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SOME NEW ČEBYŠEV TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES BELONGS TO L_p SPACES

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Abstract. In this paper, we obtain some new Čebyšev type inequalities for functions whose derivatives belongs to L_p spaces which are similar to Pachpatte's results (see [1]). These results are more general results and give some new estimations.

1. INTRODUCTION

There are several classical and analytic inequalities for functions. One of them was established by Čebyšev in [2] as following;

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions whose derivatives $f',g' \in L_\infty[a,b]$ and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.2}$$

which is called the Čebyšev functional, provided the integrals in (1.2) exist.

Several results related to the inequalities (1.1) and (1.2) can be found in the literature, see [3]-[8]. In [1], Pachpatte proved some inequalities similar to the inequality (1.1) as followings;

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions whose derivatives $f',g'\in L_p[a,b]$, $p>1$. Then, we have the inequalities*

$$|T(f,g)| \leq \frac{1}{(b-a)^3}\|f'\|_p \|g'\|_p \int_a^b (B(x))^{\frac{2}{q}}\,dx, \tag{1.3}$$

$$|T(f,g)| \leq \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\,\|f'\|_p + |f(x)|\,\|g'\|_p\right](B(x))^{\frac{1}{q}}\,dx, \tag{1.4}$$

where

$$B(x) = \frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1} \tag{1.5}$$

for $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

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Theorem 2. *Let $f,g:[a,b]\rightarrow\mathbb{R}$ be absolutely continuous functions whose derivatives $f',g'\in L_p[a,b]$, $p>1$. Then, we have the inequalities*

$$|S(f,g)|\leq\frac{1}{(b-a)^2}M^{\frac{2}{q}}\|f'\|_p\|g'\|_p, \tag{1.6}$$

$$|H(f,g)|\leq\frac{1}{(b-a)^2}M^{\frac{1}{q}}\int_a^b\left[|g(x)|\|f'\|_p+|f(x)|\|g'\|_p\right]dx, \tag{1.7}$$

where

$$M=\frac{\left(2^{q+1}+1\right)(b-a)^{q+1}}{3(q+1)6^q}, \tag{1.8}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

In the above Theorems Pachpatte used following notations for simplicity. For suitable functions $f,g:[a,b]\rightarrow\mathbb{R}$,

$$F=\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right],$$

$$G=\frac{1}{3}\left[\frac{g(a)+g(b)}{2}+2g\left(\frac{a+b}{2}\right)\right],$$

$$\begin{aligned}S(f,g) &= FG-\frac{1}{b-a}\left[F\int_a^b g(x)dx+G\int_a^b f(x)dx\right]\\ &\quad+\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right),\end{aligned}$$

$$\begin{aligned}H(f,g) &= \frac{1}{b-a}\left[F\int_a^b g(x)dx+G\int_a^b f(x)dx\right]\\ &\quad-2\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right),\end{aligned}$$

and

$$\|f\|_p=\left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}},\qquad p>0.$$

The main purpose of this paper is to prove some new inequalities similar to Pachpatte's results, but now by using new kernels. We obtain several new generalizations for Čebyšev type inequalities involving functions whose derivatives belong to L_p spaces.

2. MAIN RESULTS

In the sequel of the paper, we will use following notations, for the functions $f, g : [a, b] \to \mathbb{R}$;

$$\begin{aligned} T_{\lambda}(f,g) &= \frac{(1-\lambda)^2}{b-a}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(x)\,g(x)\,dx-(1-\lambda)\left[\left(\frac{1}{b-a}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\right. \\ &\quad\left.+\left(\frac{1}{b-a}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} g(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\right] \\ &\quad+(1-\lambda)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right), \end{aligned}$$

$$\begin{aligned} J_{\lambda}(f,g) &= 2(1-\lambda)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} g(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &-\left(\frac{1}{b-a}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right), \end{aligned}$$

