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NORM AND NUMERICAL RADIUS INEQUALITIES FOR SUMS OF OPERATORS IN HILBERT SPACES

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ABSTRACT. By the use of some nonnegative Hermitian forms defined for n-tuple of bounded linear operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$ we establish new numerical radius and operator norm inequalities for sum of products of operators. We obtain among others that

$$w^2\left(\sum_{j=1}^{n} V_j^* T_j\right) \le \left\|\sum_{j=1}^{n} |T_j|^2\right\| \left\|\sum_{j=1}^{n} |V_j|^2\right\|$$

and

$$w\left(\sum_{j=1}^{n} V_j^* T_j\right) \le \left\|\sum_{j=1}^{n} \frac{|T_j|^2 + |V_j|^2}{2}\right\|$$

for any n-tuple of operators $(T_1,...,T_n)$, $(V_1,...,V_n)$. Applications for power series of normal operators with examples are given as well.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by [15, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [15, p. 9]:

Theorem 1 (Equivalent norm)**.** *For any $T \in \mathcal{B}(H)$ one has*

$$w(T) \le \|T\| \le 2w(T). \tag{1.2}$$

Some improvements of (1.2) are as follows:

Theorem 2 (Kittaneh, 2003 [20])**.** *For any operator $T \in \mathcal{B}(H)$ we have the following refinement of the first inequality in (1.2)*

$$w(T) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.3}$$

From a different perspective, we have the following result as well:

Theorem 3 (Dragomir, 2007 [6])**.** *For any operator $T \in \mathcal{B}(H)$ we have*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{1.4}$$

The following general result for the product of two operators holds [15, p. 37]:

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Theorem 4 (Holbrook, 1969 [17])**.** *If A, B are two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then $w(AB) \leq 4w(A)w(B)$. In the case that $AB = BA$, then $w(AB) \leq 2w(A)w(B)$. The constant 2 is best possible here.*

The following results are also well known [15, p. 38].

Theorem 5 (Holbrook, 1969 [17])**.** *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \leq w(B). \tag{1.5}$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

For other results on numerical radius inequalities see [1], [3]-[7], [9]-[12], [14] and [18]-[23].

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X, i.e., the mapping $(\cdot,\cdot) : X \times X \to \mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions

(i) $(x, x) \geq 0$ for all x in X;
(ii) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$;
(iii) $(y, x) = \overline{(x, y)}$ for all $x, y \in X$.

If $(\cdot,\cdot) \in \mathcal{H}(X)$, then the following equivalent versions of *Schwarz's inequality* hold:

$$\|x\|^2 \|y\|^2 \geq |(x, y)|^2 \quad \text{or} \quad \|x\| \|y\| \geq |(x, y)| \tag{1.6}$$

for any $x, y \in X$.

A simple consequence of the Schwarz inequality is the functional $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X, i.e. we have the properties

(n) $\|x\| \geq 0$ for all x in X;
(nn) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$;
(nnn) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (the triangle inequality).

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{H}(X)$ implies that $(\cdot,\cdot)_1 + (\cdot,\cdot)_2 \in \mathcal{H}(X)$;
(ee) $\alpha \geq 0$ and $(\cdot,\cdot) \in \mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot) \in \mathcal{H}(X)$.

For various properties and new results for nonnegative Hermitian forms, see the book [8]

In this paper, by the use of some nonnegative Hermitian forms defined for n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ we establish some new numerical radius and operator norm inequalities for sums and sum of products of operators. Applications for functions defined by power series of operators are given. Some applications related for the exponential function, trigonometric functions and the functions $f(z) := (1 - z)^{-1}$ and $g(z) := \ln(1 - z)^{-1}$ are also provided.

2. Inequalities for Sums

Following Popescu's work [24], we can consider the following norm on $B(H)^{(n)} := B(H) \times \cdots \times B(H)$, by setting

$$\|(T_1, \ldots, T_n)\|_e := \sup_{(\lambda_1,\ldots,\lambda_n) \in \mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\| \tag{2.1}$$

where B_n is the closed unit ball in $\mathbb{C}^n$.

Notice that $\|\cdot\|_e$ is a norm on $B(H)^{(n)}$ and

$$\|(T_1,\ldots,T_n)\|_e = \|(T_1^*,\ldots,T_n^*)\|_e . \tag{2.2}$$

Now, if we denote by $\|[T_1,\ldots,T_n]\|$ the square root of the norm $\|\sum_{i=1}^n T_i T_i^*\|$, i.e.,

$$\|[T_1,\ldots,T_n]\| := \left\|\sum_{i=1}^n T_i T_i^*\right\|^{\frac{1}{2}}, \tag{2.3}$$

then we can present the following result due to G. Popescu [24] concerning some sharp inequalities between the norms $\|[T_1,\ldots,T_n]\|$ and $\|(T_1,\ldots,T_n)\|_e$:

Theorem 6 (Popescu, 2004)**.** *If $(T_1,\ldots,T_n)\in B(H)^{(n)}$, then*

$$\frac{1}{\sqrt{n}}\|[T_1,\ldots,T_n]\| \le \|(T_1,\ldots,T_n)\|_e \le \|[T_1,\ldots,T_n]\|, \tag{2.4}$$

where the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible in (2.4).

Let $(T_1,...,T_n)\in\mathcal{B}(H)\times...\times\mathcal{B}(H):=\mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$ and $(p_1,...,p_n)\in\mathbb{R}_+^{*n}$ an n-tuple of nonnegative weights not all of them equal to zero. For an $x\in H$, $x\neq 0$ we define

$$\langle T,V\rangle_{p,x} := \sum_{j=1}^n p_j\langle T_j x, V_j x\rangle = \left\langle\left(\sum_{j=1}^n p_j V_j^* T_j\right)x, x\right\rangle \tag{2.5}$$

where $T=(T_1,...,T_n), V=(V_1,...,V_n)\in\mathcal{B}^{(n)}(H)$.

