i)\Big]\Big| \\ &\le \frac{1}{4}(b-a)(\Gamma-\gamma).\end{aligned}$$

**Theorem 4.** *Under the conditions of Theorem 3, we have*

$$\begin{aligned}(1.4)\qquad &\Big|\frac{1}{b-a}\sum_{i=0}^{k}(m_{i+1}-m_i)f(x_i) - \frac{1}{b-a}\int_a^b f(t)dt \\ &- \frac{f(b)-f(a)}{(b-a)^2}\Big[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1}-x_i)\Big]\Big| \\ &\le \frac{1}{8}(b-a)(\Gamma-\gamma),\end{aligned}$$

*provided that one of the following two conditions holds.*

(i) *there exist $[x_{i_{j-1}}, x_{i_j})$, $j = 1,2,\dots,l_1$ such that $[x_{i_{j-1}}, x_{i_j}) \cap I^+_{[a,b]} \neq \emptyset$, $[x_{i_{j-1}}, x_{i_j}) \cap I^-_{[a,b]} \neq \emptyset$ for $j = 1,2,\dots,l_1$, $I^+_{[a,b]} \subset \bigcup_{j=1}^{l_1}[x_{i_{j-1}}, x_{i_j})$, and $m(I^+_{[a,b]}) \le \frac{b-a}{2}$,*

(ii) *there exist $[x_{n_{j-1}}, x_{n_j})$, $j = 1,2,\dots,l_2$ such that $[x_{n_{j-1}}, x_{n_j}) \cap I^+_{[a,b]} \neq \emptyset$, $[x_{n_{j-1}}, x_{n_j}) \cap I^-_{[a,b]} \neq \emptyset$ for $j = 1,2,\dots,l_2$, $I^-_{[a,b]} \subset \bigcup_{j=1}^{l_1}[x_{i_{j-1}}, x_{i_j})$, and $m(I^-_{[a,b]}) \le \frac{b-a}{2}$, where $m(\cdot)$ denotes the measure of a Lebesgue measurable set.*

The constant (1.4) is sharp in the sense that the constant $\frac{1}{8}$ cannot replaced by a smaller one. It seems that the conditions in Theorem 4 are complex. In [12], they also extend Theorem 1 to two dimensions case, but the sharpness is unresolved.

The main purpose of this article is, by using a new approach, to generalize and simplify inequality (1.4) and to obtain a sharp integral inequality of Ostrowski-Grüss type for two variables involving $(k-1)\times(l-1)$ interior points. That is, the open problem given by Feng and Meng is solved. Also, applying the obtained results, a new numerical integral formula will be pointed out.

## 2. Main Results

The following two Lemmas are necessary to prove the main results.

In [13], Cheng and Sun have established the following variant of Grüss's inequality.

**Lemma 1.** *Let $f, g : [a,b] \to R$ be two integrable functions such that*

$$-\infty < \delta \le g(x) \le \Delta < \infty \quad \textit{for all} \quad x \in [a,b],$$

*where $\delta, \Delta$ are constants. Then we have inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx\right| \le \frac{(\Delta-\delta)}{2(b-a)}\int_a^b \left|f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt\right|dx. \tag{2.1}$$

For extensions in the general case of the Lebesgue integral on measurable spaces, the sharpness of the constant $\frac{1}{2}$ as well as the corresponding discrete version, see [14].

Similar to the Lemma 1 in [13], we have the following Grüss type inequality for the function of two variables. It can be seen as a particular case of the general Lebsgue integral considered in [14] that is considered for general measurable sets.

**Lemma 2.** *Let $f, g : [a,b]\times[c,d] \to R$ be two integrable functions such that*

$$-\infty < \delta \le g(x,y) \le \Delta < \infty \quad \text{for all} \quad (x,y) \in [a,b]\times[c,d],$$

*for all , where $\delta, \Delta$ are constants. Then we have inequality*

$$\left|\int_a^b\int_c^d f(x,y)g(x,y)dx - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\int_a^b\int_c^d g(x,y)dxdy\right| \le \frac{\Delta-\delta}{2}\int_a^b\int_c^d\left|f(x,y) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt\right|dxdy. \tag{2.2}$$

The Theorem 4 will be generalized as follows.

