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COMPARING TWO INTEGRAL MEANS FOR FUNCTIONS WHOSE ABSOLUTE VALUE OF THE DERIVATIVE ARE QUASI-CONVEX AND APPLICATIONS

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Abstract. Some new estimates of the difference between the integral means of a function whose absolute value of the first derivative is quasi-convex compared to its mean over a subinterval are established and new applications for special means and probability density functions are also given.

1. Introduction

The classical Ostrowski type integral inequality [1] stipulates a bound for the difference between a function evaluated at an interior point x and the average of the function f over an interval. That is,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x)\right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$, where $f' \in L_\infty(a,b)$, that is,

$$\|f'\|_\infty = ess \sup_{t\in[a,b]} |f'(t)| < \infty,$$

and $f : [a,b] \to R$ is a differentiable mapping on (a,b). Here, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For the results and generalizations concerning Ostrowski's inequality, see [2-13] and the references therein.

In [14], Barnett et al. compared the difference of two integral means as in the following Theorem 1 in which the function has the first derivative bounded where is defined. The obtained results are also a generalization of (1.1), and has been applied to probability density functions, special means, Jeffreys divergence in Information Theorem and the sampling of continuous streams in Statistics.

Theorem 1. *Let $f : [a,b] \to R$ be an absolutely continuous mapping with the property that $f' \in L_\infty[a,b]$. Then, for $a \le x < y \le b$, we have the inequality*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\le \left\{\frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{x+y}{2}}{b-a-y+x}\right]^2\right\}(b-a-y+x)\|f'\|_\infty. \\ &\le \frac{1}{2}(b-a-y+x)\|f'\|_\infty. \end{aligned} \tag{1.2}$$

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The constant $\frac{1}{4}$ is best possible in the first inequality and $\frac{1}{2}$ is best in the second one.

In [15], Hwang and Dragomir obtained some inequalities of type (1.2) for the functions whose absolute value of the first derivative are convex.

In what following, we recall the definition of quasi-convex. In [16], this class is defined in the following way : $f : [a, b] \to R$ is a quasi-convex functions if

$$f(\alpha x + (1 - \alpha)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in [a, b], \alpha \in [0, 1]$. Clearly, any convex function is a quasi-convex function, and there exist quasi-convex functions which are not convex.

Recently, Ion [17] and Alomari and Darus [18] introduced some Ostrowski type inequalities (1.1) for the functions whose first derivatives in absolute value are quasi-convex.

The purpose of this article is to establish some new results related to the inequality (1.2) for the functions whose absolute value of the first derivative are quasi-convex. The corresponding versions in the case that the power of the absolute value of the first derivative is quasi-convex are obtained. Applying the obtained results, some new inequalities for special means and the probability density functions will be also given.

For convenience, we use the following notations :

$$A = \frac{(y - x)(b - a - y + x)}{b - a}, \ B = \frac{(x - a)(y - x)}{b - a},$$

where $a \leq x < y \leq b$.

2. The mapping $|f'|$ and $|f'|^q$ are Quasi-Convex

Theorem 2. *Let $f : [a, b] \to R$ be an absolutely continuous and $|f'|$ is convex on $[a, b]$. Then, for $a \leq x < y \leq b$, we have the inequality*

$$\begin{aligned}(2.1)\quad &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\leq \frac{(x-a)^2}{2(b-a)}\max\{|f'(a)|, |f'(x)|\} + \Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)\max\{|f'(x)|, |f'(y)|\} \\ &+ \frac{(b-y)^2}{2(b-a)}\max\{|f'(y)|, |f'(b)|\}.\end{aligned}$$

