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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES IN ABSOLUTE VALUE ARE QUASI-CONVEX WITH APPLICATIONS

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Abstract. In this paper some new Hadamard-type inequalities for functions whose derivatives in absolute values are quasi-convex are established. Some applications to special means of real numbers and applications for P.D.F.'s are given. We also give some applications of our obtained results to get new error bounds for the sum of the midpoint and trapezoidal formulae.

1. Introduction

The following definition for convex functions is well known in the mathematical literature: A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard inequality [23] has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see [5]-[29] and the references therein.

We recall that the notion of quasi-convex functions generalizes the notion of convex functions.

A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in [a, b].$$

It is to be noted that any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see e.g. [1]-[5]).

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In [1], Alomari *et al.* proved following inequalities of Hermite-Hadamard-type inequalities for quasi-convex functions;

Theorem 1. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is an quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, \left|f'(a)\right|\right\} + \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, \left|f'(b)\right|\right\}\right]. \tag{1.2}$$

Theorem 2. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is an quasi-convex on $[a, b]$, for $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}, \left|f'(a)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} + \left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right]. \tag{1.3}$$

Theorem 3. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an quasi-convex on $[a, b]$, for $q \ge 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}, \left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}} + \left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}, \left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}\right]. \tag{1.4}$$

The main aim of this paper is to establish some new Hermite-Hadamard-type inequalities for functions whose derivatives in absolute value are quasi-convex. The interesting features of our results are that they give an estimate of the difference between twice the middle and sum of rightmost and leftmost terms connected with the Hermite-Hadamard inequalities given above by (1.1).

2. Main Results

To prove our results we need the following lemma:

Lemma 1. *(See [25]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° , the interior of I, where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality* *holds:*

$$
\begin{aligned}
& f(x)+\frac{(b-x) f(b)+(x-a) f(a)}{b-a}-\frac{2}{b-a} \int_a^b f(u)\, du \qquad (2.1)\\
& =\frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1+t}{2} x+\frac{1-t}{2} a\right) dt \\
& -\frac{(x-a)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1-t}{2} x+\frac{1+t}{2} a\right) dt \\
& -\frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1+t}{2} x+\frac{1-t}{2} b\right) dt \\
& +\frac{(b-x)^2}{b-a} \int_0^1 \frac{t}{2} f'\left(\frac{1-t}{2} x+\frac{1+t}{2} b\right) dt,
\end{aligned}
$$

for all $x \in [a, b]$.

A simple proof of this equality can be done by integrating by parts on the right hand side. The details are left to the interested reader.

Using the Lemma 1 the following results can be obtained:

Theorem 4. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$
\begin{aligned}
& \left|f(x)+\frac{(b-x) f(b)+(x-a) f(a)}{b-a}-\frac{2}{b-a} \int_a^b f(u)\, du\right| \qquad (2.2)\\
& \leq \frac{(x-a)^2}{4(b-a)} \max \left\{\left|f'\left(\frac{x+a}{2}\right)\right|,\left|f'(x)\right|\right\} \\
& +\frac{(x-a)^2}{4(b-a)} \max \left\{\left|f'\left(\frac{x+a}{2}\right)\right|,\left|f'(a)\right|\right\} \\
& +\frac{(b-x)^2}{4(b-a)} \max \left\{\left|f'\left(\frac{x+b}{2}\right)\right|,\left|f'(x)\right|\right\} \\
& +\frac{(b-x)^2}{4(b-a)} \max \left\{\left|f'\left(\frac{x+b}{2}\right)\right|,\left|f'(b)\right|\right\},
\end{aligned}
$$

for all $x \in [a, b]$.

Proof. Using Lemma 1 and taking the modulus, we have

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.3}$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|dt$$
$$+\frac{(x-a)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|dt.$$

Using the quasi-convexity of $|f'|$ on $[a,b]$, we get from the inequality (2.3) that;

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.4}$$
$$\leq \frac{(x-a)^2}{b-a}\max\left\{\left|f'\left(\frac{x+a}{2}\right)\right|,|f'(x)|\right\}\int_0^1 \frac{t}{2}dt$$
$$+\frac{(x-a)^2}{b-a}\max\left\{\left|f'\left(\frac{x+a}{2}\right)\right|,|f'(a)|\right\}\int_0^1 \frac{t}{2}dt$$
$$+\frac{(b-x)^2}{b-a}\max\left\{\left|f'\left(\frac{x+b}{2}\right)\right|,|f'(x)|\right\}\int_0^1 \frac{t}{2}dt$$
$$+\frac{(b-x)^2}{b-a}\max\left\{\left|f'\left(\frac{x+b}{2}\right)\right|,|f'(b)|\right\}\int_0^1 \frac{t}{2}dt,$$

holds for all $x\in[a,b]$.
Since

$$\int_0^1 \frac{t}{2}dt=\frac{1}{4},$$

we get from (2.4) the inequality (2.2) . This completes the proof of the theorem. □

