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ON LEVINSON'S INEQUALITY

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ABSTRACT. We give a very simple proof of the classical Levinson inequality and generalise the result by Mercer.

In 1964 Norman Levinson ([3]) used Taylor expansion to prove the following inequality.

Theorem 1. *Suppose that* $f:[0,c]\to\mathbf{R}$ *has a nonnegative third derivative, for* $i=1,\dots,n$ $p_i>0$, $0\leqslant x_i\leqslant c/2$, $y_i=c-x_i$ *and* $\sum_{i=1}^n p_i=1$, *then the inequality*

$$\sum_{i=1}^n p_i f(x_i)-f(\overline{x})\leqslant\sum_{i=1}^n p_i f(y_i)-f(\overline{x}) \tag{1}$$

holds. Here $\overline{x}=\sum_{i=1}^n p_ix_i$ *and* $\overline{y}=\sum_{i=1}^n p_iy_i$ *denote the weighted arithmetic means.*

The same year Tiberiu Popoviciu generalised it by showing that for (1) to hold it is enough that f be 3-convex ([5]), and Peter S. Bullen ([2]) gave an alternative proof using mathematical induction. By rescaling axes, the Levinson inequality can be restated in the following way.

Theorem 2. *If* $f:[a,b]\to\mathbf{R}$ *is 3-convex,* $a\leqslant x_i,y_i\leqslant b$, $x_i+y_i=c$, $p_i>0$ *for* $i=1,\dots,n$, $\sum_{i=1}^n p_i=1$ *and*

$$\max(x_1,\dots,x_n)\leqslant\min(y_1,\dots,y_n), \tag{2}$$

then (1) *holds.*

We shall give here a really simple proof of Levinson's inequality using only basic properties of convex functions.

As we can see, both versions assume that x's and y's add up to the same number. Recently, Mercer made a significant improvement in [4].

Theorem 3. *If* $f:[a,b]\to\mathbf{R}$ *satisfies* $f'''\geqslant 0$, $p_i>0$ *for* $i=1,\dots,n$, $\sum_{i=1}^n p_i=1$, *and* $a\leqslant x_i,y_i\leqslant b$ *are such that* (2) *holds and*

$$\sum_{i=1}^n p_i(x_i-\overline{x})^2=\sum_{i=1}^n p_i(y_i-\overline{y})^2, \tag{3}$$

then (1) *holds.*

It is natural to try to get similar result for 3-convex function. We provide the proof weakening the condition (3).

Let us remind some properties of convex and 3-convex functions.

Property 1. *The function* g *is convex if and only if its divided difference* $h(x,y)=\frac{g(x)-g(y)}{x-y}$, $x\neq y$ *increases in both variables.*

This implies immediately

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Property 2. *If g is convex, then its right and left derivatives exist in the interior of its domain and for $x<y$ the inequalities*

$$g'_-(x)\leqslant g'_+(x)\leqslant g'_-(y)\leqslant g'_+(y)$$

hold.

Another useful property of symmetric sum follows also from Property 1

Property 3. *If g is convex, then the symmetric sum $g(a+x)+g(a-x)$ increases for $x>0$.*

Boas and Widder proved the characterisation of 3-convex functions.

Property 4 ([1])**.** *If g is 3-convex, then its derivative exists and is convex.*

Simple proof of Theorem 1. We can rewrite (1) as

$$f\left(c-\sum_{i=1}^n p_iy_i\right)-f\left(\sum_{i=1}^n p_ix_i\right)\leqslant\sum_{i=1}^n p_if(c-y_i)-\sum_{i=1}^n p_if(x_i), \tag{4}$$

which is the Jensen inequality for the function $g(x)=f(c-x)-f(x)$. The function g is diffrentiable by Property 4. To show its convexity, note that

$$g'(x)=-[f'(c-x)+f'(x)]=-[f'(c/2+(c/2-x))+f'(c/2-(c/2-x))].$$

As x increases, $c/2-x$ decreases and by Property 3, the expression in square brackets decreases. Thus g' increases, so g is convex and we are done. □

Let us take care of Mercer's version of the Levinson inequality.
In virtue of Property 4, the convexity of the derivative of a 3-convex functions implies that

$$f''_-(\max x_i)\leqslant f''_+(\min y_i),$$

(here $f''_\pm$ denote the right and left derivative of f'), which means that there are three possible cases: or f is concave in the gap between x's and y's, or f is convex there, or it changes its convexity from concave to convex. This observation allows for the following generalisation Theorem 3.

Theorem 4. *Let $f:(a,b)\to\mathbb{R}$ be 3-convex, $p_i>0$ for $i=1,\dots,n$, $\sum_{i=1}^n p_i=1$, and $a<x_i,y_i<b$ are such that (2) holds and*

(a) $f''_-(\max x_i)>0$ *and* $\sum_{i=1}^n p_i(x_i-\overline{x})^2\leqslant\sum_{i=1}^n p_i(y_i-\overline{y})^2,$
or
(b) $f''_+(\min y_i)<0$ *and* $\sum_{i=1}^n p_i(x_i-\overline{x})^2\geqslant\sum_{i=1}^n p_i(y_i-\overline{y})^2,$
or
(c) $f''_-(\max x_i)\leqslant 0\leqslant f''_+(\min y_i),$
then (1) holds.