We can then state the following result:

Lemma 1. *For any $x\in H$, $x\neq 0$ and $(p_1,...,p_n)\in\mathbb{R}_+^{*n}$ we have that $\langle\cdot,\cdot\rangle_{p,x}$ is a nonnegative Hermitian form on $\mathcal{B}^{(n)}(H)$.*

Proof. We have that

$$\langle T,T\rangle_{p,x} = \left\langle\left(\sum_{j=1}^n p_j T_j^* T_j\right)x,x\right\rangle = \left\langle\left(\sum_{j=1}^n p_j |T_j|^2\right)x,x\right\rangle \ge 0, \tag{2.6}$$

for any $T=(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$, where the *operator modulus* is defined by $|A|^2=A^*A$, $A\in\mathcal{B}(H)$.

The functional $\langle\cdot,\cdot\rangle_{p,x}$ is linear in the first variable and

$$\begin{aligned}\overline{\langle V,T\rangle}_{p,x} &= \overline{\left\langle\left(\sum_{j=1}^n p_j T_j^* V_j\right)x,x\right\rangle} = \left\langle x,\left(\sum_{j=1}^n p_j T_j^* V_j\right)x\right\rangle \\ &= \left\langle\left(\sum_{j=1}^n p_j T_j^* V_j\right)^* x,x\right\rangle = \left\langle\left(\sum_{j=1}^n p_j V_j^* T_j\right)x,x\right\rangle = \langle T,V\rangle_{p,x}\end{aligned} \tag{2.7}$$

for any $T=(T_1,...,T_n), V=(V_1,...,V_n)\in\mathcal{B}^{(n)}(H)$. □

If $p=(1,...,1)$, then we denote $\langle\cdot,\cdot\rangle_{p,x}$ by $\langle\cdot,\cdot\rangle_x$.

Theorem 7. *For any* $T=(T_1,...,T_n)$,$V=(V_1,...,V_n)\in\mathcal{B}^{(n)}(H)$ *we have the inequalities*

$$w^2\left(\sum_{j=1}^n V_j^*T_j\right)\le\left\|\sum_{j=1}^n |T_j|^2\right\|\left\|\sum_{j=1}^n |V_j|^2\right\| \tag{2.8}$$

and

$$\left\|\sum_{j=1}^n |T_j+V_j|^2\right\|^{1/2}\le\left\|\sum_{j=1}^n |T_j|^2\right\|^{1/2}+\left\|\sum_{j=1}^n |V_j|^2\right\|^{1/2}. \tag{2.9}$$

Proof. Let $x\in H$, $x\neq 0$. Writing the Schwarz inequality for the nonnegative Hermitian form $\langle\cdot,\cdot\rangle_x$ we have

$$\left|\left\langle\left(\sum_{j=1}^n V_j^*T_j\right)x,x\right\rangle\right|^2\le\left\langle\left(\sum_{j=1}^n |T_j|^2\right)x,x\right\rangle\left\langle\left(\sum_{j=1}^n |V_j|^2\right)x,x\right\rangle. \tag{2.10}$$

Taking the supremum over $x\in H,\|x\|=1$ we have

$$\begin{aligned}
w^2\left(\sum_{j=1}^n V_j^*T_j\right)&=\sup_{\|x\|=1}\left|\left\langle\left(\sum_{j=1}^n V_j^*T_j\right)x,x\right\rangle\right|^2\\
&\le\sup_{\|x\|=1}\left\{\left\langle\left(\sum_{j=1}^n |T_j|^2\right)x,x\right\rangle\left\langle\left(\sum_{j=1}^n |V_j|^2\right)x,x\right\rangle\right\}\\
&\le\sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^n |T_j|^2\right)x,x\right\rangle\sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^n |V_j|^2\right)x,x\right\rangle\\
&=\left\|\sum_{j=1}^n |T_j|^2\right\|\left\|\sum_{j=1}^n |V_j|^2\right\|
\end{aligned}$$

and the inequality (2.8) is obtained.

Let $x\in H$, $x\neq 0$. Utilising the triangle inequality for the nonnegative Hermitian form $\langle\cdot,\cdot\rangle_x$ we also have

$$\begin{aligned}
&\left\langle\left(\sum_{j=1}^n |T_j+V_j|^2\right)x,x\right\rangle^{1/2}\\
&\le\left\langle\left(\sum_{j=1}^n |T_j|^2\right)x,x\right\rangle^{1/2}+\left\langle\left(\sum_{j=1}^n |V_j|^2\right)x,x\right\rangle^{1/2}.
\end{aligned} \tag{2.11}$$

Taking the supremum over $x \in H, \|x\| = 1$ we have

$$\begin{aligned}
\left\| \sum_{j=1}^{n} |T_j + V_j|^2 \right\|^{1/2} &= \sup_{\|x\|=1} \left\langle \left(\sum_{j=1}^{n} |T_j + V_j|^2 \right) x, x \right\rangle^{1/2} \\
&\le \sup_{\|x\|=1} \left[\left\langle \left(\sum_{j=1}^{n} |T_j|^2 \right) x, x \right\rangle^{1/2} + \left\langle \left(\sum_{j=1}^{n} |V_j|^2 \right) x, x \right\rangle^{1/2} \right] \\
&\le \sup_{\|x\|=1} \left\langle \left(\sum_{j=1}^{n} |T_j|^2 \right) x, x \right\rangle^{1/2} + \sup_{\|x\|=1} \left\langle \left(\sum_{j=1}^{n} |V_j|^2 \right) x, x \right\rangle^{1/2} \\
&= \left\| \sum_{j=1}^{n} |T_j|^2 \right\|^{1/2} + \left\| \sum_{j=1}^{n} |V_j|^2 \right\|^{1/2}
\end{aligned}$$

which proves the inequality (2.9). □

Corollary 1. *For any $T = (T_1, ..., T_n)$,$V = (V_1, ..., V_n) \in \mathcal{B}^{(n)}(H)$ with $\sum_{j=1}^{n} |T_j + V_j|^2 \neq 0$ we have the following refinement of the triangle inequality*

$$\begin{aligned}
(2.12) \qquad & \left\| \sum_{j=1}^{n} |T_j + V_j|^2 \right\|^{1/2} \\
&\le \frac{w\left(\sum_{j=1}^{n} |T_j|^2 + \sum_{j=1}^{n} V_j^* T_j\right) + w\left(\sum_{j=1}^{n} T_j^* V_j + \sum_{j=1}^{n} |V_j|^2\right)}{\left\| \sum_{j=1}^{n} |T_j + V_j|^2 \right\|^{1/2}} \\
&\le \left\| \sum_{j=1}^{n} |T_j|^2 \right\|^{1/2} + \left\| \sum_{j=1}^{n} |V_j|^2 \right\|^{1/2}.
\end{aligned}$$