An immediate consequence of Theorem 4 is the following:

Corollary 1. *If all the assumptions of Theorem 4 are satisfied and if we choose* $x=\frac{a+b}{2}$*, we get the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.5}$$
$$\leq \frac{b-a}{16}\left[\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|,|f'(a)|\right\}\right.$$
$$\left.+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|,|f'(b)|\right\}\right]$$

Additionally,

(1) *If $|f'|$ is increasing, then*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.6}$$
$$\leq \frac{b-a}{16}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|+|f'(b)|\right]$$

(2) *If $|f'|$ is decreasing, then*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.7}$$
$$\leq \frac{b-a}{16}\left[|f'(a)|+\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right].$$

Remark 1. *If all the assumptions of Theorem 4 are satisfied and if we choose $x=a$ or $x=b$, we obtain the inequality of (1.2).*

Remark 2. *We note that the inequalities (2.6) and (2.7) are two new inequalities for the sum of the trapezoid and the midpoint inequalities for qusi-convex functions, and thus for convex functions.*

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following theorem.

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$ for some fixed $q>1$, then the following inequality holds:*

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.8}$$
$$\leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^2}{b-a}\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^2}{b-a}\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^2}{b-a}\left(\max\left\{|f'(b)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},$$

for all $x \in [a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the well-known Hölder integral inequality, we have

$$\left| f(x) + \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{2}{b-a} \int_a^b f(u)\,du \right| \tag{2.9}$$
$$\leq \frac{(x-a)^2}{b-a} \left(\int_0^1 \left(\frac{t}{2} \right)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{1+t}{2} x + \frac{1-t}{2} a \right) \right|^q dt \right)^{\frac{1}{q}}$$
$$+ \frac{(x-a)^2}{b-a} \left(\int_0^1 \left(\frac{t}{2} \right)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{1-t}{2} x + \frac{1+t}{2} a \right) \right|^q dt \right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^2}{b-a} \left(\int_0^1 \left(\frac{t}{2} \right)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{1+t}{2} x + \frac{1-t}{2} b \right) \right|^q dt \right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^2}{b-a} \left(\int_0^1 \left(\frac{t}{2} \right)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{1-t}{2} x + \frac{1+t}{2} b \right) \right|^q dt \right)^{\frac{1}{q}},$$

for all $x \in [a,b]$.
Since $|f'|^q$ is quasi-convex on $[a,b]$, we have

$$\int_0^1 \left| f'\left(\frac{1+t}{2} x + \frac{1-t}{2} a \right) \right|^q dt \leq \max\left\{ |f'(x)|^q, \left| f'\left(\frac{x+a}{2} \right) \right|^q \right\} \tag{2.10}$$

Similarly,

$$\int_0^1 \left| f'\left(\frac{1-t}{2} x + \frac{1+t}{2} a \right) \right|^q dt \leq \max\left\{ |f'(a)|^q, \left| f'\left(\frac{x+a}{2} \right) \right|^q \right\}, \tag{2.11}$$

$$\int_0^1 \left| f'\left(\frac{1+t}{2} x + \frac{1-t}{2} b \right) \right|^q dt \leq \max\left\{ |f'(x)|^q, \left| f'\left(\frac{x+b}{2} \right) \right|^q \right\} \tag{2.12}$$

and

$$\int_0^1 \left| f'\left(\frac{1-t}{2} x + \frac{1+t}{2} b \right) \right|^q dt \leq \max\left\{ |f'(b)|^q, \left| f'\left(\frac{x+b}{2} \right) \right|^q \right\}. \tag{2.13}$$

Using the inequalities (2.10)-(2.13) and by using the fact that;

$$\int_0^1 \left(\frac{t}{2} \right)^p dt = \frac{1}{2^p} \frac{1}{p+1}$$

in (2.9), we get the inequality (2.8), which completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5:

Corollary 2. *Suppose all the assumptions of Theorem 5 are satisfied and if we choose* $x = \frac{a+b}{2}$, *we get the following inequality:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(u)\,du \right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{b-a}{8}\right) \left\{ \left(\max\left\{ \left| f'\left(\frac{a+b}{2}\right)\right|^q, \left| f'\left(\frac{3a+b}{4}\right)\right|^q \right\}\right)^{\frac{1}{q}} \right. \\ &+ \left(\max\left\{ \left| f'(a)\right|^q, \left| f'\left(\frac{3a+b}{4}\right)\right|^q \right\}\right)^{\frac{1}{q}} \\ &+ \left(\max\left\{ \left| f'\left(\frac{a+b}{2}\right)\right|^q, \left| f'\left(\frac{a+3b}{4}\right)\right|^q \right\}\right)^{\frac{1}{q}} \\ &\left. + \left(\max\left\{ \left| f'(b)\right|^q, \left| f'\left(\frac{a+3b}{4}\right)\right|^q \right\}\right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{2.14}$$

Additionally,

(1) *If* $|f'|^q$ *is increasing, then*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(u)\,du \right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{b-a}{8}\right) \left[\left| f'\left(\frac{a+b}{2}\right)\right| + \left| f'\left(\frac{3a+b}{4}\right)\right| \right. \\ &\left. + \left| f'\left(\frac{a+3b}{4}\right)\right| + \left| f'(b)\right| \right]. \end{aligned} \tag{2.15}$$

(2) *If* $|f'|^q$ *is decreasing, then*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(u)\,du \right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{b-a}{8}\right) \left[\left| f'(a)\right| + \left| f'\left(\frac{3a+b}{4}\right)\right| \right. \\ &\left. + \left| f'\left(\frac{a+b}{2}\right)\right| + \left| f'\left(\frac{a+3b}{4}\right)\right| \right]. \end{aligned} \tag{2.16}$$

Remark 3. *If all the assumptions of Theorem 5 are satisfied and if we choose* $x = a$ *or* $x = b$, *we obtain the inequality of (1.3).*

A more general inequality is given as follows:

Theorem 6. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is quasi-convex on* $[a,b]$ *for some* *fixed $q\geq 1$, then the following inequality holds:*

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.17}$$
$$\leq\frac{1}{4}\left\{\frac{(x-a)^2}{b-a}\left[\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]$$
$$+\frac{(b-x)^2}{b-a}\left[\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\max\left\{|f'(b)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]\right\},$$

for all $x\in[a,b]$.

Proof. Suppose that $q\geq 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \tag{2.18}$$
$$\leq\frac{(x-a)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\frac{t}{2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}},$$

for all $x\in[a,b]$.

Since $|f'|^q$ is quasi-convex on $[a,b]$, we have

$$\int_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt$$
$$\leq\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\int_0^1\frac{t}{2}dt$$
$$=\frac{1}{4}\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\},$$

holds for all $x\in[a,b]$.

Similarly, we have that the following inequalities hold for all $x\in[a,b]$:

$$\int_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\leq\frac{1}{4}\max\left\{|f'(a)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\},$$

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt \le \frac{1}{4}\max\left\{\left|f'(x)\right|^q, \left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}$$

and

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt \le \frac{1}{4}\max\left\{\left|f'(b)\right|^q, \left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}.$$

By making use of the last four inequalities in (2.18), we get (2.17). Hence the proof of the theorem is complete. □

Corollary 3. *If all the assumptions of Theorem 6 are satisfied and if we choose* $x=\frac{a+b}{2}$*, we get the inequality:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int_a^b f(u)\,du\right| \qquad (2.19)\\
&\le \frac{1}{16}\left\{\left(\max\left\{\left|f'(a)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\
&+\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{\left|f'(b)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

Additionally,

1. *If* $|f'|^q$ *is increasing, then (2.6) holds.*
   And
2. *If* $|f'|^q$ *is decreasing, then (2.7) holds.*

Remark 4. *Since for* $p>1$, $\frac{1}{2}\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\le 1$*. Hence the approach via the power-mean integral inequality is better than that of the approach via the Hölder inequality.*

Remark 5. *If all the assumptions of Theorem 6 are satisfied and if we choose* $x=a$ *or* $x=b$*, we obtain the inequality of (1.4).*

3. Applications to Special Means

Now, we consider the applications of our Theorems to the special means. We consider the means for arbitrary real numbers a, $b\in\mathbb{R}$. We take

1. The arithmetic mean:
$$A(a,b)=\frac{a+b}{2};\ a,\ b\in\mathbb{R}.$$
2. The harmonic mean:
$$H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}};a,\ b\in\mathbb{R},a,b\neq 0$$