**Theorem 5.** *Let $I \subset R$ be an open interval, $a, b \in I$, $a < b$, $f : I \to R$ is a differential function such that there exist constants $\gamma, \Gamma \in R$ with $\gamma \le f'(x) \le \Gamma$, $x \in [a,b]$. Furthermore, suppose that $x_i \in [a,b]$, $i = 0,1,\ldots,k$, $I_k : a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a,b]$, and $m_i \in [x_{i-1}, x_i]$, $i = 1,2,\ldots,k$, $m_0 = a$, $m_{k+1} = b$. Then we have*

$$\left|\frac{1}{b-a}\sum_{i=0}^{k}(m_{i+1}-m_i)f(x_i) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{(b-a)^2}\left[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1}-x_i)\right]\right| \le \frac{(\Gamma-\gamma)}{2(b-a)}\int_a^b \left|P(t,I_k)\right|dt, \tag{2.3}$$

*where*

$$P(t,I_k) = \begin{cases} t - m_1 - C, & t \in [x_0, x_1), \\ t - m_2 - C, & t \in [x_1, x_2), \\ \vdots & \vdots \\ t - m_{k-1} - C, & t \in [x_{k-2}, x_{k-1}), \\ t - m_k - C, & t \in [x_{k-1}, x_k), \end{cases}$$

*and*

$$C = \frac{a+b}{2} - \frac{1}{b-a}\sum_{i=0}^{k-1} m_{i+1}(x_{i+1}-x_i).$$

*The constant $\frac{1}{2}$ in inequality (2.3) is sharp.*

*Proof.* By the proof of Theorem 2.4 in [12], we have the following two identities:

$$\int_a^b P(t, I_k)dt = 0, \text{ and}$$

$$\int_a^b P(t, I_k)f'(t)dt$$
$$= \sum_{i=0}^{k}(m_{i+1} - m_i)f(x_i) - \int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\Big[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1}-x_i)\Big]$$

By the Lemma 1, taking $f(t) = P(t, I_k)$ and $g(t) = f'(t)$ in (2.1), we have the desired inequality (2.3).

To prove the sharpness of (2.3), we take $k = 2$, $m_0 = m_1 = a$, $m_2 = m_3 = b$, $x_1 = \frac{a+3b}{4}$, $\Gamma = 1$, $\gamma = -1$, and

$$f(t) = \begin{cases} -t + a, & a \le t < \frac{3a+b}{4}, \\ t + \frac{a+b}{2}, & \frac{3a+b}{4} \le t < \frac{a+3b}{4}, \\ -t + b, & \frac{3a+b}{4} \le t \le b. \end{cases}$$

Then

$$C = \frac{b-a}{4}, \qquad P(t, I_2) = \begin{cases} t - \frac{3a+b}{4}, & a \le t < \frac{a+3b}{4}, \\ t - \frac{-a+5b}{4}, & a \le \frac{a+3b}{4} \le t \le b, \end{cases}$$

$$f'(t) = \begin{cases} -1, & t \in \left[a, \frac{3a+b}{4}\right] \cup \left[\frac{a+3b}{4}, b\right], \\ 1, & t \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right), \end{cases}$$

and then the identity of (2.3) holds obviously. This completes the proofs of Theorem 5. ∎

**Remark 1.** *In Theorem 5, if the division of* $[a, b]$ *satisfies the conditions (i) or (ii) in Theorem 4, then*

$$\int_a^b |P(t, I_k)|dt = \int_{I^+[a,b]} P(t, I_k)dt - \int_{I^-[a,b]} P(t, I_k)dt$$
$$\le \frac{(b-a)^2}{8} + \frac{(b-a)^2}{8} = \frac{(b-a)^2}{4}.$$

*That is, Theorem 5 reduces to Theorem 4.*

In the following, the Ostrowski-Grüss type inequality for the function of two variables involving $(k-1) \times (l-1)$ interior points is established, and we note that the inequality is sharp.