Proof. Utilising (2.8) we have

$$\begin{aligned}
(2.13) \quad \left\| \sum_{j=1}^{n} |T_j + V_j|^2 \right\| &= w\left(\sum_{j=1}^{n} |T_j + V_j|^2 \right) = w\left(\sum_{j=1}^{n} \left(T_j^* + V_j^*\right)(T_j + V_j) \right) \\
&= w\left(\sum_{j=1}^{n} (T_j + V_j)^* T_j + \sum_{j=1}^{n} (T_j + V_j)^* V_j \right) \\
&\le w\left(\sum_{j=1}^{n} (T_j + V_j)^* T_j \right) + w\left(\sum_{j=1}^{n} (T_j + V_j)^* V_j \right) \\
&\le \left\| \sum_{j=1}^{n} |T_j + V_j|^2 \right\|^{1/2} \left(\left\| \sum_{j=1}^{n} |T_j|^2 \right\|^{1/2} + \left\| \sum_{j=1}^{n} |V_j|^2 \right\|^{1/2} \right)
\end{aligned}$$

and since

$$w\left(\sum_{j=1}^{n}(T_j+V_j)^*T_j\right)=w\left(\sum_{j=1}^{n}|T_j|^2+\sum_{j=1}^{n}V_j^*T_j\right)$$

and

$$w\left(\sum_{j=1}^{n}(T_j+V_j)^*V_j\right)=w\left(\sum_{j=1}^{n}T_j^*V_j+\sum_{j=1}^{n}|V_j|^2\right),$$

then by dividing (2.13) with $\left\|\sum_{j=1}^{n}|T_j+V_j|^2\right\|\neq 0$ we deduce the desired result (2.12) □

Corollary 2. *For any $T=(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$ we have the inequalities*

$$w^2\left(\sum_{j=1}^{n}T_j^2\right)\leq\left\|\sum_{j=1}^{n}|T_j|^2\right\|\left\|\sum_{j=1}^{n}|T_j^*|^2\right\| \tag{2.14}$$

and

$$\left\|\sum_{j=1}^{n}\left(T_j+T_j^*\right)^2\right\|^{1/2}\leq\left\|\sum_{j=1}^{n}|T_j|^2\right\|^{1/2}+\left\|\sum_{j=1}^{n}|T_j^*|^2\right\|^{1/2}. \tag{2.15}$$

Remark 1. *If we take $V_j=I$ for all $j\in\{1,...,n\}$ in (2.8) then we get*

$$w^2\left(\sum_{j=1}^{n}T_j\right)\leq n\left\|\sum_{j=1}^{n}|T_j|^2\right\|, \tag{2.16}$$

for any $(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$.

Also, if we take $T_j:=\frac{A_j+A_j^*}{2}$ and $V_j:=\frac{A_j-A_j^*}{2}$ for all $j\in\{1,...,n\}$, in (2.9), then we get

$$\left\|\sum_{j=1}^{n}|A_j|^2\right\|^{1/2}\leq\left\|\sum_{j=1}^{n}\left(\frac{A_j+A_j^*}{2}\right)^2\right\|^{1/2}+\left\|\sum_{j=1}^{n}\left(\frac{A_j-A_j^*}{2}\right)^2\right\|^{1/2} \tag{2.17}$$

for any $(A_1,...,A_n)\in\mathcal{B}^{(n)}(H)$.

Remark 2. *If we take $T=(A,B)$ and $V=(B^*,\pm A^*)$ in (2.8), where $A,B\in\mathcal{B}(H)$, then we have*

$$w(BA\pm AB)\leq\left\||A|^2+|B|^2\right\|^{1/2}\left\||A^*|^2+|B^*|^2\right\|^{1/2}. \tag{2.18}$$

In particular, for $B=A^*$ we get

$$w(A^*A\pm AA^*)\leq\left\||A|^2+|A^*|^2\right\|. \tag{2.19}$$

If we take $T=(A,A^*)$ and $V=(A^*,A)$ in (2.9), where $A,B\in\mathcal{B}(H)$, then we have

$$\|A+A^*\|\leq\sqrt{2}\left\||A|^2+|A^*|^2\right\|^{1/2}$$

which is equivalent with

$$\left\| \frac{A+A^*}{2} \right\|^2 \le \left\| \frac{|A|^2 + |A^*|^2}{2} \right\|. \tag{2.20}$$

Similarly, if we take $T = (A, -A^*)$ and $V = (-A^*, A)$ in (2.9), where $A, B \in \mathcal{B}(H)$, then we have

$$\left\| \frac{A-A^*}{2} \right\|^2 \le \left\| \frac{|A|^2 + |A^*|^2}{2} \right\|. \tag{2.21}$$

Now, if we apply the inequality (2.8) for $T = (C, I)$ and $V = (I, \pm D^*)$ where $C, D \in \mathcal{B}(H)$, then we get

$$w(C \pm D) \le \left\| |C|^2 + I \right\|^{1/2} \left\| |D^*|^2 + I \right\|^{1/2} \tag{2.22}$$

and by taking in this inequality $C = BA$ and $D = AB$ where $A, B \in \mathcal{B}(H)$, then we get

$$w(BA \pm AB) \le \left\| |BA|^2 + I \right\|^{1/2} \left\| |B^*A^*|^2 + I \right\|^{1/2} \tag{2.23}$$

and, in particular, for $B = A^*$

$$w(A^*A \pm AA^*) \le \left\| |A|^4 + I \right\|^{1/2} \left\| |A^*|^4 + I \right\|^{1/2}.$$

If we choose $T = (B, I)$ and $V = \left(A^*, (AB)^*\right)$ in (2.8), where $A, B \in \mathcal{B}(H)$, then we get

$$4w^2(AB) \le \left\| |B|^2 + I \right\| \left\| |A^*|^2 + \left|(AB)^*\right|^2 \right\|.$$

Since

$$|A^*|^2 + \left|(AB)^*\right|^2 = AA^* + ABB^*A^* = A\left(I + |B^*|^2\right)A^*$$

then we have

$$w(AB) \le \left\| \frac{I + |B|^2}{2} \right\|^{1/2} \left\| A\left(\frac{I + |B^*|^2}{2}\right)A^* \right\|^{1/2} \tag{2.24}$$

for any $A, B \in \mathcal{B}(H)$.