(3) The logarithmic mean:

$$L(a,b)=\frac{\ln|b|-\ln|a|}{b-a};a,b\in\mathbb{R},|a|\neq|b|,a,b\neq 0.$$

(4) Generalized log-mean:

$$L_n(a,b)=\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}};\ a,\ b\in\mathbb{R},\ n\in\mathbb{Z}\backslash\{-1,0\},\ a\neq b.$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let $a,\ b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\geq 2$. Then*

$$\begin{aligned}&|A^n(a,b)+A(a^n,b^n)-2L_n^n(a,b)| \\ &\leq n\left(\frac{b-a}{16}\right)\left[\max\left\{\left|\frac{3a+b}{4}\right|^{n-1},\left|\frac{a+b}{2}\right|^{n-1}\right\}\right. \\ &+\max\left\{\left|\frac{3a+b}{4}\right|^{n-1},|a|^{n-1}\right\} \\ &+\max\left\{\left|\frac{3a+b}{4}\right|^{n-1},\left|\frac{a+b}{2}\right|^{n-1}\right\} \\ &\left.+\max\left\{\left|\frac{3a+b}{4}\right|^{n-1},|b|^{n-1}\right\}\right].\end{aligned}\tag{3.1}$$

Proof. The assertion follows from Corollary 1 when applied to the function $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{N}$, $n\geq 2$. □

Proposition 2. *Let $a,\ b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\geq 2$. Then for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\begin{aligned}&|A^n(a,b)+A(a^n,b^n)-2L_n^n(a,b)| \\ &\leq n\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{8}\right)\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)},\left|\frac{a+b}{2}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right. \\ &+\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)},|a|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\ &+\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)},\left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\ &\left.+\left(\max\left\{|b|^{q(n-1)},\left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.2}$$

Proof. The assertion follows from Corollary 2 when applied to the function $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{N}$, $n\geq 2$. □

Proposition 3. *Let a, $b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then $q \geq 1$, we have*

$$\begin{aligned}
&\left|A^{n}(a,b)+A(a^{n},b^{n})-2L_{n}^{n}(a,b)\right| \qquad (3.3)\\
&\leq n\left(\frac{b-a}{16}\right)\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)},\left|\frac{a+b}{2}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right.\\
&+\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)},|a|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\\
&+\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)},\left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{|b|^{q(n-1)},\left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. The assertion follows from Corollary 3 when applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$. □

Proposition 4. *Let a, $b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$. Then*

$$\begin{aligned}
&\left|A^{-1}(a,b)+H^{-1}(a,b)-2L(a,b)\right| \qquad (3.4)\\
&\leq\left(\frac{b-a}{16}\right)\left[\max\left\{\left|\frac{3a+b}{4}\right|^{-2},\left|\frac{a+b}{2}\right|^{-2}\right\}+\max\left\{\left|\frac{3a+b}{4}\right|^{-2},|a|^{-2}\right\}\right.\\
&\left.+\max\left\{\left|\frac{3a+b}{4}\right|^{-2},\left|\frac{a+b}{2}\right|^{-2}\right\}+\max\left\{\left|\frac{3a+b}{4}\right|^{-2},|b|^{-2}\right\}\right].
\end{aligned}$$

Proof. It is a direct consequence of Corollary 1 when applied to the function, $f(x) = \frac{1}{x}$, $x \in [a, b] \setminus \{0\}$. □

Proposition 5. *Let a, $b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$, then for all $p > 1$, we have*

$$\begin{aligned}
&\left|A^{-1}(a,b)+H^{-1}(a,b)-2L(a,b)\right| \qquad (3.5)\\
&\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{8}\right)\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q(n-1)},\left|\frac{a+b}{2}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\right.\\
&+\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q(n-1)},|a|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\\
&+\left(\max\left\{\left|\frac{a+b}{2}\right|^{-2q(n-1)},\left|\frac{a+3b}{4}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{|b|^{-2q(n-1)},\left|\frac{a+3b}{4}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. It follows directly from Corollary 2 for the function, $f(x) = \frac{1}{x}$, $x \in [a, b] \setminus \{0\}$. □

