Received 18/10/12

ON HADAMARD TYPE INEQUALITIES FOR s-GEOMETRICALLY CONVEX FUNCTIONS

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ABSTRACT. In the present paper, we establish inequalities for s-geometrically and geometrically convex functions which are connected with the famous Hermite-Hadamard inequality holding for convex functions. Some applications to special means of positive real numbers are given.

1. INTRODUCTION

In this section, we firstly list several definitions and some known results.

Definition 1. *Let I be an interval in $\mathbb{R}$. Then $f: I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$ is said to be convex if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$.

The famous Hermite-Hadamard inequality which was first published in [1] gives us an estimate of the mean value of a convex function $f: I \to R$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Definition 2. [2] *Let $s \in (0,1]$. A function $f: I \subset \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be $s-$convex in the second sense if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

It can be easily checked for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Recently, In [3], the concept of geometrically and s-geometrically convex functions was introduced as follows.

Definition 3. [3] *A function $f: I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if*

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^t [f(y)]^{1-t} \tag{1.4}$$

for all $x, y \in I$ and $t \in [0,1]$.

Date: October 08, 2012.
2000 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. geometrically convex, s-geometrically convex, special means.
This paper is in final form and no version of it will be submitted for publication elsewhere.

Definition 4. [3] *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a s-geometrically convex function if*

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s} \tag{1.5}$$

for some $s \in (0,1]$, where $x, y \in I$ and $t \in [0,1]$.

If $s = 1$, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

Example 1. [3] *Let $f(x) = x^s/s$, $x \in (0,1]$, $0 < s < 1$, $q \geq 1$, and then the function*

$$|f'(x)|^q = x^{(s-1)q} \tag{1.6}$$

is monotonically decreasing on $(0,1]$. For $t \in [0,1]$, we have

$$(s-1)q(t^s - t) \leq 0, \quad (s-1)q((1-t)^s - (1-t)) \leq 0. \tag{1.7}$$

Hence, $|f'(x)|^q$ is s-geometrically convex on $(0,1]$ for $0 < s < 1$.

Let

$$\begin{aligned} A(a,b) &= \frac{a+b}{2},\ L(a,b) = \frac{b-a}{\ln b - \ln a} \qquad (a \neq b), \\ L_p(a,b) &= \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p},\ a \neq b,\ p \in \mathbb{R},\ p \neq -1, 0 \end{aligned}$$

be the arithmetic, logarithmic, generalized logatihmic means for $a, b > 0$ respectively.

In the present paper, based on the above works, we establish several new inequalities of Hermite–Hadamard type for geometrically and s-geometrically convex functions. What's more, as applications, some special means of positive real numbers are deduced.

2. Mean results for s-geometrically convexity and applications

Theorem 1. *Let $s \in (0,1]$. If $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is s-geometrically convex and monotonically decreasing on $[a,b]$, then one has*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \Psi(s, \psi_1(\alpha), \psi_1(\beta)) \tag{2.1}$$

where

$$\psi_1(\alpha) = \begin{cases} 1, \alpha = 1, \\ \frac{\alpha - 1}{\ln \alpha}, \alpha \neq 1 \end{cases}, \qquad \psi_1(\beta) = \begin{cases} 1, \beta = 1, \\ \frac{\beta - 1}{\ln \beta}, \beta \neq 1 \end{cases} \tag{2.2}$$

$$\alpha(u,v) = [f(a)]^u [f(b)]^{-v},\ \beta(u,v) = [f(a)]^{-u} [f(b)]^v,\ u, v > 0 \tag{2.3}$$

$$\begin{aligned} &\Psi(s, g_1(\alpha), g_1(\beta)) \\ &= \begin{cases} \frac{1}{2}\left[[f(b)]^s \psi_1(\alpha(s,s)) + [f(a)]^s \psi_1(\beta(s,s))\right], & f(a) \leq 1 \\ \frac{1}{2}\left[[f(b)]^s \psi_1\left(\alpha\left(\frac{1}{s}, s\right)\right) + [f(a)]^{1/s} \psi_1\left(\beta\left(\frac{1}{s}, s\right)\right)\right], & f(b) \leq 1 \leq f(a) \\ \frac{1}{2}\left[[f(b)]^{1/s} \psi_1\left(\alpha\left(\frac{1}{s}, \frac{1}{s}\right)\right) + [f(a)]^{1/s} \psi_1\left(\beta\left(\frac{1}{s}, \frac{1}{s}\right)\right)\right], & 1 \leq f(b) \end{cases} \end{aligned} \tag{2.4}$$