Proof. Using the following identity given in [14],

$$\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du = \int_a^b K_{x,y}(s)f'(s)ds$$

where

$$K_{x,y}(s) = \begin{cases} \frac{a-s}{b-a}, & \text{if } s \in [a, x], \\ \frac{s-x}{y-x} + \frac{a-s}{b-a}, & \text{if } s \in (x, y), \\ \frac{b-s}{b-a}, & \text{if } s \in [y, b], \end{cases}$$

and by suitable substitution of variables, we have the following identity,

$$
\begin{aligned}
(2.2)\qquad & \int_a^b K_{x,y}(s)f'(s)ds \\
&= \frac{-(x-a)^2}{b-a}\int_0^1 tf'((1-t)a+tx)dt \\
&+ \int_0^1 \Big(\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\Big)f'((1-t)x+ty)dt \\
&+ \frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt.
\end{aligned}
$$

From (2.2), we obtain

$$
\begin{aligned}
(2.3)\qquad & \left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\
&+ \int_0^1 \left|\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right| \cdot \big|f'((1-t)x+ty)\big|dt \\
&+ \frac{(b-y)^2}{b-a}\int_0^1 (1-t)\big|f'((1-t)y+tb)\big|dt.
\end{aligned}
$$

Using the quasi-convexity of $|f'|$, we get

$$
(2.4)\qquad \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \leq \frac{(x-a)^2}{2(b-a)}\max\{|f'(a)|,|f'(x)|\},
$$

$$
\begin{aligned}
(2.5)\qquad & \int_0^1 \left|\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right| \cdot \big|f'((1-t)x+ty)\big|dt \\
&\leq \Big[\int_0^{B/A}(B-At)dt + \int_{B/A}^1 (At-B)dt\Big]\max\{|f'(x)|,|f'(y)|\} \\
&= \left(\frac{A}{2} - B + \frac{B^2}{A}\right)\max\{|f'(y)|,|f'(y)|\}
\end{aligned}
$$

and

$$
(2.6)\qquad \frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt \leq \frac{(b-y)^2}{2(b-a)}\max\{|f'(y)|,|f'(b)|\}.
$$

By combining inequalities (2.3), (2.4), (2.5) and (2.6), we deduce the desire inequality (2.1). This completes the proof of Theorem 2. ∎

By the Theorem 2, we have following two Corollarys immediately.

Corollary 1. *Assume that hypotheses in Theorem 2 hold and $|f'|$ is increasing. Then we have*

$$\begin{aligned}(2.7)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\leq \frac{(x-a)^2}{2(b-a)}|f'(x)| + \Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)|f'(y)| + \frac{(b-y)^2}{2(b-a)}|f'(b)|.\end{aligned}$$

Corollary 2. *Assume that the hypotheses in Theorem 2 hold and $|f'|$ is decreasing. Then we have*

$$\begin{aligned}(2.8)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\leq \frac{(x-a)^2}{2(b-a)}|f'(a)| + \Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)|f'(x)| + \frac{(b-y)^2}{2(b-a)}|f'(y)|.\end{aligned}$$

Remark 1. *By* (2.2)*, definition of $\|f'\|_\infty$, and simple computation, we have*

$$\begin{aligned}&\frac{(x-a)^2}{2(b-a)}\max\{|f'(a)|,|f'(x)|\} + \Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)\max\{|f'(x)|,|f'(y)|\} \\ &+ \frac{(b-y)^2}{2(b-a)}\max\{|f'(y)|,|f'(b)|\}| \\ &\leq \left[\frac{(x-a)^2}{b-a}\int_0^1 tdt + \int_0^1\left|\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right|dt\right. \\ &\left.+ \frac{(b-y)^2}{b-a}\int_0^1(1-t)dt\right]\cdot\|f'\|_\infty \\ &= \left\{\frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{x+y}{2}}{b-a-y+x}\right]^2\right\}(b-a-y+x)\cdot\|f'\|_\infty,\end{aligned}$$

where the last identity had been given in the proofs of Theorem 2.2 in [14]*. Therefore, for the strict quasi-convex mapping, we have the bound in* (2.1) *is smaller than the one in* (1.2)*. Similarly, the bound in* (2.7)*and* (2.8) *is also smaller than the one in* (1.2)*.*