**Theorem 6.** *Let* $f : [a, b] \times [c, d] \to R$ *be an absolutely continuous function such that the partial derivative of order 2 exists and there exist constants* $K_1$ *and* $K_2$ *with* $K_1 \le \frac{\partial^2 f(s,t)}{\partial s \partial t} \le K_2$. *Suppose* $x_i \in [a, b]$, $y_i \in [c, d]$, $i = 0, 1, \ldots, k$, $j = 0, 1, \ldots, l$, $I_k : a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b$ *is a division of the interval* $[a, b]$, *while* $J_l : c = y_0 < y_1 < y_2 < \cdots < y_{k-1} < y_k = d$ *is a division of the interval* $[c, d]$, *and* $\alpha_{i+1} \in [x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, k-1$, $\beta_{j+1} \in [x_j, x_{j+1}]$, $j = 0, 1, 2, \dots, l-1$, $\alpha_0 = \alpha$, $\alpha_{k+1} = b$, $\beta_0 = c$, $\beta_{i+1} = d$. *Then we have*

(2.4)
$$
\begin{aligned}
&\Big|\sum_{i=1}^{k-1}\sum_{j=1}^{l-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j) + (y_l-\beta_l-D)\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i,y_l)\\
&-(y_0-\beta_1-D)\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i,y_0) + (x_k-\alpha_k-C)\sum_{j=1}^{l-1}[(\beta_{j+1}-\beta_j)f(x_k,y_j)]\\
&+(x_k-\alpha_k-C)(y_l-\beta_l-D)f(x_k,y_l)-(x_k-\alpha_k-C)(y_0-\beta_1-D)f(x_k,y_0)\\
&-(x_0-\alpha_1-C)\sum_{j=1}^{l-1}[(\beta_{j+1}-\beta_j)f(x_0,y_j)]\\
&-(x_0-\alpha_1-C)(y_l-\beta_l-D)f(x_0,y_l)+(x_0-\alpha_1-C)(y_0-\beta_1-D)f(x_0,y_0)\\
&-\sum_{j=1}^{l-1}\Big[(\beta_{j+1}-\beta_j)\int_a^b f(s,y_j)ds\Big]-(y_l-\beta_l-D)\int_a^b f(s,y_l)ds\\
&+(y_0-\beta_1-D)\int_a^b f(s,y_0)ds-\sum_{i=1}^{k-1}\Big[(\alpha_{i+1}-\alpha_i)\int_c^d f(x_i,t)dt\Big]\\
&-(x_k-\alpha_k-C)\int_c^d f(x_k,t)dt\\
&+(x_0-\alpha_1-C)\int_c^d f(x_0,t)dt+\int_a^b\int_c^d f(s,t)dtds\Big|\\
&\le \frac{(K_2-K_1)}{2}\int_a^b\int_c^d |H(s,t,I_k,J_l)|dtds,
\end{aligned}
$$

*where* $H(s,t,I_k,J_l) = (s-\alpha_{i+1}-C)(t-\beta_{j+1}-D)$, $(s,t) \in [x_i, x_{i+1}) \times [y_i, y_{j+1})$, $i = 0, 1, 2, \dots, k-1$, $j = 0, 1, 2, \dots, l-1$,

$$C = \frac{a+b}{2} - \frac{1}{b-a}\sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}-x_i) \quad \text{or} \quad D = \frac{c+d}{2} - \frac{1}{d-c}\sum_{j=0}^{l-1}\beta_{j+1}(y_{j+1}-y_j).$$

