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APPLICATIONS OF KATO'S INEQUALITY TO OPERATOR-VALUED INTEGRALS ON HILBERT SPACES

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Abstract. By the use of the celebrated Kato's inequality we obtain in this paper some inequalities for operator-valued integrals on a complex Hilbert space H. Among others, we show that

$$\int_E p(t) |\langle V_t x, y\rangle| \, d\mu(t)$$
$$\leq \left\langle \left(\int_E p(t) |V|_t^{2\alpha} d\mu(t) \right) x, x \right\rangle^{1/2} \left\langle \left(\int_E p(t) |V^*|_t^{2(1-\alpha)} d\mu(t) \right) y, y \right\rangle^{1/2}$$

for any $x, y \in H$, provided $V_{(\cdot)} : E \to \mathcal{B}(H)$ and $p : E \to [0, \infty)$ are μ-measurable functions on E and such that $p |V|_{(\cdot)}^{2\alpha}$ and $p |V^*|_{(\cdot)}^{2(1-\alpha)}$ are Bochner integrable on E for some $\alpha \in [0, 1]$.

Natural applications for various norms and numerical radii associated with the Bochner integral of operator-valued functions and some examples for the operator exponential are presented as well.

1. Introduction

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$.

In 1952, Kato [16] proved the following celebrated generalization of Schwarz inequality for any bounded linear operator T on H:

$$|\langle Tx, y\rangle|^2 \leq \langle (T^*T)^\alpha x, x\rangle \left\langle (TT^*)^{1-\alpha} y, y \right\rangle, \tag{1.1}$$

for any $x, y \in H$, $\alpha \in [0, 1]$. Utilizing the modulus notation, we can write (1.1) as follows

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2\alpha} x, x \right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y \right\rangle \tag{1.2}$$

for any $x, y \in H$, $\alpha \in [0, 1]$.

It is useful to observe that, if $T = N$, a normal operator, i.e., we recall that $NN^* = N^*N$, then the inequality (1.2) can be written as

$$|\langle Nx, y\rangle|^2 \leq \left\langle |N|^{2\alpha} x, x \right\rangle \left\langle |N|^{2(1-\alpha)} y, y \right\rangle, \tag{1.3}$$

and in particular, for selfadjoint operators A we can state it as

$$|\langle Ax, y\rangle| \leq \left\| |A|^\alpha x \right\| \left\| |A|^{1-\alpha} y \right\| \tag{1.4}$$

for any $x, y \in H$, $\alpha \in [0, 1]$.

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If $T=U$, a unitary operator, i.e., we recall that $UU^{*}=U^{*}U=1_{H}$, then the inequality (1.2) becomes

$$|\langle Ux,y\rangle|\leq\|x\|\,\|y\|$$

for any $x,y\in H$, which provides a natural generalization for the Schwarz inequality in H.

The symmetric powers in the inequalities above are natural to be considered, so if we choose in (1.2), (1.3) and in (1.4) $\alpha=1/2$ then we get for any $x,y\in H$

$$|\langle Tx,y\rangle|^{2}\leq\langle|T|\,x,x\rangle\,\langle|T^{*}|\,y,y\rangle\,, \tag{1.5}$$

$$|\langle Nx,y\rangle|^{2}\leq\langle|N|\,x,x\rangle\,\langle|N|\,y,y\rangle\,, \tag{1.6}$$

and

$$|\langle Ax,y\rangle|\leq\left\||A|^{1/2}\,x\right\|\left\||A|^{1/2}\,y\right\| \tag{1.7}$$

respectively.

It is also worthwhile to observe that, if we take the supremum over $y\in H$, $\|y\|=1$ in (1.2) then we get

$$\|Tx\|^{2}\leq\|T\|^{2(1-\alpha)}\left\langle|T|^{2\alpha}\,x,x\right\rangle \tag{1.8}$$

for any $x\in H$, or in an equivalent form

$$\|Tx\|\leq\||T|^{\alpha}\,x\|\,\|T\|^{1-\alpha} \tag{1.9}$$

for any $x\in H$.

If we take $\alpha=1/2$ in (1.8), then we get

$$\|Tx\|^{2}\leq\|T\|\,\langle|T|\,x,x\rangle \tag{1.10}$$

for any $x\in H$.

For various interesting generalizations, extension and Kato related results, see the papers [3]-[12], [19]-[23] and [26].

By the use of the celebrated Kato's inequality we obtain in this paper some inequalities for operator-valued integrals on a complex Hilbert space H. Natural applications for various norms and numerical radii associated with the Bochner integral of operator-valued functions and some examples for the operator exponential are presented as well.

2. Some Facts on Bochner Integral

Let $\mathcal{F}(B;E,\mathcal{A},\mu)$ be the linear space of functions $x(t)$, $t\in E$, with values in a real or complex Banach space B, given on a measurable space $(E,\mathcal{A},\mu)$ endowed with a countably-additive scalar measure μ on a σ-algebra $\mathcal{A}$ of subsets of E.