$$\begin{aligned} S_r(f,g) &= F_rG_r-F_r\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)-G_r\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \\ &\quad+\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \end{aligned}$$

and

$$\begin{aligned} H_r(f,g) &= \frac{1}{b-a}\int_a^b \left[F_rg(x)+G_rf(x)\right]dx \\ &\quad-2\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \end{aligned}$$

where

$$\begin{aligned} F_r &= \frac{r-1}{r+1}f\left(\frac{a+b}{2}\right)+\frac{1}{r+1}\left(f(a)+f(b)\right) \\ G_r &= \frac{r-1}{r+1}g\left(\frac{a+b}{2}\right)+\frac{1}{r+1}\left(g(a)+g(b)\right). \end{aligned}$$

We will start with the following Theorem.

Theorem 3. *Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions whose derivatives $f',g'\in L_p[a,b]$, $p>1$ and $\lambda\in[0,1]$. Then we have the inequalities*

$$\begin{aligned}
&|T_\lambda(f,g)| \\
\leq\ & \frac{1}{(b-a)^3}\|f'\|_p\|g'\|_p\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}(B(x))^{\frac{2}{q}}dx \\
&+\frac{\lambda}{(b-a)^2}\left[\frac{|f(a)|+|f(b)|}{2}\|g'\|_p+\frac{|g(a)|+|g(b)|}{2}\|f'\|_p\right]\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}(B(x))^{\frac{1}{q}}dx \\
&+\lambda^2(1-\lambda)\left(\frac{|f(a)|+|f(b)|}{2}\right)\left(\frac{|g(a)|+|g(b)|}{2}\right)
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
|J_\lambda(f,g)| \leq\ & \frac{1}{(b-a)^2}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left[|g(x)|\|f'\|_p+|f(x)|\|g'\|_p\right](B(x))^{\frac{1}{q}}dx \\
&+\frac{\lambda}{b-a}\left[\left(\frac{|f(a)|+|f(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|g(x)|dx\right. \\
&\left.+\left(\frac{|g(a)|+|g(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|f(x)|dx\right]
\end{aligned}\tag{2.2}$$

where $B(x)$ is as in (1.5) for $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. For the proof of results, we use the following identities (see [8]);

$$(1-\lambda)f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b p(x,t)f'(t)\,dt-\lambda\frac{f(a)+f(b)}{2}\tag{2.3}$$

and

$$(1-\lambda)g(x)-\frac{1}{b-a}\int_a^b g(t)\,dt=\frac{1}{b-a}\int_a^b p(x,t)g'(t)\,dt-\lambda\frac{g(a)+g(b)}{2}\tag{2.4}$$

for $x\in\left[a+\lambda\frac{b-a}{2},b-\lambda\frac{b-a}{2}\right]$ and $\lambda\in[0,1]$, where

$$p(x,t)=\begin{cases} t-\left[a+\lambda\frac{b-a}{2}\right], & t\in[a,x] \\ t-\left[b-\lambda\frac{b-a}{2}\right], & t\in(x,b]\end{cases}.$$

By multiplying both sides of the inequalities (2.3) and (2.4), we have

$$\begin{aligned}
&(1-\lambda)^2 f(x)\, g(x) - (1-\lambda) f(x) \left(\frac{1}{b-a} \int_a^b g(t)\, dt \right) \\
&- (1-\lambda)\, g(x) \left(\frac{1}{b-a} \int_a^b f(t)\, dt \right) + \left(\frac{1}{b-a} \int_a^b f(t)\, dt \right) \left(\frac{1}{b-a} \int_a^b g(t)\, dt \right) \\
= {} & \frac{1}{(b-a)^2} \left(\int_a^b p(x,t)\, f'(t)\, dt \right) \left(\int_a^b p(x,t)\, g'(t)\, dt \right) - \frac{\lambda}{b-a} \left(\frac{f(a)+f(b)}{2} \right) \left(\int_a^b p(x,t)\, g'(t)\, dt \right) \\
&- \frac{\lambda}{b-a} \left(\frac{g(a)+g(b)}{2} \right) \left(\int_a^b p(x,t)\, f'(t)\, dt \right) + \lambda^2 \left(\frac{f(a)+f(b)}{2} \right) \left(\frac{g(a)+g(b)}{2} \right).
\end{aligned}$$