If we take in (2.24) $A = I$, then we have

$$w(B) \le \left\| \frac{I + |B|^2}{2} \right\|^{1/2} \left\| \frac{I + |B^*|^2}{2} \right\|^{1/2} \tag{2.25}$$

for any $B \in \mathcal{B}(H)$.

If we choose $A = B$ in (2.24) then we get

$$w\left(B^2\right) \le \left\| \frac{I + |B|^2}{2} \right\|^{1/2} \left\| B\left(\frac{I + |B^*|^2}{2}\right)B^* \right\|^{1/2} \tag{2.26}$$

while for $A = B^*$ we get

$$\|B\|^2 \le \left\| \frac{I + |B|^2}{2} \right\|^{1/2} \left\| B^*\left(\frac{I + |B^*|^2}{2}\right)B \right\|^{1/2} \tag{2.27}$$

for any $B \in \mathcal{B}(H)$.

Remark 3. *Let $C = A + iB$ be the Cartesian decomposition of the operator C. Then A and B are selfadjoint and*

$$A^2 + B^2 = \frac{1}{2}(C^*C + CC^*) = \frac{1}{2}\left(|C|^2 + |C^*|^2\right).$$

Moreover

$$A = \frac{C + C^*}{2} \text{ and } B = \frac{C - C^*}{2i}.$$

If we apply the inequality (2.8) for $T = (A, iI), V = (I, B)$ then we get

$$w(C) \le \left\| |A|^2 + I \right\|^{1/2} \left\| |B|^2 + I \right\|^{1/2}. \tag{2.28}$$

For an n-tuple of operators $T = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$ we consider the weighted *s-norms* defined as follows:

$$\|T\|_{p,s} := \left(\sum_{j=1}^{n} p_j \|T_j\|^s \right)^{1/s} \quad \text{if } s \ge 1 \tag{2.29}$$

where $\|\cdot\|$ is the usual operator norm on $\mathcal{B}(H)$ and $p = (p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ is a given sequence.

Proposition 1. *The mapping $\|(\cdot, ..., \cdot)\|_{p,\ell} : \mathcal{B}^{(n)}(H) \to [0, \infty)$ defined by*

$$\|T\|_{p,\ell} := \|(T_1, ..., T_n)\|_{p,\ell} := \left\| \sum_{j=1}^{n} p_j |T_j|^2 \right\|^{1/2}$$

is a norm on $\mathcal{B}^{(n)}(H)$ and

$$\max_{k \in \{1,...,n\}} \left\{ p_k^{1/2} \|T_k\| \right\} \le \|T\|_{p,\ell} \le \|T\|_{p,2}. \tag{2.30}$$

Proof. The fact that $\|\alpha T\|_{p,\ell} := |\alpha| \|T\|_{p,\ell}$ follows by the definition while the triangle inequality can be obtained from (2.9) written for the sequences $\sqrt{p_k} T_k$ and $\sqrt{p_k} V_k$, $k \in \{1, ..., n\}$.

We observe that

$$\sum_{j=1}^{n} p_j |T_j|^2 \ge p_k |T_k|^2$$

in the operator order of $\mathcal{B}(H)$ for any $k \in \{1, ..., n\}$. This implies

$$\left\| \sum_{j=1}^{n} p_j |T_j|^2 \right\| \ge \left\| p_k |T_k|^2 \right\| = p_k \|T_k\|^2$$

for any $k \in \{1, ..., n\}$, which implies that

$$\left\| \sum_{j=1}^{n} p_j |T_j|^2 \right\|^{1/2} \ge \max_{k \in \{1,...,n\}} \left\{ p_k^{1/2} \|T_k\| \right\}$$

and the first inequality in (2.30).

By the triangle inequality for the operator norm we also have

$$\left\| \sum_{j=1}^{n} p_j |T_j|^2 \right\| \le \sum_{j=1}^{n} \left\| p_j |T_j|^2 \right\| = \sum_{j=1}^{n} p_j \|T_j\|^2$$

which implies that

$$\left\| \sum_{j=1}^{n} p_j |T_j|^2 \right\|^{1/2} \le \left(\sum_{j=1}^{n} p_j \|T_j\|^2 \right)^{1/2}$$

and the corollary is proved. □

Remark 4. *We observe that for* $p = (1, ..., 1)$ *we have*

$$\|(T_1, ..., T_n)\|_{p,\ell} := \left\| \sum_{j=1}^{n} |T_j|^2 \right\|^{1/2} = \|[T_1^*, \ldots, T_n^*]\|$$

with the notation from (2.3), which satisfies the inequalities (2.4).

We have the alternative inequality as follows:

Theorem 8. *For any* $T = (T_1, ..., T_n), V = (V_1, ..., V_n) \in \mathcal{B}^{(n)}(H)$ *we have the inequalities*

$$w\left(\sum_{j=1}^{n} V_j^* T_j \right) \le \left\| \sum_{j=1}^{n} \frac{|T_j|^2 + |V_j|^2}{2} \right\| \tag{2.31}$$

and

$$\left\| \sum_{j=1}^{n} \left| \frac{T_j + V_j}{2} \right|^2 \right\| \le \left\| \sum_{j=1}^{n} \frac{|T_j|^2 + |V_j|^2}{2} \right\|. \tag{2.32}$$

Proof. Let $x \in H$, $x \neq 0$. Taking the square root in (2.10) and utilising the *arithmetic mean-geometric mean inequality* we have

$$\begin{aligned}
\left| \left\langle \left(\sum_{j=1}^{n} V_j^* T_j \right) x, x \right\rangle \right| &\le \left\langle \left(\sum_{j=1}^{n} |T_j|^2 \right) x, x \right\rangle^{1/2} \left\langle \left(\sum_{j=1}^{n} |V_j|^2 \right) x, x \right\rangle^{1/2} \\
&\le \left\langle \left(\sum_{j=1}^{n} \frac{|T_j|^2 + |V_j|^2}{2} \right) x, x \right\rangle.
\end{aligned}$$

Taking the supremum over $x \in H, \|x\| = 1$ we obtain the desired inequality (2.31).