A function $x_0\in\mathcal{F}$ is called *simple* if can be defined as, see [25]

$$x_{0}(t):=\begin{cases} x_{i}\in B, & t\in A_{i}\in\mathcal{A},\ \mu(A_{i})<\infty, i\in\{1,...,n\} \\ & A_{k}\cap A_{j}=\emptyset, k\neq j, k,j\in\{1,...,n\}\,, \\ 0, & t\in E\setminus\cup_{i=1}^{n}A_{i},\ n\in\mathbb{N}. \end{cases}$$

A function $x\in\mathcal{F}$ is called *strongly measurable* if there exists a sequence $\{x_n\}$ of simple functions with $\|x_n-x\|\to0$ almost-everywhere with respect to the measure μ on E. As a consequence of this, the scalar function $\|x\|$ is $\mathcal{A}$-measurable.

For the simple function $x_0 \in \mathcal{F}$ as above we define the integral by

$$\int_E x_0(t)\, d\mu(t) := \sum_{i=1}^{n} x_i \mu(A_i).$$

A function $x \in \mathcal{F}$ is said to be *Bochner integrable* if it is strongly measurable and if for some approximating sequence $\{x_n\}$ of simple functions we have

$$\lim_{n\to\infty} \int_E \|x(t) - x_n(t)\|\, d\mu(t) = 0.$$

The *Bochner integral* of such a function over a set $A \in \mathcal{A}$ is defined as

$$\int_A x(t)\, d\mu(t) = \lim_{n\to\infty} \int_E \chi_A(t)\, x_n(t)\, d\mu(t),$$

where χ_A is the *characteristic function* of A, and the limit is understood in the sense of strong convergence in the Banach space E. This limit exists, and is independent of the choice of the approximation sequence of simple functions.

It is well-known that, for a strongly-measurable function to be Bochner integrable it is necessary and sufficient for the norm of this function to be integrable, i.e.

$$\int_A \|x(t)\|\, d\mu(t) < \infty.$$

The set of Bochner-integrable functions forms a vector subspace $\mathcal{L}(B; E, \mathcal{A}, \mu)$ of $\mathcal{F}(B; E, \mathcal{A}, \mu)$, and the Bochner integral is a linear operator on this subspace.

Some fundamental properties of Bochner integrals are as follows [25], see also [27], [2], [14], [15] and [28]:

(1) For any $x \in \mathcal{L}(B; E, \mathcal{A}, \mu)$ we have the norm inequality

$$\left\| \int_A x(t)\, d\mu(t) \right\| \le \int_A \|x(t)\|\, d\mu(t).$$

(2) Bochner integral is a countably-additive μ-absolutely continuous set-function on the σ-algebra $\mathcal{A}$, i.e.

$$\int_{\cup_{i=1}^{\infty} A_i} x(t)\, d\mu(t) = \sum_{i=1}^{\infty} \int_{A_i} x(t)\, d\mu(t)$$

if $A_i \in \mathcal{A}$, $\mu(A_i) < \infty, i \in \mathbb{N}$, $A_k \cap A_j = \emptyset, k \ne j, k, j \in \mathbb{N}$, and

$$\left\| \int_A x(t)\, d\mu(t) \right\| \to 0 \text{ if } \mu(A) \to 0,$$

uniformly over $A \in \mathcal{A}$.

(3) If $x_n \in F, x_n \to x$ almost-everywhere with respect to the measure μ on $A \in \mathcal{A}$, if $\|x_n\| \le f$ almost-everywhere with respect to μ on A, and if $\int_A f(t)\, d\mu(t) < \infty$, then $x \in \mathcal{L}(B; E, \mathcal{A}, \mu)$ and

$$\int_A x_n(t)\, d\mu(t) \to \int_A x(t)\, d\mu(t).$$

(4) The space is complete with respect to the norm

$$\|x\| := \int_A \|x(t)\|\, d\mu(t).$$

(5) If T is a closed linear operator from a Banach space X into a Banach space Y and if $x \in \mathcal{L}(X; E, \mathcal{A}, \mu)$ and $Tx \in \mathcal{L}(Y; E, \mathcal{A}, \mu)$, then

$$\int_A Tx(t)\, d\mu(t) = T\left(\int_A x(t)\, d\mu(t)\right).$$

If T is bounded, the condition $Tx \in \mathcal{L}(Y; E, \mathcal{A}, \mu)$ is automatically satisfied.

3. Applications of Kato's Inequality

In this section we consider a measurable space $(E, \mathcal{A}, \mu)$ and operator-valued μ-measurable functions $E \ni t \longmapsto V_t \in \mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. We can define the adjoint function by $V_t^* := (V_t)^*$ and the modulus function by $|V|_t := \sqrt{(V_t)^* V_t}$, $t \in E$.