Proof. For $0\leqslant t\leqslant 1$ let $x_i(t)=\overline{x}+t(x_i-\overline{x})$ and $y_i(t)=\overline{y}+t(y_i-\overline{y})$. The function

$$U(t)=\sum_{i=1}^n p_if(y_i(t))-f(\overline{y})-\sum_{i=1}^n p_if(x_i(t))+f(\overline{x})$$

is differentiable. Since f' is convex, for arbitrary $A \in [f''_-(\max x_i), f''_+(\min y_i)]$ we have

$$
\begin{aligned}
\frac{U'(t)-U'(s)}{t-s} &= \sum_{i=1}^n p_i(y_i-\overline{y})\frac{f'(y_i(t))-f'(y_i(s))}{t-s} - \sum_{i=1}^n p_i(x_i-\overline{x})\frac{f'(x_i(t))-f'(x_i(s))}{t-s} \\
&= \sum_{i=1}^n p_i(y_i-\overline{y})^2\frac{f'(y_i(t))-f'(y_i(s))}{y_i(t)-y_i(s)} - \sum_{i=1}^n p_i(x_i-\overline{x})^2\frac{f'(x_i(t))-f'(x_i(s))}{x_i(t)-x_i(s)} \\
&= \sum_{i=1}^n p_i(y_i-\overline{y})^2\left[\frac{f'(y_i(t))-f'(y_i(s))}{y_i(t)-y_i(s)} - A\right] + A\sum_{i=1}^n p_i(y_i-\overline{y})^2 \\
&\quad + \sum_{i=1}^n p_i(x_i-\overline{x})^2\left[A - \frac{f'(x_i(t))-f'(x_i(s))}{x_i(t)-x_i(s)}\right] - A\sum_{i=1}^n p_i(x_i-\overline{x})^2 \\
&\geqslant A\sum_{i=1}^n p_i[(y_i-\overline{y})^2-(x_i-\overline{x})^2] \geqslant 0. \qquad (5)
\end{aligned}
$$

(in case (c) we set $A = 0$ in (5)). So U' is nondecreasing, thus U is convex and since $U(0) = U'(0) = 0$, we conclude $U(0) \leqslant U(1)$, which is equivalent to (1). □

For the 3-concave functions Theorem 4 reads as follows:

Theorem 5. *Let $f : (a, b) \to \mathbb{R}$ be 3-concave, $p_i > 0$ for $i = 1, \dots, n$, $\sum_{i=1}^n p_i = 1$, and $a < x_i, y_i < b$ are such that (2) holds and*

(a') $f''_-(\max x_i) < 0$ and $\sum_{i=1}^n p_i(x_i-\overline{x})^2 \leqslant \sum_{i=1}^n p_i(y_i-\overline{y})^2$,

or

(b') $f''_+(\min y_i) > 0$ and $\sum_{i=1}^n p_i(x_i-\overline{x})^2 \geqslant \sum_{i=1}^n p_i(y_i-\overline{y})^2$,

or

(c') $f''_+(\min y_i) \leqslant 0 \leqslant f''_-(\max x_i)$,

then inverse to (1) holds.

The proof is exactly the same, but in this case the function $U(t)$ is concave.

Applying the Hermite-Hadamard inequality to the function U, we obtain the following refinement of the Levinson inequality.

Corollary 6. *Under the assumptions of Theorem 4 the following inequalities are valid*

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^n p_i f\left(\frac{y_i+\overline{y}}{2}\right) - f(\overline{y}) - \sum_{i=1}^n p_i f\left(\frac{x_i+\overline{x}}{2}\right) + f(\overline{x}) \\
&\leqslant \sum_{i=1}^n p_i \frac{\int_{\overline{y}}^{y_i} f(t)dt}{y_i-\overline{y}} - f(\overline{y}) - \sum_{i=1}^n p_i \frac{\int_{\overline{x}}^{x_i} f(t)dt}{x_i-\overline{x}} + f(\overline{x}) \\
&\leq \frac{1}{2}\left[\sum_{i=1}^n p_i f(y_i) - f(\overline{y}) - \sum_{i=1}^n p_i f(x_i) + f(\overline{x})\right].
\end{aligned}
$$

Under the assumptions of Theorem 5 inverse inequalities hold.

Note that the rightmost inequality can be rewritten in a nice symmetric form

$$\sum_{i=1}^{n} p_i \left(\frac{f(x_i)+f(\overline{x})}{2} - \frac{\int_{\overline{x}}^{x_i} f(t)dt}{x_i-\overline{x}} \right) \leqslant \sum_{i=1}^{n} p_i \left(\frac{f(y_i)+f(\overline{y})}{2} - \frac{\int_{\overline{y}}^{y_i} f(t)dt}{y_i-\overline{y}} \right), \tag{6}$$

while the leftmost inequality is

$$\sum_{i=1}^{n} p_i \left(\frac{\int_{\overline{x}}^{x_i} f(t)dt}{x_i-\overline{x}} - f\left(\frac{x_i+\overline{x}}{2}\right) \right) \leqslant \sum_{i=1}^{n} p_i \left(\frac{\int_{\overline{y}}^{y_i} f(t)dt}{y_i-\overline{y}} - f\left(\frac{y_i+\overline{y}}{2}\right) \right). \tag{7}$$

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