Now, if we take the square in the inequality (2.11) we get

$$\begin{aligned}
&\left\langle \left(\sum_{j=1}^{n} |T_j + V_j|^2 \right) x, x \right\rangle \\
&\le \left[\left\langle \left(\sum_{j=1}^{n} |T_j|^2 \right) x, x \right\rangle^{1/2} + \left\langle \left(\sum_{j=1}^{n} |V_j|^2 \right) x, x \right\rangle^{1/2} \right]^2 \\
&= \left\langle \left(\sum_{j=1}^{n} \left[|T_j|^2 + |V_j|^2 \right] \right) x, x \right\rangle + 2 \left\langle \left(\sum_{j=1}^{n} |T_j|^2 \right) x, x \right\rangle^{1/2} \left\langle \left(\sum_{j=1}^{n} |V_j|^2 \right) x, x \right\rangle^{1/2}
\end{aligned}$$

and since, as above,

$$2\left\langle\left(\sum_{j=1}^{n}|T_j|^2\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^{n}|V_j|^2\right)x,x\right\rangle^{1/2}$$
$$\leq\left\langle\left(\sum_{j=1}^{n}\left[|T_j|^2+|V_j|^2\right]\right)x,x\right\rangle.$$

then we get

$$\left\langle\left(\sum_{j=1}^{n}|T_j+V_j|^2\right)x,x\right\rangle\leq 2\left\langle\left(\sum_{j=1}^{n}\left[|T_j|^2+|V_j|^2\right]\right)x,x\right\rangle$$

for any $x\in H$, $x\neq 0$.

Taking the supremum over $x\in H,\|x\|=1$ we obtain the desired inequality (2.32). □

Corollary 3. *For any $T=(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$ we have the inequalities*

$$w\left(\sum_{j=1}^{n}T_j^2\right)\leq\left\|\sum_{j=1}^{n}\frac{|T_j|^2+|T_j^*|^2}{2}\right\| \tag{2.33}$$

and

$$\left\|\sum_{j=1}^{n}\left|\frac{T_j\pm T_j^*}{2}\right|^2\right\|\leq\left\|\sum_{j=1}^{n}\frac{|T_j|^2+|T_j^*|^2}{2}\right\|. \tag{2.34}$$

Remark 5. *If we take $V_j=I$ for all $j\in\{1,...,n\}$ in (2.31) then we get*

$$w\left(\sum_{j=1}^{n}T_j\right)\leq\left\|\sum_{j=1}^{n}\frac{|T_j|^2+I}{2}\right\| \tag{2.35}$$

for any $(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$.

Remark 6. *If we take $T=(A,B)$ and $V=(B^*,\pm A^*)$ in the inequality (2.31), where $A,B\in\mathcal{B}(H)$, then we get*

$$w(BA\pm AB)\leq\left\|\frac{|A|^2+|A^*|^2+|B|^2+|B^*|^2}{2}\right\|, \tag{2.36}$$

which was obtained in [21] *as a particular case of a different result. In particular, for $B=A^*$ we recapture (2.19).*

Now, if we apply the inequality (2.31) for $T=(C,I)$ and $V=(I,\pm D^*)$, where $C,D\in\mathcal{B}(H)$, then we get

$$w(C\pm D)\leq\left\|\frac{|C|^2+|D^*|^2}{2}+I\right\| \tag{2.37}$$

and by taking in this inequality $C = BA$ and $D = AB$, where $A, B \in \mathcal{B}(H)$, then we get

$$w(BA \pm AB) \leq \left\| \frac{|BA|^2 + |B^*A^*|^2}{2} + I \right\| \tag{2.38}$$

and, in particular, for $B = A^*$

$$w(A^*A \pm AA^*) \leq \left\| \frac{|A|^4 + |A^*|^4}{2} + I \right\|. \tag{2.39}$$

If we choose $T = (B, I)$ and $V = \left(A^*, (AB)^*\right)$ in (2.31), where $A, B \in \mathcal{B}(H)$, then we get

$$w(AB) \leq \left\| \frac{|B|^2 + I + |A^*|^2 + \left|(AB)^*\right|^2}{4} \right\| \tag{2.40}$$

or, equivalently

$$w(AB) \leq \left\| \frac{I + |B|^2 + A\left(I + |B^*|^2\right)A^*}{4} \right\|, \tag{2.41}$$

for any $A, B \in \mathcal{B}(H)$.

If we choose $A = I$ in (2.41), then we get

$$w(B) \leq \left\| \frac{1}{2}I + \frac{|B|^2 + |B^*|^2}{4} \right\|, \tag{2.42}$$

for any $B \in \mathcal{B}(H)$, while, if we choose $B = I$ in the same inequality, then we have

$$w(A) \leq \left\| \frac{I + |A^*|^2}{2} \right\|,$$

for any $A \in \mathcal{B}(H)$. If in this inequality we take A^* instead of A and use the fact that $w(A) = w(A^*)$, then we get

$$w(A) \leq \min\left\{ \left\| \frac{I + |A^*|^2}{2} \right\|, \left\| \frac{I + |A|^2}{2} \right\| \right\} \tag{2.43}$$

for any $A \in \mathcal{B}(H)$.

Moreover, if we take $A = B$ in (2.41), then we get

$$w\left(B^2\right) \leq \left\| \frac{I + |B|^2 + B\left(I + |B^*|^2\right)B^*}{4} \right\|, \tag{2.44}$$

while if we take $A = B^*$ in the same inequality, then we have

$$\|B\| \leq \left\| \frac{I + |B|^2 + B^*\left(I + |B^*|^2\right)B}{4} \right\|^{1/2}, \tag{2.45}$$

for any $B \in \mathcal{B}(H)$.