Integrating both sides of the above equality with respect to x over $\left[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}\right]$ and dividing both sides of the result equality by $(b-a)$, we get

$$\begin{aligned}
T_\lambda(f,g) = {} & \frac{1}{(b-a)^3} \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \left(\int_a^b p(x,t)\, f'(t)\, dt \right) \left(\int_a^b p(x,t)\, g'(t)\, dt \right) dx \\
& - \frac{\lambda}{(b-a)^2} \left(\frac{f(a)+f(b)}{2} \right) \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \left(\int_a^b p(x,t)\, g'(t)\, dt \right) dx \\
& - \frac{\lambda}{(b-a)^2} \left(\frac{g(a)+g(b)}{2} \right) \int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} \left(\int_a^b p(x,t)\, f'(t)\, dt \right) dx \\
& + \lambda^2 (1-\lambda) \left(\frac{f(a)+f(b)}{2} \right) \left(\frac{g(a)+g(b)}{2} \right).
\end{aligned}$$

By using the properties of modulus and Hölder's integral inequality, we can write

$$\begin{aligned}
&\left|T_\lambda(f,g)\right| \\
\leq\ & \frac{1}{(b-a)^3}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left\{\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}}\right\} \\
&\times\left\{\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |g'(t)|^p\,dt\right)^{\frac{1}{p}}\right\}dx \\
&+\frac{\lambda}{(b-a)^2}\left(\frac{|f(a)|+|f(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left\{\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |g'(t)|^p\,dt\right)^{\frac{1}{p}}\right\}dx \\
&+\frac{\lambda}{(b-a)^2}\left(\frac{|g(a)|+|g(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left\{\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}}\right\}dx \\
&+\lambda^2(1-\lambda)\left(\frac{|f(a)|+|f(b)|}{2}\right)\left(\frac{|g(a)|+|g(b)|}{2}\right).
\end{aligned}$$

That is

$$\begin{aligned}
&(2.5) \\
&\left|T_\lambda(f,g)\right| \\
\leq\ & \frac{1}{(b-a)^3}\left\|f'\right\|_p\left\|g'\right\|_p\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left[\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\right]^2 dx \\
&+\frac{\lambda}{(b-a)^2}\left[\frac{|f(a)|+|f(b)|}{2}\left\|g'\right\|_p+\frac{|g(a)|+|g(b)|}{2}\left\|f'\right\|_p\right] \\
&\times\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}dx+\lambda^2(1-\lambda)\left(\frac{|f(a)|+|f(b)|}{2}\right)\left(\frac{|g(a)|+|g(b)|}{2}\right).
\end{aligned}$$

By a simple computation, one can see that

$$\int_a^b |p(x,t)|^q\,dt=\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}=B(x) \tag{2.6}$$

By using (2.6) in (2.5), we obtain the inequality (2.1). For the proof of the inequality (2.2), if we multiply both sides of (2.3) and (2.4) by $g(x)$ and $f(x)$, respectively, we have

$$(2.7)\quad \begin{aligned} & (1-\lambda) f(x) g(x) - g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ = & \; g(x)\left(\frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt\right) - \lambda g(x)\left(\frac{f(a)+f(b)}{2}\right) \end{aligned}$$

and

$$(2.8)\quad \begin{aligned} & (1-\lambda) f(x) g(x) - f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) \\ = & \; f(x)\left(\frac{1}{b-a}\int_a^b p(x,t) g'(t)\,dt\right) - \lambda f(x)\left(\frac{g(a)+g(b)}{2}\right). \end{aligned}$$