If we set $y = x + h$ with $x + h \in (a,b)$, then by (2.2)*, we get*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{h}\int_x^{x+h} f(t)dt\right| \\ &\leq \frac{(x-a)^2}{2(b-a)}\max\{|f'(a)|,|f'(x)|\} + h\Big[\frac{(a+b-2x-h)(b-a-h)+2(x-a)^2}{2(b-a)(b-a-h)}\Big] \\ &\times\max\{|f'(x)|,|f'(x+h)|\} + \frac{(b-y)^2}{2(b-a)}\max\{|f'(x+h)|,|f'(b)|\}.\end{aligned}$$

Now, letting $h \to 0^+$, we have

$$\begin{aligned} (2.9) \qquad & \left| \frac{1}{b-a} \int_a^b f(u)du - f(x) \right| \\ & \leq \frac{(x-a)^2}{2(b-a)} \max\{|f'(a)|, |f'(x)|\} + \frac{(b-x)^2}{2(b-a)} \max\{|f'(x)|, |f'(b)|\} \\ & \leq (b-a) \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty . \end{aligned}$$

We note that (2.9) is also given in [18], and the bound is better than the one in (1.1). Also, for $x = \frac{a+b}{2}$, we note that (2.9) reduces to (2.4) in [19] and the result of (2.4) improves the Theorem 2.2 in [20].

In the following theorem, the estimates of the difference between the integral means of a function compared to its mean over a subinterval for the quasi-convex mapping $|f'|^{p/(p-1)}$ are given.

Theorem 3. *Let $f : [a, b] \to R$ be an absolutely continuous and $|f'|^{p/p-1}$ is quasi-convex on $[a, b]$, for $p, q > 1, 1/p + 1/q = 1$. Then, for $a \leq x < y \leq b$, we have the inequality*

(2.10)
$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{y-x} \int_x^y f(u)du \right| \\ & \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{(x-a)^2}{b-a} \Big(\max\{|f'(a)|^q, |f'(x)|^q\} \Big)^{\frac{1}{q}} + \left(\frac{B^{p+1} + (A-B)^{p+1}}{A} \right)^{\frac{1}{p}} \right. \\ & \left. \times \Big(\max\{|f'(x)|^q, |f'(y)|^q\} \Big)^{\frac{1}{q}} + \frac{(b-y)^2}{b-a} \Big(\max\{|f'(y)|^q, |f'(b)|^q\} \Big)^{\frac{1}{q}} \right] \end{aligned}$$

for $a \leq x < y \leq b$.

Proof. Using (2.3) and the Hölder inequality, we have that

$$\begin{aligned} (2.11) \qquad & \left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{y-x} \int_x^y f(u)du \right| \\ & \leq \frac{(x-a)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'((1-t)a + tx)|^q dt \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 \left| \frac{(y-x)(b-a-y+x)}{b-a} t - \frac{(x-a)(y-x)}{b-a} \right|^p dt \right)^{\frac{1}{p}} \\ & \times \left(\int_0^1 |f'((1-t)x + ty)|^q dt \right)^{\frac{1}{q}} + \frac{(b-y)^2}{b-a} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \\ & \times \left(\int_0^1 |f'((1-t)y + tb)|^q dt \right)^{\frac{1}{q}} . \end{aligned}$$

Now, using the quasi-convexity of $|f'|^{p/p-1}$, we get

$$\int_0^1 |f'((1-t)a+tx)|^q dt \le \max\{|f'(a)|^q, |f'(x)|^q\}, \tag{2.12}$$

$$\int_0^1 |f'((1-t)x+ty)|^q dt \le \max\{|f'(x)|^q, |f'(y)|^q\} \tag{2.13}$$

and

$$\int_0^1 |f'((1-t)y+tb)|^q dt \le \max\{|f'(y)|^q, |f'(b)|^q\}. \tag{2.14}$$

By simple computation, we obtain

$$\int_0^1 t^p dt = \frac{1}{p+1}, \tag{2.15}$$

$$\int_0^1 (1-t)^p dt = \frac{1}{p+1} \tag{2.16}$$

and

$$\begin{aligned} &\int_0^1 \left| \frac{(y-x)(b-a-y+x)}{b-a} t - \frac{(x-a)(y-x)}{b-a} \right|^p dt \\ &\le \int_0^{B/A} (B-At)^p dt + \int_{B/A}^1 (At-B)^p dt \\ &= \frac{1}{p+1}\left[\frac{B^{p+1}+(A-B)^{p+1}}{A}\right], \end{aligned} \tag{2.17}$$