*The inequality* (2.4) *is sharp.*

*Proof.* Since

(2.5)
$$
\begin{aligned}
&\int_a^b\int_c^d H(s,t,I_k,J_l)dtds\\
&=\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}(s-\alpha_{i+1}-C)\sum_{j=0}^{l-1}\int_{y_i}^{y_{i+1}}(t-\beta_{j+1}-D)dtds\\
&=\sum_{i=0}^{k-1}\Big[\frac{(x_{i+1}-\alpha_{i+1}-C)^2}{2}-\frac{(x_i-\alpha_{i+1}-C)^2}{2}\Big]\\
&\times\sum_{j=0}^{l-1}\Big[\frac{(y_{j+1}-\beta_{i+1}-D)^2}{2}-\frac{(y_j-\beta_{j+1}-D)^2}{2}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=0}^{k-1}[(x_{i+1}-x_i)(x_{i+1}+x_i-2\alpha_{i+1}-2C)] \\
&\times \frac{1}{2}\sum_{j=0}^{l-1}[(y_{i+1}-y_i)(y_{i+1}+y_i-2\beta_{i+1}-2D)] \\
&= \Big[\frac{1}{2}\sum_{i=0}^{k-1}(x_{i+1}^2-x_i^2)-\sum_{i=0}^{k-1}(x_{i+1}-x_i)\alpha_{i+1}-C\sum_{i=0}^{k-1}(x_{i+1}-x_i)\Big] \\
&\times \Big[\frac{1}{2}\sum_{j=0}^{l-1}(y_{j+1}^2-y_j^2)-\sum_{j=0}^{l-1}(y_{i+1}-y_i)\beta_{i+1}-D\sum_{j=0}^{l-1}(y_{i+1}-y_i)\Big] \\
&= \Big[\frac{1}{2}(d^2-c^2)-\sum_{j=0}^{l-1}(y_{j+1}-y_j)\beta_{j+1}-D(d-c)\Big] \\
&\times \Big[\frac{1}{2}(b^2-a^2)-\sum_{i=0}^{k-1}(x_{i+1}-x_i)\alpha_{i+1}-C(b-a)\Big] \\
&= 0
\end{aligned}
$$

and

(2.6)

$$
\begin{aligned}
&\int_a^b\int_c^d H(s,t,I_k,J_l)\frac{\partial^2 f(s,t)}{\partial s\partial t}dtds \\
&= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}(s-\alpha_{i+1}-C)\sum_{j=0}^{l-1}\int_{y_i}^{y_{i+1}}(t-\beta_{j+1}-D)\frac{\partial^2 f(s,t)}{\partial s\partial t}dtds \\
&= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}(s-\alpha_{i+1}-C)\sum_{j=0}^{l-1}\Big[(y_{j+1}-\beta_{j+1}-D)\frac{\partial f(s,y_{j+1})}{\partial s} \\
&-(y_j-\beta_{j+1}-D)\frac{\partial f(s,y_j)}{\partial s}-\int_{y_i}^{y_{i+1}}\frac{\partial f(s,t)}{\partial s}dt\Big]ds \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\Big\{(y_{j+1}-\beta_{j+1}-D)\Big[(x_{i+1}-\alpha_{i+1}-C)f(x_{i+1},y_{j+1}) \\
&-(x_i-\alpha_{i+1}-C)f(x_i,y_{j+1})-\int_{x_i}^{x_{i+1}}f(s,y_{j+1})ds\Big] \\
&-(y_j-\beta_{j+1}-D)\Big[(x_{i+1}-\alpha_{i+1}-C)f(x_{i+1},y_j) \\
&-(x_i-\alpha_{i+1}-C)f(x_i,y_j)-\int_{x_i}^{x_{i+1}}f(s,y_j)ds\Big] \\
&-\int_{y_j}^{y_{j+1}}\Big[(x_{i+1}-\alpha_{i+1}-C)f(x_{i+1},t) \\
&-(x_i-\alpha_{i+1}-C)f(x_i,y_{j+1})-\int_{x_i}^{x_{i+1}}f(s,t)ds\Big]dt\Big\}
\end{aligned}
$$

$$= \sum_{j=0}^{l-1} \Big\{ (y_{j+1} - \beta_{j+1} - D) \Big[ \Big( \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i, y_{i+1}) \Big)$$

$$+ (x_k - \alpha_k - C) f(x_k, y_{j+1}) - (x_0 - \alpha_1 - C) f(x_0, y_{j+1}) - \int_a^b f(s, y_{j+1}) ds \Big]$$

$$- (y_j - \beta_{j+1} - D) \Big[ \Big( \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i, y_i) \Big)$$

$$+ (x_k - \alpha_k - C) f(x_k, y_j) - (x_0 - \alpha_1 - C) f(x_0, y_j) - \int_a^b f(s, y_j) ds \Big]$$

$$- \int_{y_i}^{y_{i+1}} \Big[ \Big( \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i, t) \Big)$$

$$+ (x_k - \alpha_k - C) f(x_k, t) - (x_0 - \alpha_1 - C) f(x_0, t) - \int_a^b f(s, t) ds \Big] dt \Big\}$$