Proof. Since f is s-geometrically convex and monotonically decreasing on $[a,b]$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq \frac{1}{2}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right] \\ &\leq \frac{1}{2}\left[f\left(a^{t}b^{(1-t)}\right)+f\left(a^{(1-t)}b^{t}\right)\right] \\ &\leq \frac{1}{2}\left[\left[f(a)\right]^{t^{s}}\left[f(b)\right]^{(1-t)^{s}}+\left[f(a)\right]^{(1-t)^{s}}\left[f(b)\right]^{t^{s}}\right]. \end{aligned} \tag{2.5}$$

By integrating this inequality (2.5) over t on $[0,1]$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_{a}^{b}f(x)\,dx \\ &\leq \frac{1}{2}\int_{0}^{1}\left[\left[f(a)\right]^{t^{s}}\left[f(b)\right]^{(1-t)^{s}}+\left[f(a)\right]^{(1-t)^{s}}\left[f(b)\right]^{t^{s}}\right]dt \end{aligned} \tag{2.6}$$

If $0<\rho\leq 1\leq\sigma$, $0<t,s\leq 1$, then

$$\rho^{t^{s}}\leq\rho^{ts},\ \ \sigma^{t^{s}}\leq\sigma^{t/s}. \tag{2.7}$$

Case i) If $f(a)\leq 1$, by (2.7), we get that

$$\begin{aligned} &\frac{1}{2}\int_{0}^{1}\left[\left[f(a)\right]^{t^{s}}\left[f(b)\right]^{(1-t)^{s}}+\left[f(a)\right]^{(1-t)^{s}}\left[f(b)\right]^{t^{s}}\right]dt \\ \leq\ &\frac{1}{2}\int_{0}^{1}\left[\left[f(a)\right]^{st}\left[f(b)\right]^{s(1-t)}+\left[f(a)\right]^{s(1-t)}\left[f(b)\right]^{st}\right]dt \\ =\ &\frac{1}{2}\int_{0}^{1}\left[\left[f(b)\right]^{s}\left[\frac{f(a)}{f(b)}\right]^{st}+\left[f(a)\right]^{s}\left[\frac{f(b)}{f(a)}\right]^{st}\right]dt \\ =\ &\frac{1}{2}\left[\left[f(b)\right]^{s}g_{1}\left(\alpha(s,s)\right)+\left[f(a)\right]^{s}g_{1}\left(\beta(s,s)\right)\right]. \end{aligned} \tag{2.8}$$

Case ii) If $f(b)\leq 1\leq f(a)$, by (2.7), we get that

$$\begin{aligned} &\frac{1}{2}\int_{0}^{1}\left[\left[f(a)\right]^{t^{s}}\left[f(b)\right]^{(1-t)^{s}}+\left[f(a)\right]^{(1-t)^{s}}\left[f(b)\right]^{t^{s}}\right]dt \\ \leq\ &\frac{1}{2}\int_{0}^{1}\left[\left[f(a)\right]^{t/s}\left[f(b)\right]^{s(1-t)}+\left[f(a)\right]^{(1-t)/s}\left[f(b)\right]^{st}\right] \\ =\ &\frac{1}{2}\int_{0}^{1}\left[\left[f(b)\right]^{s}\left[\frac{f(a)^{1/s}}{f(b)^{s}}\right]^{t}+\left[f(a)\right]^{1/s}\left[\frac{f(b)^{s}}{f(a)^{1/s}}\right]^{t}\right]dt \\ =\ &\frac{1}{2}\left[\left[f(b)\right]^{s}g_{1}\left(\alpha\left(\frac{1}{s},s\right)\right)+\left[f(a)\right]^{1/s}g_{1}\left(\beta\left(\frac{1}{s},s\right)\right)\right]. \end{aligned} \tag{2.9}$$