and combining inequalities (2.11)-(2.17), we deduce the desire inequality (2.10). This completes the proofs of Theorem 3. ∎

By the Theorem 3, we have following two Corollarys immediately.

Corollary 3. *Assume that the hypotheses in Theorem 3 hold and $|f'|$ is increasing. Then we have*

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du \right| \\ &\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\frac{(x-a)^2}{b-a}|f'(x)| + \left(\frac{B^{p+1}+(A-B)^{p+1}}{A}\right)^{\frac{1}{p}} |f'(y)| \right. \\ &\quad \left. + \frac{(b-y)^2}{b-a}|f'(b)| \right]. \end{aligned} \tag{2.18}$$

Corollary 4. *Assume that the hypotheses in Theorem 3 hold and $|f'|$ is decreasing. Then we have*

$$\begin{aligned}(2.19)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{y-x}\int_x^y f(u)du\right|\\ \le&\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^2}{b-a}|f'(a)|+\Big(\frac{B^{p+1}+(A-B)^{p+1}}{A}\Big)^{\frac{1}{p}}|f'(x)|\\ &+\frac{(b-y)^2}{b-a}|f'(y)|\Big].\end{aligned}$$

Remark 2. *If we set $y=x+h$ with $x+h\in(a,b)$, then by* (2.10)*, we get*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{h}\int_x^{x+h} f(t)dt\right|\\ &\le\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^2}{b-a}\Big(\max\{|f'(a)|^q,|f'(x)|^q\}\Big)^{\frac{1}{q}}\\ &+\frac{h}{b-a}\Big(\frac{(x-a)^{p+1}+(b-x-h)^{p+1}}{b-a-h}\Big)^{\frac{1}{p}}\Big(\max\{|f'(x)|^q,|f'(x+h)|^q\}\Big)^{\frac{1}{q}}\\ &+\frac{(b-x-h)^2}{b-a}\Big(\max\{|f'(x+h)|^q,|f'(b)|^q\}\Big)^{\frac{1}{q}}\Big].\end{aligned}$$

Now, letting $h\to0^+$, we have

$$\begin{aligned}(2.20)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du-f(x)\right|\\ &\le\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^2}{b-a}\Big(\max\{|f'(a)|^q,|f'(x)|^q\}\Big)^{\frac{1}{q}}\\ &+\frac{(b-x)^2}{b-a}\Big(\max\{|f'(x)|^q,|f'(b)|^q\}\Big)^{\frac{1}{q}}\Big].\end{aligned}$$

The Theorem 3, Corollary 3 and Corollary 4 are all the new type of Theorem 1,respcetively. The bound in (2.20) *is also given in (2.9) in* [18]*, and, for $x=\frac{a+b}{2}$,* (2.20) *reduces to (2.9) in* [19].

In the following Theorem, the estimates of the difference between the integral means of a function compared to its mean over a subinterval for the quasi-convex mapping $|f'|^q$ are given.

Theorem 4. *Let $f:[a,b]\to R$ be an absolutely continuous and $|f'|^q$ is quasi-convex on $[a,b]$, for $q\ge1$. Then, for $a\le x<y\le b$, we have the inequality*

$$\begin{aligned}(2.21)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{y-x}\int_x^y f(u)du\right|\\ &\le\frac{1}{2}\Big[\frac{(x-a)^2}{b-a}\Big(\max\{|f'(a)|^q,|f'(x)|^q\}\Big)^{\frac{1}{q}}+\Big(\frac{B^2+(A-B)^2}{A}\Big)\\ &\times\Big(\max\{|f'(x)|^q,|f'(y)|^q\}\Big)^{\frac{1}{q}}+\frac{(b-y)^2}{b-a}\Big(\max\{|f'(y)|^q,|f'(b)|^q\}\Big)^{\frac{1}{q}}\Big]\end{aligned}$$

for $a\le x<y\le b$.