$$= \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) \Big\{ \sum_{j=0}^{l-1} [(y_{j+1} - \beta_{j+1} - D) f(x_i, y_{j+1}) - (y_j - \beta_{j+1} - D) f(x_i, y_j)] \Big\}$$

$$+ (x_k - \alpha_k - C) \Big\{ \sum_{j=0}^{l-1} [(y_{j+1} - \beta_{j+1} - D) f(x_i, y_{j+1}) - (y_j - \beta_{j+1} - D) f(x_i, y_j)] \Big\}$$

$$- (x_0 - \alpha_1 - C) \Big\{ \sum_{j=0}^{l-1} [(y_{j+1} - \beta_{j+1} - D) f(x_0, y_{j+1}) - (y_j - \beta_{j+1} - D) f(x_0, y_j)] \Big\}$$

$$- \sum_{j=0}^{l-1} [(y_{j+1} - \beta_{j+1} - D) \int_a^b f(s, y_{j+1}) ds - (y_j - \beta_{j+1} - D) \int_a^b f(s, y_j) ds]$$

$$- \sum_{i=1}^{k-1} \Big[ (\alpha_{i+1} - \alpha_i) \int_c^d f(x_i, t) dt \Big] - (x_k - \alpha_k - C) \int_c^d f(x_k, t) dt$$

$$+ (x_0 - \alpha_1 - C) \int_c^d f(x_0, t) dt + \int_a^b \int_c^d f(s, t) dt ds$$

$$= \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) \Big\{ \sum_{j=1}^{l-1} [(\beta_{j+1} - \beta_j) f(x_i, y_j)] + (y_l - \beta_l - D) f(x_i, y_l)$$

$$- (y_0 - \beta_l - D) f(x_i, y_0) \Big\} + (x_k - \alpha_k - C) \Big\{ \sum_{j=1}^{l-1} [(\beta_{j+1} - \beta_j) f(x_k, y_j)]$$

$$+ (y_l - \beta_l - D) f(x_k, y_l) - (y_0 - \beta_l - D) f(x_k, y_0) \Big\}$$

$$- (x_0 - \alpha_1 - C) \Big\{ \sum_{j=1}^{l-1} [(\beta_{j+1} - \beta_j) f(x_0, y_j)] + (y_j - \beta_l - D) f(x_0, y_l)$$

$$- (y_0 - \beta_l - D) f(x_0, y_0) \Big\} - \sum_{j=1}^{l-1} \Big[ (\beta_{j+1} - \beta_j) \int_a^b f(s, y_j) ds \Big]$$

$$- (y_l - \beta_l - D)\int_a^b f(s, y_l)ds + (y_0 - \beta_1 - D)\int_a^b f(s, y_0)ds$$
$$- \sum_{i=1}^{k-1}\left[(\alpha_{i+1} - \alpha_i)\int_c^d f(x_i, t)dt\right] - (x_k - \alpha_k - C)\int_c^d f(x_k, t)dt$$
$$+ (x_0 - \alpha_1 - C)\int_c^d f(x_0, t)dt + \int_a^b\int_c^d f(s, t)dtds$$
$$= \sum_{i=1}^{k-1}\sum_{j=1}^{l-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i, y_j) + (y_l-\beta_l-D)\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i, y_l)$$
$$- (y_0 - \beta_l - D)\sum_{i=1}^{k-1}(\alpha_{i+1} - \alpha_i)f(x_i, y_0)$$
$$+ (x_k - \alpha_k - C)\sum_{j=1}^{l-1}[(\beta_{j+1}-\beta_j)f(x_k, y_j)]$$
$$+(x_k - \alpha_k - C)(y_l-\beta_l-D)f(x_k, y_l) - (x_k - \alpha_k - C)(y_0-\beta_l-D)f(x_k, y_0)$$
$$+ (x_0 - \alpha_l - C)\sum_{j=1}^{l-1}[(\beta_{j+1}-\beta_j)f(x_0, y_j)] - (x_0 - \alpha_1 - C)(y_l-\beta_l-D)f(x_0, y_l)$$
$$+ (x_0 - \alpha_1 - C)(y_0-\beta_1-D)f(x_0, y_0) - \sum_{j=1}^{l-1}\left[(\beta_{j+1} - \beta_j)\int_a^b f(s, y_j)ds\right]$$
$$- (y_l - \beta_l - D)\int_a^b f(s, y_l)ds + (y_0 - \beta_1 - D)\int_a^b f(s, y_0)ds$$
$$- \sum_{i=1}^{k-1}\left[(\alpha_{i+1} - \alpha_i)\int_c^d f(x_i, t)dt\right] - (x_k - \alpha_k - C)\int_c^d f(x_k, t)dt$$
$$+ (x_0 - \alpha_1 - C)\int_c^d f(x_0, t)dt + \int_a^b\int_c^d f(s, t)dtds.$$