Case iii) If $1 \leq f(b)$, by (2.7), we get that

$$\frac{1}{2}\int_0^1 \left[[f(a)]^{t^s} [f(b)]^{(1-t)^s} + [f(a)]^{(1-t)^s} [f(b)]^{t^s} \right] dt \tag{2.10}$$

$$\leq \frac{1}{2}\int_0^1 \left[[f(a)]^{t/s} [f(b)]^{(1-t)/s} + [f(a)]^{(1-t)/s} [f(b)]^{t/s} \right]$$

$$= \frac{1}{2}\int_0^1 \left[[f(b)]^{1/s} \left[\frac{f(a)}{f(b)}\right]^{t/s} + [f(a)]^{1/s} \left[\frac{f(b)}{f(a)}\right]^{t/s} \right] dt$$

$$= \frac{1}{2}\left[[f(b)]^{1/s} g_1\left(\alpha\left(\frac{1}{s},\frac{1}{s}\right)\right) + [f(a)]^{1/s} g_1\left(\beta\left(\frac{1}{s},\frac{1}{s}\right)\right) \right].$$

From (2.6)-(2.10), (2.1) holds. □

Corollary 1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be geometrically convex and monotonically decreasing on $[a, b]$, then one has*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \Psi(1, \psi_1(\alpha), \psi_1(\beta)), \tag{2.11}$$

where $\alpha(u, v)$, $\beta(u, v)$, $\Psi(s, g_1(\alpha), g_1(\beta))$, $g_1(\alpha)$ and $g_1(\beta)$ are the same as in Theorem 1.

Proposition 1. *Let $0 < a < b \leq 1$, $0 < s \leq 1$. Then*

$$[A(a,b)]^s \leq [L_s(a,b)]^s \leq \frac{s^{1-1/s}}{2} L(a,b) \tag{2.12}$$

Proof. The proof is obvious from Theorem 1 applied $f(x) = x^s/s$, $x \in (0, 1]$, $0 < s < 1$. Then $f(a) = a^s/s > b^s/s = f(b) \geq 1$ and

$$\frac{\left(\frac{a+b}{2}\right)^s}{s} \leq \frac{1}{s(b-a)}\int_a^b x^s dx$$

$$\leq \frac{1}{2}\left[\left[\frac{b^s}{s}\right]^{1/s} \psi_1\left(\alpha\left(\frac{1}{s},\frac{1}{s}\right)\right) + \left[\frac{a^s}{s}\right]^{1/s} \psi_1\left(\beta\left(\frac{1}{s},\frac{1}{s}\right)\right) \right]$$

$$= \frac{1}{2s^{1/s}}\left[\left[\frac{b}{s^{1/s}}\frac{a-b}{b(\ln a - \ln b)}\right] + \left[\frac{a}{s^{1/s}}\frac{b-a}{a(\ln b - \ln a)}\right] \right]$$

By Theorem 1, Proposition 1 is thus proved. □

Theorem 2. *Let $s \in (0, 1]$. If $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is s-geometrically convex and monotonically decreasing on $[a, b]$, then one has*

$$f^{2^s}(G(a,b)) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \leq \Psi(s, f) \tag{2.13}$$

where, $G(a, b)$ is geometric mean, and

$$\Psi(s,f) = \begin{cases} [f(a)f(b)]^s, & f(a) \leq 1 \\ [f(a)]^{1/s}[f(b)]^s, & f(b) \leq 1 \leq f(a) \\ [f(a)f(b)]^{1/s}, & 1 \leq f(b) \end{cases}.$$

Proof. Since f is s-geometrically convex and monotonically decreasing on $[a,b]$, we have

$$\begin{aligned} f\left(a^t b^{(1-t)}\right) &\leq [f(a)]^{t^s}[f(b)]^{(1-t)^s} \\ f\left(a^{(1-t)} b^t\right) &\leq [f(a)]^{(1-t)^s}[f(b)]^{t^s} \end{aligned}$$

for all $t\in[0,1]$, $s\in(0,1]$. If we multiply the above inequalities, we obtain

$$f\left(a^t b^{(1-t)}\right) f\left(a^{(1-t)} b^t\right) \leq [f(a) f(b)]^{t^s+(1-t)^s}.$$