Proof. Using (2.3), the Hölder inequality and the quasi-convexity of $|f'|^q$, we have that

(2.22)

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{\frac{q-1}{q}}\left(\int_0^1 t|f'((1-t)a+tx)|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\frac{(y-x)(b-a-y+x)}{b-a}t-\frac{(x-a)(y-x)}{b-a}\right|dt\right)^{\frac{q-1}{q}} \\
&\times\left(\int_0^1\left|\frac{(y-x)(b-a-y+x)}{b-a}t-\frac{(x-a)(y-x)}{b-a}\right|\cdot|f'((1-t)x+ty)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-y)^2}{b-a}\left(\int_0^1(1-t)dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)|f'((1-t)y+tb)|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\frac{(y-x)(b-a-y+x)}{b-a}t-\frac{(x-a)(y-x)}{b-a}\right|dt\right) \\
&\times\left(\max\{|f'(y)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}} \\
&+\frac{(b-y)^2}{b-a}\left(\int_0^1(1-t)dt\right)\left(\max\{|f'(y)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}
\end{aligned}$$

Combining for $p=1$ in (2.15), (2.16) and (2.17), we deduce the desire inequality (2.21). This completes the proofs of Theorem 4. ∎

By the Theorem 4, we have following two Corollarys immediately.

Corollary 5. *Assume that the hypotheses in Theorem 4 hold and $|f'|$ is increasing. Then we have* (2.7).

Corollary 6. *Assume that the hypotheses in Theorem 4 hold and $|f'|$ is decreasing. Then we have* (2.8).

Remark 3. *If we set $y=x+h$ with $x+h\in(a,b)$, then by* (2.21)*, we get*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{h}\int_x^{x+h} f(t)dt\right| \\
&\le \frac{1}{2}\left[\frac{(x-a)^2}{b-a}\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}+\frac{h}{b-a}\left(\frac{(x-a)^2+(b-x-h)^2}{b-a-h}\right)\right. \\
&\left.\times\left(\max\{|f'(x)|^q,|f'(x+h)|^q\}\right)^{\frac{1}{q}}+\frac{(b-x-h)^2}{b-a}\left(\max\{|f'(x+h)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Now, letting $h \to 0^+$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x)\right| \le \frac{1}{2}\left[\frac{(x-a)^2}{b-a}\Big(\max\{|f'(a)|^q, |f'(x)|^q\}\Big)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a}\Big(\max\{|f'(x)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}\right]. \tag{2.23}$$

In Theorem 4, for $q = 1$, this reduces to Theorem 2, and for $q = \frac{p}{p-1}$, $p > 1$, we have an improvement of the constants in Theorem 3, since $2 > (p+1)^{\frac{1}{p}}$ if $p > 1$ and accordingly $\frac{1}{2} < \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$. The bound in (2.23) *is also given in (2.16) in* [18], *and the* (2.23) *reduces to (2.14) in* [19] *for $x = \frac{a+b}{2}$.*

3. Applications to Special Means

In the following, we shall consider logarithmic, identric and generalized logarithmic means from two positive real numbers. We define

$$\begin{aligned} L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln\beta - \ln\alpha}, && \alpha,\beta \in R^+, \alpha \neq \beta,\\ I(\alpha,\beta) &= \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, && \alpha,\beta \in R^+, \alpha \neq \beta,\\ L_p(\alpha,\beta) &= \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, && p \in R\setminus\{-1,0\}, \alpha,\beta \in R^+, \alpha \neq \beta, \end{aligned}$$

where R is real numbers.