Proposition 6. *Let $a,\ b\in\mathbb{R},\ a<b,\ 0\notin[a,b]$. Then for all $q\geq 1$, we have the inequality*

$$\begin{aligned}&\left|A^{-1}(a,b)+H^{-1}(a,b)-2L(a,b)\right| \qquad (3.6)\\ &\leq\left(\frac{b-a}{16}\right)\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q(n-1)},\left|\frac{a+b}{2}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\right.\\ &+\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q(n-1)},|a|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\\ &+\left(\max\left\{\left|\frac{a+b}{2}\right|^{-2q(n-1)},\left|\frac{a+3b}{4}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\\ &\left.+\left(\max\left\{|b|^{-2q(n-1)},\left|\frac{a+3b}{4}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. It follows directly from Corollary 3 for the function, $f(x)=\frac{1}{x}$, $x\in[a,b]\setminus\{0\}$. □

4. Application to the Midpoint and Trapezoidal Formulae

Let d be a division of the interval $[a,b]$, *i.e.* $a=x_0<x_1<...<x_{n-1}<x_n=b$, and consider the quadrature formulae

$$\int_a^b f(x)dx=M(f,d)+E(f,d),$$

and

$$\int_a^b f(x)dx=T'(f,d)+E'(f,d),$$

where

$$T(f,d)=\sum_{i=0}^{n-1}(x_{i+1}-x_i)\,f\left(\frac{x_i+x_{i+1}}{2}\right)$$

and

$$T'(f,d)=\sum_{i=0}^{n-1}(x_{i+1}-x_i)\,\frac{f(x_i)+f(x_{i+1})}{2},$$

are the midpoint and trapezoidal versions and the approximation errors $E(f,d)$ and $E'(f,d)$ of the integral $\int_a^b f(x)dx$ by the midpoint formula and trapezoidal formula satisfy

$$|E(f,d)|\leq\frac{K}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \qquad (4.1)$$

and

$$|E'(f,d)|\leq\frac{K}{12}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3, \qquad (4.2)$$

respectively. If f is not twice differentiable (or the second derivative of f is not bounded on (a,b) then (4.1) and (4.2) cannot be applied. Following results give some new estimates for the sum of remainders $E(f,d)$ and $E'(f,d)$ in terms of the first derivative.

Proposition 7. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then for every division d of $[a,b]$, we have:*

$$
\begin{aligned}
&|E(f,d)+E'(f,d)| \qquad (4.3)\\
&\le \frac{1}{16}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\max\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|,\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}\right.\\
&+\max\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|,\left|f'(x_i)\right|\right\}\\
&+\max\left\{\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|,\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}\\
&\left.+\max\left\{\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|,\left|f'(x_{i+1})\right|\right\}\right].
\end{aligned}
$$

Proof. By applying Corollary 1 on the subinterval $[x_i,x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the division d, we have

$$
\begin{aligned}
&\left|f\left(\frac{x_i+x_{i+1}}{2}\right)+\frac{f(x_{i+1})+f(x_i)}{2}-\frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\,dx\right| \qquad (4.4)\\
&\le \frac{1}{16}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left[\max\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|,\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}\right.\\
&+\max\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|,\left|f'(x_i)\right|\right\}\\
&+\max\left\{\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|,\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}\\
&\left.+\max\left\{\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|,\left|f'(x_{i+1})\right|\right\}\right].
\end{aligned}
$$

Now

$$
\begin{aligned}
&|E(f,d)+E'(f,d)| \qquad (4.5)\\
&=\left|\sum_{i=0}^{n-1}(x_{i+1}-x_i)\,f\left(\frac{x_i+x_{i+1}}{2}\right)\right.\\
&\left.+\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)-2\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}f(x)dx\right|\\
&\le \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|f\left(\frac{x_i+x_{i+1}}{2}\right)+\frac{f(x_i)+f(x_{i+1})}{2}\right.\\
&\left.-\frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right|.
\end{aligned}
$$

Using (4.4) in (4.5), we get (4.3). This completes the proof of the proposition. □

Corollary 4. *Suppose all the assumptions of Proposition 7 are satisfied. Additionally,*

(1) *If* $|f'|$ *is increasing, then*

$$|E(f,d)+E'(f,d)| \tag{4.6}$$
$$\leq \frac{1}{16}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right.$$
$$\left.+\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|+\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|+\left|f'\left(x_{i+1}\right)\right|\right].$$

(2) *If* $|f'|$ *is decreasing, then*

$$|E(f,d)+E'(f,d)| \tag{4.7}$$
$$\leq \frac{1}{16}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|\right.$$
$$\left.+\left|f'\left(x_i\right)\right|+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|\right].$$