Integrating this inequality on $[0,1]$ over t, we get

$$\int_0^1 f\left(a^t b^{(1-t)}\right) f\left(a^{(1-t)} b^t\right) dt \leq \int_0^1 [f(a) f(b)]^{t^s+(1-t)^s} dt. \tag{2.14}$$

If we change the variable $x=a^t b^{(1-t)}$, $t\in[0,1]$, $dx=a^t b^{(1-t)}\ln\frac{a}{b}dt$, we obtain

$$\int_0^1 f\left(a^t b^{(1-t)}\right) f\left(a^{(1-t)} b^t\right) dt = \frac{1}{\ln b-\ln a}\int_a^b f(x) f\left(\frac{ab}{x}\right)\frac{dx}{x} \tag{2.15}$$

and, by (2.7), we have

Case i) If $f(a)\leq 1$, by (2.7), we get that

$$\int_0^1 [f(a) f(b)]^{t^s+(1-t)^s} dt \leq \int_0^1 [f(a) f(b)]^{st+s(1-t)} dt = [f(a) f(b)]^s. \tag{2.16}$$

Case ii) If $f(b)\leq 1\leq f(a)$, by (2.7), we get that

$$\begin{aligned} \int_0^1 [f(a) f(b)]^{t^s+(1-t)^s} dt &\leq \int_0^1 [f(a)]^{t/s+(1-t)/s}[f(b)]^{st+s(1-t)} dt \\ &= [f(a)]^{1/s}[f(b)]^s. \end{aligned} \tag{2.17}$$

Case iii) If $1\leq f(b)$, by (2.7), we get that

$$\int_0^1 [f(a) f(b)]^{t^s+(1-t)^s} dt \leq \int_0^1 [f(a) f(b)]^{t/s+(1-t)/s} dt = [f(a) f(b)]^{1/s}. \tag{2.18}$$

and from (2.14)-(2.18), the second inequality in (2.13) is proved.

Secondly, by (1.5), for $t=1/2$, we have that

$$\begin{aligned} f(\sqrt{xy}) &\leq [f(x) f(y)]^{1/2^s} \\ f^{2^s}(\sqrt{xy}) &\leq f(x) f(y) \end{aligned}$$

for all $x,y\in I$.If we choose $x=a^t b^{1-t}$, $y=a^{1-t}b^t$, we get the inequality

$$f^{2^s}(G(a,b)) \leq f\left(a^t b^{1-t}\right) f\left(a^{1-t} b^t\right).$$

Integrating this inequality on $[0,1]$ over t, we obtain the first inequality in (2.13). □

Corollary 2. *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be geometrically convex and monotonically decreasing on $[a,b]$, then one has*

$$f^2(G(a,b)) \leq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx \leq \Psi(1,f), \tag{2.19}$$

where $\Psi(s,f)$ is the same as in Theorem 2.

Proposition 2. *Let* $0 < a < b \leq 1$, $0 < s \leq 1$. *Then*

$$\left(\frac{[G(a,b)]^s}{s}\right)^{(2^s)} \leq \frac{[G(a,b)]^{2s}}{s^2} \leq \frac{[G(a,b)]^2}{s^{2/s}}, \tag{2.20}$$

with equality if and only if $s = 1$.

Proof. The proof is obvious from Theorem 1 applied $f(x) = x^s/s$, $x \in (0,1]$, $0 < s < 1$. Then $f(a) = a^s/s > b^s/s = f(b) \geq 1$ and

$$\begin{aligned} f^{2^s}(G(a,b)) &= \left(\frac{(ab)^{\frac{s}{2}}}{s}\right)^{(2^s)} \\ \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} f\left(\frac{ab}{x}\right) dx &= \frac{(ab)^s}{s^2}\frac{1}{\ln b - \ln a}\int_a^b \frac{dx}{x} \\ [f(a) f(b)]^{1/s} &= \frac{ab}{s^{\frac{2}{s}}}. \end{aligned}$$

By Theorem 2.2, Proposition 2.2 is thus proved. □

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