Proposition 1. *Let $a, b, x, y \in R$, $0 < a \le x < y \le b$ and $p \in R \setminus \{-1, 0\}$. Then, if $p \ge 1$, we have*

$$\left|L_p^p(a,b) - L_p^p(x,y)\right| \le \frac{p(x-a)^2x^{p-1}}{2(b-a)} + p\Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)y^{p-1} + \frac{p(b-y)^2b^{p-1}}{2(b-a)}, \tag{3.1}$$

and, if $p < 1$, we have

$$\left|L_p^p(a,b) - L_p^p(x,y)\right| \le \frac{|p|(x-a)^2a^{p-1}}{2(b-a)} + |p|\Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)x^{p-1} + \frac{|p|(b-y)^2y^{p-1}}{2(b-a)}. \tag{3.2}$$

Proof. The proof is immediate from Corollary 1 and Corollary 2 with $f(x) = x^p$, $x \in R^+$, $p \in R \setminus \{-1, 0\}$, respectively. ∎

Proposition 2. *Suppose $a, b, x, y \in R$, and $0 < a \le x < y \le b$. Then we have*

$$\left|L^{-1}(a,b) - L^{-1}(x,y)\right| \le \frac{(x-a)^2}{2(b-a)a^2} + \Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)\frac{1}{x^2} + \frac{(b-y)^2}{2(b-a)y^2}. \tag{3.3}$$

Proof. The result follows from Corollary 2 with $f(x) = \frac{1}{x}, x \in R^+$. ∎

Proposition 3. *Suppose $a, b, x, y \in R$, and $0 < a \le x < y \le b$. Then we have*

$$\left|\ln\Big[\frac{I(a,b)}{I(x,y)}\Big]\right| \le \frac{(x-a)^2}{2(b-a)a} + \Big(\frac{A}{2} - B + \frac{B^2}{A}\Big)\frac{1}{x} + \frac{(b-y)^2}{2(b-a)y}.$$

Proof. The result follows from Corollary 2 with $f(x) = \ln x$. ∎

Remark 4. *We note that the bounds in* (3.1)*,* (3.2) *and* (3.3) *are better than the ones in* (4.1)*,* (4.2) *and* (4.3) *given in* [14]*, respectively.*

Proposition 4. *Let $a, b, x, y, \alpha \in R, 0 < a \le x < y \le b$, and $q \ge 1$. Then, for $\alpha \ge 0$, we have*

$$\left|L_{(\alpha+q)/q}^{(\alpha+q)/q}(a,b) - L_{(\alpha+q)/q}^{(\alpha+q)/q}(x,y)\right| \le \frac{1}{2}\left[\frac{(x-a)^2}{(b-a)x^{\frac{1}{q}}} + \Big(\frac{B^2 + (A-B)^2}{A}\Big)\frac{1}{y^{\frac{1}{q}}} + \frac{(b-y)^2}{(b-a)b^{\frac{1}{q}}}\right], \tag{3.4}$$

and, for $\alpha < 0$ and $\alpha + p \neq 0$, we have

$$\left|L_{(\alpha+q)/q}^{(\alpha+q)/q}(a,b) - L_{(\alpha+q)/q}^{(\alpha+q)/q}(x,y)\right| \le \frac{1}{2}\left[\frac{(x-a)^2}{(b-a)a^{\frac{1}{q}}} + \Big(\frac{B^2 + (A-B)^2}{A}\Big)\frac{1}{x^{\frac{1}{q}}} + \frac{(b-y)^2}{(b-a)y^{\frac{1}{q}}}\right]. \tag{3.5}$$

Proof. Set $f(x) = \frac{q}{\alpha+q}x^{\frac{\alpha+q}{q}}$, we obtain $|f'(x)|^q = x^\alpha$ is quasi-convex obviously. If $\alpha \ge 0$, by (2.7), we have (3.4), and if $\alpha < 0$, by (2.8), we have (3.5), immediately. ∎

Remark 5. *The inequality* (3.4) *and inequality* (3.5) *are both the new type for comparing two generalized logarithmic means.*

4. Applications for Probability Density Functions

In the following, assume that $f : [a,b] \to R^+$ is a probability density function of a certain random variable X and $F : [a,b] \to R^+$, $F(t) = \int_a^t f(x)dx$ is its cumulative distribution function.