3. Applications for Functions of Normal Operators

From the inequalities (2.8) and (2.9) we have the *weighted inequalities*

$$w^{2}\left(\sum_{j=1}^{n}p_{j}V_{j}T_{j}\right)\leq\left\|\sum_{j=1}^{n}p_{j}\left|T_{j}\right|^{2}\right\|\left\|\sum_{j=1}^{n}p_{j}\left|V_{j}^{*}\right|^{2}\right\| \tag{3.1}$$

and

$$\left\|\sum_{j=1}^{n}p_{j}\left|T_{j}+V_{j}^{*}\right|^{2}\right\|^{1/2}\leq\left\|\sum_{j=1}^{n}p_{j}\left|T_{j}\right|^{2}\right\|^{1/2}+\left\|\sum_{j=1}^{n}p_{j}\left|V_{j}^{*}\right|^{2}\right\|^{1/2}, \tag{3.2}$$

while from the inequalities (2.31) and (2.32) we have

$$w\left(\sum_{j=1}^{n}p_{j}V_{j}T_{j}\right)\leq\left\|\sum_{j=1}^{n}p_{j}\left(\frac{\left|T_{j}\right|^{2}+\left|V_{j}^{*}\right|^{2}}{2}\right)\right\| \tag{3.3}$$

and

$$\left\|\sum_{j=1}^{n}p_{j}\left|\frac{T_{j}+V_{j}^{*}}{2}\right|^{2}\right\|\leq\left\|\sum_{j=1}^{n}p_{j}\left(\frac{\left|T_{j}\right|^{2}+\left|V_{j}^{*}\right|^{2}}{2}\right)\right\|, \tag{3.4}$$

where $T=(T_{1},...,T_{n}),V=(V_{1},...,V_{n})\in\mathcal{B}^{(n)}(H),(p_{1},...,p_{n})\in\mathbb{R}_{+}^{*n}$.

These inequalities will be used in the follows to obtain some results for functions of normal operators.

Theorem 9. *Let $f(z):=\sum_{j=0}^{\infty}p_{j}z^{j}$ a power series with nonnegative coefficients and convergent on the open disk $D(0,R),R>0$. If T and V are two normal and commuting operators with $\|T\|^{2},\|V\|^{2}<R$, then we have the inequalities*

$$w(f(VT))\leq\left\|f\left(|T|^{2}\right)\right\|^{1/2}\left\|f\left(|V|^{2}\right)\right\|^{1/2} \tag{3.5}$$

and

$$\begin{aligned}&\left\|f\left(|T|^{2}\right)+f(VT)+f(V^{*}T^{*})+f\left(|V|^{2}\right)\right\|^{1/2}\\&\leq\left\|f\left(|T|^{2}\right)\right\|^{1/2}+\left\|f\left(|V|^{2}\right)\right\|^{1/2},\end{aligned} \tag{3.6}$$

and the inequalities

$$w(f(VT))\leq\left\|\frac{f\left(|T|^{2}\right)+f\left(|V|^{2}\right)}{2}\right\| \tag{3.7}$$

and

$$\left\|\frac{f\left(|T|^{2}\right)+f(VT)+f(V^{*}T^{*})+f\left(|V|^{2}\right)}{4}\right\|\leq\left\|\frac{f\left(|T|^{2}\right)+f\left(|V|^{2}\right)}{2}\right\|. \tag{3.8}$$

Proof. If we use the inequality (3.1) for the powers of operators we have

$$w^{2}\left(\sum_{j=0}^{m}p_{j}V^{j}T^{j}\right)\leq\left\|\sum_{j=0}^{m}p_{j}\left|T^{j}\right|^{2}\right\|\left\|\sum_{j=0}^{m}p_{j}\left|(V^{*})^{j}\right|^{2}\right\| \tag{3.9}$$

for any $m \geq 1$.

Since V and T are normal we have $\left|T^j\right|^2 = |T|^{2j}$ and $\left|(V^*)^j\right|^2 = |V|^{2j}$ for any $j \in \{0, ..., m\}$ and by the commutativity of V and T we also have $V^j T^j = (VT)^j$ for any $j \in \{0, ..., m\}$.

Therefore from (3.9) we have

$$w^2\left(\sum_{j=0}^{m} p_j (VT)^j\right) \leq \left\|\sum_{j=0}^{m} p_j |T|^{2j}\right\| \left\|\sum_{j=0}^{m} p_j |V|^{2j}\right\| \tag{3.10}$$

for any $m \geq 1$.

Since all the series whose partial sums are involved in the inequality (3.10) are convergent then by letting $m \to \infty$ in (3.10) we get (3.5).

By the normality and commutativity of V and T we have

$$\sum_{j=0}^{m} p_j \left|T^j + (V^*)^j\right|^2 = \sum_{j=0}^{m} p_j \left[|T|^{2j} + (VT)^j + (V^*T^*)^j + |V|^{2j}\right].$$

Then from (3.6) we have

$$\begin{aligned} &\left\|\sum_{j=0}^{m} p_j \left[|T|^{2j} + (VT)^j + (V^*T^*)^j + |V|^{2j}\right]\right\|^{1/2} \\ &\leq \left\|\sum_{j=0}^{m} p_j |T|^{2j}\right\|^{1/2} + \left\|\sum_{j=0}^{m} p_j |V|^{2j}\right\|^{1/2} \end{aligned} \tag{3.11}$$

for any $m \geq 1$.

Since all the series whose partial sums are involved in the inequality (3.11) are convergent then by letting $m \to \infty$ in (3.11) we get (3.6).