Proposition 8. *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$*, where* $a,b\in I$ *with* $a<b$*. If* $|f'|^q$ *is quasi-convex on* $[a,b]$ *for some fixed* $q>1$*, then for every division* d *of* $[a,b]$*, we have*

$$|E(f,d)+E'(f,d)| \tag{4.8}$$
$$\leq \frac{1}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$+\left(\max\left\{\left|f'\left(x_i\right)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$\left.+\left(\max\left\{\left|f'\left(x_{i+1}\right)\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The proof is similar to that of Proposition 7 and using Corollary 2. □

Corollary 5. *Suppose all the conditions of Proposition 8 are satisfied. Additionally,*

(1) *If* $|f'|^q$ *is increasing, then*

$$|E(f,d)+E'(f,d)| \tag{4.9}$$
$$\leq \frac{1}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\max\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right.$$
$$\left.+\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|+\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|+\left|f'\left(x_{i+1}\right)\right|\right\}$$

And

(2) *If* $|f'|^q$ *is decreasing, then*

$$\begin{aligned}&|E(f,d)+E'(f,d)| \qquad (4.10)\\&\leq \frac{1}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|\right.\\&\left.+\left|f'(x_i)\right|+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|\right\}.\end{aligned}$$

Proposition 9. *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$*, where* $a,b\in I$ *with* $a<b$*. If* $|f'|^q$ *is quasi-convex on* $[a,b]$ *for some fixed* $q\geq 1$*, then for every division* d *of* $[a,b]$*, we have*

$$\begin{aligned}&|E(f,d)+E'(f,d)| \qquad (4.11)\\&\leq \frac{1}{16}\left\{\left(\max\left\{\left|f'(x_i)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\&+\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\&+\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\&\left.+\left(\max\left\{\left|f'(x_{i+1})\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. The proof is similar to that of Proposition 7 and using Corollary 3. □

Corollary 6. *Under the assumptions of Proposition 9, if*

(1) $|f'|^q$ *is increasing, then (4.6) holds.*
(2) $|f'|^q$ *is decreasing, then (4.7) holds.*

5. APPLICATIONS FOR P.D.F's

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=\Pr(X\leq x)=\int_a^b f(t)dt$.

Theorem 7. *Under the assumptions of Theorem 4, we have the inequality;*

$$\begin{aligned}&\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}(b-E(x))\right|\\&\leq\frac{(x-a)^2}{4(b-a)}\max\left\{\left|f'\left(\frac{x+a}{2}\right)\right|,\left|f'(x)\right|\right\}\\&+\frac{(x-a)^2}{4(b-a)}\max\left\{\left|f'\left(\frac{x+a}{2}\right)\right|,\left|f'(a)\right|\right\}\\&+\frac{(b-x)^2}{4(b-a)}\max\left\{\left|f'\left(\frac{x+b}{2}\right)\right|,\left|f'(x)\right|\right\}\\&+\frac{(b-x)^2}{4(b-a)}\max\left\{\left|f'\left(\frac{x+b}{2}\right)\right|,\left|f'(b)\right|\right\},\end{aligned}$$

where $E(x)$ *is the expectation of* X.

Proof. The proof is immediate follows from the fact that;

$$E(x)=\int_a^b tdF(t)=b-\int_a^b F(t)\,dt.$$

□

Theorem 8. *Under the assumptions of Theorem 5, we have the inequality;*

$$\begin{aligned}
&\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}(b-E(x))\right| \\
&\leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^2}{b-a}\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&+\frac{(x-a)^2}{b-a}\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{b-a}\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\left.+\frac{(b-x)^2}{b-a}\left(\max\left\{|f'(b)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. Likewise the proof of the previous theorem, by using the fact that;

$$E(x)=\int_a^b tdF(t)=b-\int_a^b F(t)\,dt$$

the proof is completed. □

Theorem 9. *Under the assumptions of Theorem 6, we have inequality;*

$$\begin{aligned}
&\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}(b-E(x))\right| \\
&\leq \frac{1}{4}\left\{\frac{(x-a)^2}{b-a}\left[\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+a}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right] \\
&+\frac{(b-x)^2}{b-a}\left[\left(\max\left\{|f'(x)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\max\left\{|f'(b)|^q,\left|f'\left(\frac{x+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]\right\},
\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. The proof is similar to the previous theorem. □

Corollary 7. *Some new results can be obtained by choosing $x=a$, $x=b$ and $x=\frac{a+b}{2}$, seperately, in Theorem 7-9.*

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