Using Lemma 2, taking $f(s,t) = H(s,t,I_k,J_l)$ and $g(s,t) = \frac{\partial^2 f(s,t)}{\partial s \partial t}$ in (2.2), and by (2.5) and (2.6), we have the desire of the inequality (2.4). To prove the sharpness of (2.4), we take $k = 2$, $l = 2$, $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = b$, $\beta_0 = \beta_1 = c$, $\beta_2 = \beta_3 = d$,

$x_1 = \frac{a+3b}{4}$, $y_1 = \frac{c+3d}{4}$, $K_2 = 1$, $K_1 = -1$, and

$$f(s,t) = \begin{cases} (-s+a)(-t+c), & (s,t) \in [a, \frac{3a+b}{4}) \times [c, \frac{3c+d}{4}), \\ (-s+a)(t-\frac{c+d}{2}), & (s,t) \in [a, \frac{3a+b}{4}) \times [\frac{3c+d}{4}, \frac{c+3d}{4}), \\ (-s+a)(-t+d), & (s,t) \in [a, \frac{3a+b}{4}) \times [\frac{c+3d}{4}, d), \\ (s-\frac{a+b}{2})(-t+c), & (s,t) \in [\frac{3a+b}{4}, \frac{a+3b}{4}) \times [c, \frac{3c+d}{4}), \\ (s-\frac{a+b}{2})(t-\frac{c+d}{2}), & (s,t) \in [\frac{3a+b}{4}, \frac{a+3b}{4}) \times [\frac{3c+d}{4}, \frac{c+3d}{4}), \\ (s-\frac{a+b}{2})(-t+d), & (s,t) \in [\frac{3a+b}{4}, \frac{a+3b}{4}) \times [\frac{c+3d}{4}, d], \\ (-s+b)(-t+c), & (s,t) \in [\frac{a+3b}{4}, b] \times [c, \frac{3c+d}{4}), \\ (-s+b)(t-\frac{c+d}{2}), & (s,t) \in [\frac{a+3b}{4}, b] \times [\frac{3c+d}{4}, \frac{c+3d}{4}), \\ (-s+b)(-t+d), & (s,t) \in [\frac{a+3b}{4}, b] \times [\frac{c+3d}{4}, d]. \end{cases}$$

Then we get $C = \frac{b-a}{4}$, $D = \frac{d-c}{4}$, $\frac{\partial^2}{\partial s \partial t} f(t) = -1$ or $1$ for all $(s,t) \in [a,b] \times [c,d]$, and

$$H(s,t,I_2,J_2) = \begin{cases} (s-\frac{3a+b}{4})(t-\frac{3c+d}{4}), & (s,t) \in [a, \frac{a+3b}{4}) \times [c, \frac{c+3d}{4}), \\ (s-\frac{3a+b}{4})(t-\frac{-c+5d}{4}), & (s,t) \in [a, \frac{a+3b}{4}) \times [\frac{c+3d}{4}, d], \\ (s-\frac{-a+5b}{4})(t-\frac{3c+d}{4}), & (s,t) \in [\frac{a+3b}{4}, b] \times [c, \frac{c+3d}{4}), \\ (s-\frac{-a+5b}{4})(t-\frac{-c+5d}{4}), & (s,t) \in [\frac{a+3b}{4}, b] \times [\frac{c+3d}{4}, d). \end{cases}$$