By adding the equalities (2.7) and (2.8), integrating both sides of the result equality with respect to x over $\left[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}\right]$ and dividing both sides of the result equality by $(b-a)$, we get

$$\begin{aligned} & J_\lambda(f,g) \\ = & -\frac{\lambda}{b-a}\left[\left(\frac{f(a)+f(b)}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} g(x)\,dx + \left(\frac{g(a)+g(b)}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(x)\,dx\right] \\ & +\frac{1}{(b-a)^2}\left[\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} g(x)\left(\int_a^b p(x,t) f'(t)\,dt\right)dx\right] \\ & +\frac{1}{(b-a)^2}\left[\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}} f(x)\left(\int_a^b p(x,t) g'(t)\,dt\right)dx\right]. \end{aligned}$$

By using the properties of modulus and Hölder's integral inequality, we can write

$$\begin{aligned}
&|J_\lambda(f,g)| \\
\leq\ & \frac{\lambda}{b-a}\left[\left(\frac{|f(a)|+|f(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|g(x)|\,dx+\left(\frac{|g(a)|+|g(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|f(x)|\,dx\right] \\
&+\frac{1}{(b-a)^2}\left[\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|g(x)|\left\{\left(\int_a^b|p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b|f'(t)|^p\,dt\right)^{\frac{1}{p}}\right\}\right] \\
&+\frac{1}{(b-a)^2}\left[\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|f(x)|\left\{\left(\int_a^b|p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b|g'(t)|^p\,dt\right)^{\frac{1}{p}}\right\}\right] \\
=\ & \frac{\lambda}{b-a}\left[\left(\frac{|f(a)|+|f(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|g(x)|\,dx+\left(\frac{|g(a)|+|g(b)|}{2}\right)\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}|f(x)|\,dx\right] \\
&+\frac{1}{(b-a)^2}\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left[|g(x)|\,\|f'\|_p+|f(x)|\,\|g'\|_p\right][B(x)]^{\frac{1}{q}}\,dx,
\end{aligned}$$

which completes the proof. □

Remark 1. *If we choose $\lambda = 0$ in the inequalities (2.1) and (2.2), we obtain the inequalities (1.3) and (1.4), respectively.*

We obtain some new inequalities similar to the inequalities (1.6) and (1.7) in the following Theorem.

Theorem 4. *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$. Then we have the inequalities*

$$|S_r(f,g)| \leq \frac{1}{(b-a)^2}M_r^{\frac{2}{q}}\,\|f'\|_p\,\|g'\|_p\,, \tag{2.9}$$

$$|H_r(f,g)| \leq \frac{1}{(b-a)^2}M_r^{\frac{1}{q}}\int_a^b\left[|g(x)|\,\|f'\|_p+|f(x)|\,\|g'\|_p\right]dx, \tag{2.10}$$

where

$$M_r = \frac{2(b-a)^{q+1}\left(1+(r-1)^{q+1}\right)}{(q+1)(r+1)^{q+1}}, \tag{2.11}$$

for $r \geq 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. We have the following identities;

$$F_r - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b K(x,r)f'(x)\,dx \tag{2.12}$$

and

$$G_r - \frac{1}{b-a}\int_a^b g(x)\,dx = \frac{1}{b-a}\int_a^b K(x,r)g'(x)\,dx \tag{2.13}$$

where $K(x,r)$ is given as

$$K(x,r) = \begin{cases} x - \frac{ra+b}{r+1}, & x \in \left[a, \frac{a+b}{2}\right] \\ x - \frac{a+rb}{r+1}, & x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

By multiplynig both sides of the equalities (2.12) and (2.13), we have

$$\begin{aligned} S_r(f,g) &= F_r G_r - F_r\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) - G_r\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \\ &\quad + \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\ &= \frac{1}{(b-a)^2}\left(\int_a^b K(x,r)f'(x)\,dx\right)\left(\int_a^b K(x,r)g'(x)\,dx\right). \end{aligned} \tag{2.14}$$