The other two inequalities follow in a similar way and the details are omitted. □

Corollary 4. *Let $f(z) := \sum_{j=0}^{\infty} p_j z^j$ a power series with nonnegative coefficients and convergent on the open disk $D(0, R)$, $R > 0$. If T and V are two selfadjoint and commuting operators with $\|T\|^2, \|V\|^2 < R$, then we have the inequalities*

$$\|f(VT)\| \leq \left\|f\left(T^2\right)\right\|^{1/2} \left\|f\left(V^2\right)\right\|^{1/2} \tag{3.12}$$

and

$$\left\|f\left(T^2\right) + 2f(VT) + f\left(V^2\right)\right\|^{1/2} \leq \left\|f\left(T^2\right)\right\|^{1/2} + \left\|f\left(V^2\right)\right\|^{1/2}, \tag{3.13}$$

and the inequalities

$$\|f(VT)\| \leq \left\|\frac{f\left(T^2\right) + f\left(V^2\right)}{2}\right\| \tag{3.14}$$

and

$$\left\|\frac{f\left(T^2\right) + 2f(VT) + f\left(V^2\right)}{4}\right\| \leq \left\|\frac{f\left(T^2\right) + f\left(V^2\right)}{2}\right\|. \tag{3.15}$$

Proof. If T and V are two selfadjoint and commuting operators, then

$$\begin{aligned} f^{*}(VT) &= \left(\lim_{n\to\infty}\sum_{j=0}^{n}p_j(VT)^j\right)^{*}=\lim_{n\to\infty}\sum_{j=0}^{n}p_j\left((VT)^j\right)^{*} \\ &= \lim_{n\to\infty}\sum_{j=0}^{n}p_j\left((VT)^{*}\right)^j=\lim_{n\to\infty}\sum_{j=0}^{n}p_j(T^{*}V^{*})^j \\ &= \lim_{n\to\infty}\sum_{j=0}^{n}p_j(TV)^j=\lim_{n\to\infty}\sum_{j=0}^{n}p_j(VT)^j=f(VT) \end{aligned}$$

showing that $f(VT)$ is selfadjoint. Then $w(f(VT))=\|f(VT)\|$ and by Theorem 9 we get the desired results. □

4. Other Applications for Power Series

Now, by the help of power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n\geq 0$, then $f_a=f$.

Theorem 10. *Let $f(z):=\sum_{j=0}^{\infty}a_jz^j$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $A\in\mathcal{B}(H)$ is such that*

$$\left\|\frac{I+|A^{*}|^2}{2}\right\|,\left\|\frac{I+|A|^2}{2}\right\|<R \tag{4.1}$$

then

$$w(f(A))\leq\min\left\{f_a\left(\left\|\frac{I+|A^{*}|^2}{2}\right\|\right),f_a\left(\left\|\frac{I+|A|^2}{2}\right\|\right)\right\}. \tag{4.2}$$

Proof. Let $m\geq 1$. Utilising the properties of the numerical radius we have

$$w\left(\sum_{n=0}^{m}a_nA^n\right)\leq\sum_{n=0}^{m}|a_n|w(A^n)\leq\sum_{n=0}^{m}|a_n|w^n(A). \tag{4.3}$$

On making use of the inequality (2.43) we have

$$w^n(A)\leq\min\left\{\left\|\frac{I+|A^{*}|^2}{2}\right\|^n,\left\|\frac{I+|A|^2}{2}\right\|^n\right\}$$

which implies

$$\sum_{n=0}^{m}|a_n|w^n(A)\leq\sum_{n=0}^{m}|a_n|\min\left\{\left\|\frac{I+|A^{*}|^2}{2}\right\|^n,\left\|\frac{I+|A|^2}{2}\right\|^n\right\}. \tag{4.4}$$

Utilising the elementary inequality for nonnegative numbers p_j,c_j,d_j with $j\in\{0,...,m\}$, $m\geq 1$

$$\sum_{j=0}^{m}p_j\min\{c_j,d_j\}\leq\min\left\{\sum_{j=0}^{m}p_jc_j,\sum_{j=0}^{m}p_jd_j\right\},$$

we have

$$\sum_{n=0}^{m}|a_n|\min\left\{\left\|\frac{I+|A^*|^2}{2}\right\|^n,\left\|\frac{I+|A|^2}{2}\right\|^n\right\} \le \min\left\{\sum_{n=0}^{m}|a_n|\left\|\frac{I+|A^*|^2}{2}\right\|^n,\sum_{n=0}^{m}|a_n|\left\|\frac{I+|A|^2}{2}\right\|^n\right\}. \tag{4.5}$$

By the inequalities (4.3)-(4.5) we conclude that

$$w\left(\sum_{n=0}^{m}a_nA^n\right)\le \min\left\{\sum_{n=0}^{m}|a_n|\left\|\frac{I+|A^*|^2}{2}\right\|^n,\sum_{n=0}^{m}|a_n|\left\|\frac{I+|A|^2}{2}\right\|^n\right\} \tag{4.6}$$

for any $m\ge 1$.

Since all the series whose partial sums are involved in (4.6) are convergent, then by letting $m\to\infty$ in (4.6) we deduce the desired result (4.2). □

Remark 7. *Since*

$$\left\|\frac{I+|A^*|^2}{2}\right\|,\left\|\frac{I+|A|^2}{2}\right\|\le\frac{1+\|A\|^2}{2}$$

and if we assume that the convergence radius $R>1$, then by the monotonicity and convexity of the power series $f_a(z):=\sum_{n=0}^{\infty}|a_n|z^n$ on $(0,R)$ we also have the inequality

$$\max\left\{f_a\left(\left\|\frac{I+|A^*|^2}{2}\right\|\right),f_a\left(\left\|\frac{I+|A|^2}{2}\right\|\right)\right\} \le \frac{f_a(1)+f_a\left(\|A\|^2\right)}{2}, \tag{4.7}$$

provided $\|A\|^2<R$.