Further, we obtain

$$f(\frac{3a+b}{4}, \frac{c+3d}{4}) = \frac{(b-a)(d-c)}{16}, \quad f(\frac{a+3b}{4}, d) = 0, \quad f(\frac{a+3b}{4}, c) = 0,$$

$$f(b, \frac{c+3d}{4}) = 0,\ f(b,d) = 0,\ f(b,c) = 0,\ f(a, \frac{c+3d}{4}) = 0,\ f(a,d) = 0,\ f(a,c) = 0,$$

$$\int_a^b f(s, \frac{c+3d}{4})ds = 0, \ \int_a^b f(s,d)ds = 0, \ \int_a^b f(s,c)ds = 0, \ \int_c^d f(\frac{a+3b}{4}, t)dt = 0,$$

$$\int_c^d f(b,t)dt = 0, \quad \int_c^d f(a,t)dt = 0, \quad \int_a^b \int_c^d f(s,t)dsdt = 0, \quad \text{and}$$

$$\int_a^b \int_c^d H(s,t,I_2,J_2)dsdt = \frac{(b-a)^2(d-c)^2}{16}.$$

Thus, the identity of (2.4) is obtained immediately. This completes the proofs of Theorem 6. ∎

**Remark 2.** *We note that open problem in* [12] *is solved in Theorem 6.*

## 3. Applications In Numerical Integrations

In the following, a sharp error bound for new numerical integration formula will be given.

**Theorem 7.** *Suppose that $f$ satisfies the conditions of Theorem 5, $I_n : a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ is a division of $[a,b]$, and $I_{i,k} : t_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,k-1} < x_{i,k} = t_{i+1}$ is a division of the interval $[t_i, t_{i+1}]$,*

$i = 0, 1, \ldots, n-1$. *Further, let* $m_{i,j} \in [x_{i,j-1}, x_{i,j}]$, $m_{i,0} = t_i$, $m_{i,k+1} = t_{i+1}$, $j = 1, \ldots, k$, $C_i = \frac{1}{2}(t_{i+1} + t_i) - \frac{1}{t_{i+1}-t_i}\sum_{j=0}^{k-1} m_{i,j+1}(x_{i,j+1} - x_{i,j})$ *and*

$$P(t, I_{i,k}) = \begin{cases} t - m_{i,1} - C_i, & t \in [x_{i,0}, x_{i,1}), \\ t - m_{i,2} - C_i, & t \in [x_{i,1}, x_{i,2}), \\ \quad\vdots & \quad\vdots \\ t - m_{i,k-1} - C_i, & t \in [x_{i,k-2}, x_{i,k-1}), \\ t - m_{i,k} - C_i, & t \in [x_{i,k-1}, x_{i,k}), \end{cases}$$

*for* $i = 0, 1, \ldots, n-1$.

*Then for any matrices* $X = (x_{i,j})_{n\times(k+1)}$, $M = (m_{i,j})_{n\times(k+2)}$, *we have*

$$\Big|\int_a^b f(x)dx - T(f, X, M)\Big| \le \frac{\Gamma - l}{2}\int_a^b |P(t, I_k)|dt \tag{3.1}$$

*where*

$$T(f, X, M) = \sum_{i=0}^{n-1}\Big\{\frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}\Big[\frac{t_{i+1}^2 - t_i^2}{2} - \sum_{j=0}^{k-1} m_{i,j+1}(x_{i,j+1} - x_{i,j})\Big] - \sum_{j=0}^{k-1}(m_{i,j+1} - m_{i,j})f(x_{i,j})\Big\}.$$

*Proof.* Applying Theorem 5 with $[a, b]$ replaced by $[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, n-1$, we obtain

$$\begin{aligned} \Big|\sum_{j=0}^{k}(m_{i,j+1} - m_{i,j})f(x_{i,j}) - \int_{t_i}^{t_{i+1}} f(t)dt \\ - \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}\Big[\frac{t_{i+1}^2 - t_i^2}{2} - \sum_{j=0}^{k-1} m_{i,j+1}(x_{i,j+1} - x_{i,j})\Big]\Big| \\ \le \frac{(\Gamma - \gamma)}{2}\int_{t_i}^{t_{i+1}} |P(t, I_{i,k})|dt, \end{aligned} \tag{3.2}$$

and then by summing the inequality (3.2) from $i = 0$ to $n-1$ we have the desired result. This completes the proofs of Theorem 7. ∎

**Remark 3.** *The constant* $\frac{1}{2}$ *in inequality* (3.1) *is sharp, and Theorem 7 is a generalization of Theorem 3.1 in* [12].

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