From (2.14) and by applying the properties of modulus and by using the well-known Hölder's inequality, we get

(2.15)

$$\begin{aligned} |S_r(f,g)| &\le \frac{1}{(b-a)^2}\left(\int_a^b |K(x,r)|\,|f'(x)|\,dx\right)\left(\int_a^b |K(x,r)|\,|g'(x)|\,dx\right) \\ &\le \frac{1}{(b-a)^2}\left\{\left(\int_a^b |K(x,r)|^q\,dx\right)^{\frac{1}{q}}\left(\int_a^b |f'(x)|^p\,dx\right)^{\frac{1}{p}}\right\} \\ &\quad \times\left\{\left(\int_a^b |K(x,r)|^q\,dx\right)^{\frac{1}{q}}\left(\int_a^b |g'(x)|^p\,dx\right)^{\frac{1}{p}}\right\} \\ &= \frac{1}{(b-a)^2}\left(\int_a^b |K(x,r)|^q\,dx\right)^{\frac{2}{q}}\|f'\|_p\,\|g'\|_p. \end{aligned}$$

By a simple computation, one can see that

$$\int_a^b |K(x,r)|^q\,dx = \frac{2(b-a)^{q+1}\left(1+(r-1)^{q+1}\right)}{(q+1)(r+1)^{q+1}} = M_r. \tag{2.16}$$

By using (2.16) in (2.15), we obtain the inequality (2.9). For the proof of the inequality (2.10), we multiply both sides of the equalities (2.12) and (2.13) by $g(x)$

and $f(x)$, respectively and adding the results, we get

$$\begin{aligned} & F_r g(x) + G_r f(x) - g(x)\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) - f(x)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\ = & \; g(x)\left(\frac{1}{b-a}\int_a^b K(x,r)f'(x)\,dx\right) + f(x)\left(\frac{1}{b-a}\int_a^b K(x,r)g'(x)\,dx\right). \end{aligned}$$

By integrating both sides of the above equality with respect to x over $[a,b]$ and dividing bith sides of the result by $(b-a)$, we obtain

$$\begin{aligned} H_r(f,g) &= \frac{1}{b-a}\int_a^b [F_r g(x) + G_r f(x)]\,dx - 2\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\ &= \frac{1}{(b-a)^2}\int_a^b \left[g(x)\int_a^b K(x,r)f'(x)\,dx + f(x)\int_a^b K(x,r)g'(x)\,dx\right]dx. \end{aligned}$$

By applying the properties of modulus and by using the well-known Hölder's inequality, we get

$$\begin{aligned} H_r(f,g) &\leq \frac{1}{(b-a)^2}\int_a^b \left[|g(x)|\int_a^b |K(x,r)|\,|f'(x)|\,dx + |f(x)|\int_a^b |K(x,r)|\,|g'(x)|\,dx\right]dx \\ &\leq \frac{1}{(b-a)^2}\int_a^b \left[|g(x)|\left\{\left(\int_a^b |K(x,r)|^q\,dx\right)^{\frac{1}{q}}\left(\int_a^b |f'(x)|^p\,dx\right)^{\frac{1}{p}}\right\}\right. \\ &\quad \left. + |f(x)|\left\{\left(\int_a^b |K(x,r)|^q\,dx\right)^{\frac{1}{q}}\left(\int_a^b |g'(x)|^p\,dx\right)^{\frac{1}{p}}\right\}\right]dx \\ &= \frac{1}{(b-a)^2}M^{\frac{1}{q}}\int_a^b \left[|g(x)|\,\|f'\|_p + |f(x)|\,\|g'\|_p\right]dx \end{aligned}$$

which completes the proof. □

Remark 2. *If we choose $r = 5$ in the inequalities (2.9) and (2.10), we obtain the inequalities (1.6) and (1.7).*

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