Theorem 11. *Let $f(z):=\sum_{j=0}^{\infty}a_jz^j$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>1$. If $A,B\in\mathcal{B}(H)$ are such that $\|A\|^2,\|B\|^2,\|AB\|^2<R$, then*

$$\begin{aligned} w(f(AB)) &\le f_a\left(\left\|\frac{I+|B|^2+|A^*|^2+\left|(AB)^*\right|^2}{4}\right\|\right)\\ &\le \frac{1}{4}\left[f_a(1)+f_a\left(\|A\|^2\right)+f_a\left(\|B\|^2\right)+f_a\left(\|AB\|^2\right)\right]. \end{aligned} \tag{4.8}$$

Proof. Let $m\ge1$. Utilising the properties of the numerical radius we have

$$w\left(\sum_{n=0}^{m}a_n(AB)^n\right)\le\sum_{n=0}^{m}|a_n|\,w\left((AB)^n\right)\le\sum_{n=0}^{m}|a_n|\,w^n(AB).$$

On making use of the inequality (2.40) we have

$$\sum_{n=0}^{m}|a_n|\,w^n(AB)\le\sum_{n=0}^{m}|a_n|\left\|\frac{I+|B|^2+|A^*|^2+\left|(AB)^*\right|^2}{4}\right\|^n.$$

Since

$$\left\| \frac{I + |B|^2 + |A^*|^2 + \left|(AB)^*\right|^2}{4} \right\| \leq \frac{1}{4}\left(1 + \|A\|^2 + \|B\|^2 + \|AB\|^2\right) < R,$$

it follows that the series whose partial sums are involved above are convergent and the first part of (4.8) holds true.

The second part follows by the monotonicity and convexity of f_a on $(0, R)$. $\square$

Corollary 5. *Let $f(z) := \sum_{j=0}^{\infty} a_j z^j$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 1$. If $B \in \mathcal{B}(H)$ is such that $\|B\|^2 < R$, then*

$$w(f(B)) \leq f_a\left(\left\| \frac{1}{2} I + \frac{|B|^2 + |B^*|^2}{4} \right\|\right) \leq \frac{1}{2}\left[f_a(1) + f_a\left(\|B\|^2\right)\right]. \tag{4.9}$$

5. Some Examples

We recall some examples of functions defined by power series with nonnegative coefficients

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n, \ z \in D(0,1); \\ \cosh z &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \ z \in \mathbb{C}; \\ \sinh z &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \ z \in \mathbb{C}; \\ \ln \frac{1}{1-z} &= \sum_{n=1}^{\infty} \frac{1}{n!} z^n, \ z \in D(0,1). \end{aligned} \tag{5.1}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\begin{aligned} \exp(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \ z \in \mathbb{C}; \\ \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1); \\ \sin^{-1}(z) &= \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}(2n+1) n!} z^{2n+1}, \ z \in D(0,1); \\ \tanh^{-1}(z) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1); \\ {}_2F_1(\alpha, \beta, \gamma, z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha, \beta, \gamma > 0, \\ & z \in D(0,1); \end{aligned} \tag{5.2}$$

where Γ is *Gamma function*.

Now, if we use, for instance, the inequality (3.5) we have the following inequalities

$$w\left((I-VT)^{-1}\right) \leq \left\|\left(I-|T|^2\right)^{-1}\right\|^{1/2} \left\|\left(I-|V|^2\right)^{-1}\right\|^{1/2},$$

and

$$w\left(\ln (I-VT)^{-1}\right) \leq \left\|\ln\left(I-|T|^2\right)^{-1}\right\|^{1/2} \left\|\ln\left(I-|V|^2\right)^{-1}\right\|^{1/2},$$

for any T and V two normal and commuting operators with $\|T\|, \|V\| < 1$.

If T and V are normal and commuting operators, then we also have

$$w(\exp(VT)) \leq \left\|\exp\left(|T|^2\right)\right\|^{1/2} \left\|\exp\left(|V|^2\right)\right\|^{1/2},$$

and

$$w(\sinh(VT)) \leq \left\|\sinh\left(|T|^2\right)\right\|^{1/2} \left\|\sinh\left(|V|^2\right)\right\|^{1/2}.$$

If we use the inequality (3.7) then we get

$$w\left((I-VT)^{-1}\right) \leq \left\|\frac{\left(1-|T|^2\right)^{-1}+\left(I-|V|^2\right)^{-1}}{2}\right\|,$$

and

$$w\left(\ln (I-VT)^{-1}\right) \leq \left\|\frac{\ln\left(I-|T|^2\right)^{-1}+\ln\left(I-|V|^2\right)^{-1}}{2}\right\|,$$

for any T and V two normal and commuting operators with $\|T\|, \|V\| < 1$.

If T and V are normal and commuting operators, then we also have

$$w(\exp(VT)) \leq \left\|\frac{\exp\left(|T|^2\right)+\exp\left(|V|^2\right)}{2}\right\|,$$

and

$$w(\cosh(VT)) \leq \left\|\frac{\cosh\left(|T|^2\right)+\cosh\left(|V|^2\right)}{2}\right\|.$$

We notice that if

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z},\ z \in D(0,1); \tag{5.3}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z,\ z \in \mathbb{C};$$

$$h(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z,\ z \in \mathbb{C};$$

$$l(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z},\ z \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z) = \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \ln \frac{1}{1-z}, \ z \in D(0,1); \tag{5.4}$$

$$g_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \ z \in \mathbb{C};$$

$$h_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \ z \in \mathbb{C};$$

$$l_a(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ z \in D(0,1).$$

On making use of the inequality (4.2) we have

$$w\left((I \pm A)^{-1}\right) \le \min\left\{\left(1 - \left\|\frac{I + |A^*|^2}{2}\right\|\right)^{-1}, \left(1 - \left\|\frac{I + |A|^2}{2}\right\|\right)^{-1}\right\}$$

for any $A \in \mathcal{B}(H)$ with $\|A\| < 1$.

By the inequalities (4.2) and (4.7) we also have

$$w(\exp(A)) \le \min\left\{\left(\exp\left(\left\|\frac{I + |A^*|^2}{2}\right\|\right)\right), \exp\left(\left\|\frac{I + |A|^2}{2}\right\|\right)\right\}$$

and

$$w(\sin(A)) \le \min\left\{\left(\sinh\left(\left\|\frac{I + |A^*|^2}{2}\right\|\right)\right), \sinh\left(\left\|\frac{I + |A|^2}{2}\right\|\right)\right\}$$

for any $A \in \mathcal{B}(H)$.

The interested reader may state further similar results by utilising the other general inequalities obtained above or other examples of functions defined by power series. The details are omitted.

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