Proposition 5. *Let f and F be as previous assumption. Then we have*

$$\begin{aligned}\left|F(t) - \frac{t-a}{b-a}\right| &\le \frac{(b-t)(t-a)}{2(b-a)}\Big[(t-a)\max\{|f'(a)|, |f'(t)|\} \\ &\quad + (b-t)\max\{|f'(t)|, |f'(b)|\}\Big].\end{aligned} \tag{4.1}$$

provided that $|f'|$ is quasi-convex.

Proof. Taking $x = a$ and $y = t$ in (2.1), we have the desired inequality. ∎

Proposition 6. *Let f and F be as previous assumption. Then we have*

$$\begin{aligned}\left|F(t) - \frac{t-a}{b-a}\right| \\ \leq \frac{(b-t)(t-a)}{2(b-a)}\Big[(t-a)\big(\max\{|f'(a)|^q, |f'(t)|^q\}\big)^{\frac{1}{q}} \\ + (b-t)\big(\max\{|f'(t)|^q, |f'(b)|^q\}\big)^{\frac{1}{q}}\Big].\end{aligned} \tag{4.2}$$

provided that $|f'|^q$ is quasi-convex.

Proof. Taking $x = a$ and $y = t$ in (2.21), the inequality (4.2) holds obviously. ∎

Remark 6. *The bounds in* (4.1)*and* (4.2) *are both better than the one in* (3.1) *given in* [14].

Proposition 7. *Let f and F be as previous assumption and let*

$$E_t(X) = \int_a^t uf(u)du, t \in [a, b].$$

Then, for $t \in [a, b]$, we have

$$\begin{aligned}\left|\frac{(b-E(X))(t-a)}{b-a} + E_t(X) - tF(t)\right| \\ \leq \frac{(b-t)(t-a)}{2(b-a)}\Big[(t-a)\max\{|f(a)|, |f(t)|\} \\ + (b-t)\max\{|f(t)|, |f(b)|\}\Big].\end{aligned} \tag{4.3}$$

provided that $|f|$ is quasi-convex.

Proof. Taking $F = f, x = a$ and $y = t$ in (2.1), we get

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b F(x)dx - \frac{1}{t-a}\int_a^t F(u)du\right| \\ \leq \frac{(b-t)(t-a)}{2(b-a)}\Big[(t-a)\max\{|F'(a)|, |F'(t)|\} \\ + (b-t)\max\{|F'(t)|, |F'(b)|\}\Big].\end{aligned} \tag{4.4}$$

Since

$$\int_a^b F(x)dx = b - E(X)$$

and

$$\int_a^t F(u)du = tF(t) - \int_a^t uf(u)du = tF(t) - E_t(X),$$

thus, by (4.4), we have the desired inequality. ∎

Similarly, taking $F = f, x = a$ and $y = t$ in (2.21), we have the following proposition.

Proposition 8. *Let f, F and $E_t(X)$ be as defined in Proposition 6. Then, for $t \in [a,b]$, we have*

$$\left|\frac{(b-E(X))(t-a)}{b-a} + E_t(X) - tF(t)\right| \le \frac{(b-t)(t-a)}{2(b-a)}\Big[(t-a)(\max\{|f(a)|^q, |f(t)|^q\})^{\frac{1}{q}} + (b-t)\big(\max\{|f(t)|^q, |f(b)|^q\}\big)^{\frac{1}{q}}\Big]. \tag{4.5}$$

provided that $|f|^q$ is quasi-convex.

Remark 7. *The bound in* (4.3) *and* (4.5) *are both better than the one in* (3.2) *given in* [14]*, obviously.*

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