Theorem 1. *Let $V_{(\cdot)} : E \to \mathcal{B}(H)$ and $p : E \to [0, \infty)$ be μ-measurable functions on E and such that $p|V|_{(\cdot)}^2$ and $p|V^*|_{(\cdot)}^2$ are Bochner integrable on E. Then we have the inequality*

$$\begin{aligned} &\int_E p(t) |\langle V_t x, y\rangle|^2 d\mu(t) \\ &\leq \left\langle \left(\int_E p(t) |V|_t^2 d\mu(t)\right) x, x \right\rangle^{\alpha} \left\langle \left(\int_E p(t) |V^*|_t^2 d\mu(t)\right) y, y \right\rangle^{1-\alpha} \end{aligned} \tag{3.1}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0, 1]$.

Proof. Let $t \in E$. If we write Kato's inequality for the operator V_t we have

$$|\langle V_t x, y\rangle|^2 \leq \left\langle |V|_t^{2\alpha} x, x \right\rangle \left\langle |V^*|_t^{2(1-\alpha)} y, y \right\rangle \tag{3.2}$$

for any $x, y \in H$.

By Hölder-McCarthy inequality $\langle P^r x, x\rangle \leq \langle Px, x\rangle^r$ that holds for any positive operator P, for any $x \in H$ with $\|x\| = 1$ and any power $r \in (0, 1)$ we have

$$\left\langle |V|_t^{2\alpha} x, x \right\rangle \leq \left\langle |V_t|^2 x, x \right\rangle^{\alpha} \tag{3.3}$$

and

$$\left\langle |V^*|_t^{2(1-\alpha)} y, y \right\rangle \leq \left\langle |V^*|_t^2 y, y \right\rangle^{1-\alpha} \tag{3.4}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

On making use of (3.2)-(3.4) we get

$$|\langle V_t x, y\rangle|^2 \leq \left\langle |V|_t^2 x, x \right\rangle^{\alpha} \left\langle |V^*|_t^2 y, y \right\rangle^{1-\alpha} \tag{3.5}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $t \in E$.

Now, if we multiply the inequality (3.5) by $p(t) \geq 0$, integrate over $d\mu(\cdot)$ on E and use the scalar weighted Hölder inequality we have

$$\begin{aligned}
&\int_E p(t) \left|\langle V_t x, y\rangle\right|^2 d\mu(t) \\
&\leq \int_E p(t) \left\langle |V|_t^2 x, x\right\rangle^{\alpha} \left\langle |V^*|_t^2 y, y\right\rangle^{1-\alpha} d\mu(t) \\
&\leq \left(\int_E p(t) \left[\left\langle |V|_t^2 x, x\right\rangle^{\alpha}\right]^{1/\alpha} d\mu(t)\right)^{\alpha} \\
&\times \left(\int_E p(t) \left[\left\langle |V^*|_t^2 y, y\right\rangle^{1-\alpha}\right]^{1/(1-\alpha)} d\mu(t)\right)^{1-\alpha} \\
&= \left(\int_E p(t) \left\langle |V|_t^2 x, x\right\rangle d\mu(t)\right)^{\alpha} \left(\int_E p(t) \left\langle |V^*|_t^2 y, y\right\rangle d\mu(t)\right)^{1-\alpha} \\
&= \left\langle \left(\int_E p(t) |V|_t^2 d\mu(t)\right) x, x\right\rangle^{\alpha} \left\langle \left(\int_E p(t) |V^*|_t^2 d\mu(t)\right) y, y\right\rangle^{1-\alpha}
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, and the proof is complete. □

Remark 1. *The inequality (3.1) becomes for $y = x$ the following simpler result that is useful for deriving numerical radius inequalities:*

$$\begin{aligned}
&\int_E p(t) \left|\langle V_t x, x\rangle\right|^2 d\mu(t) \\
&\leq \left\langle \left(\int_E p(t) |V|_t^2 d\mu(t)\right) x, x\right\rangle^{\alpha} \left\langle \left(\int_E p(t) |V^*|_t^2 d\mu(t)\right) x, x\right\rangle^{1-\alpha} \\
&\leq \left\langle \left(\int_E p(t) \left[\alpha |V|_t^2 + (1-\alpha) |V^*|_t^2\right] d\mu(t)\right) x, x\right\rangle
\end{aligned} \tag{3.6}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 2. *In addition to the assumptions of Theorem 1, if the values of the function $V_{(\cdot)}$ are normal operators for μ-almost every (a.e.) $t \in E$, i.e., $|V|_t^2 = |V^*|_t^2$ for μ-a.e. $t \in E$ we have*

$$\begin{aligned}
&\int_E p(t) \left|\langle V_t x, y\rangle\right|^2 d\mu(t) \\
&\leq \left\langle \left(\int_E p(t) |V|_t^2 d\mu(t)\right) x, x\right\rangle^{\alpha} \left\langle \left(\int_E p(t) |V|_t^2 d\mu(t)\right) y, y\right\rangle^{1-\alpha}
\end{aligned} \tag{3.7}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

This inequality implies the simpler result

$$\int_E p(t) \left|\langle V_t x, x\rangle\right|^2 d\mu(t) \leq \left\langle \left(\int_E p(t) |V|_t^2 d\mu(t)\right) x, x\right\rangle \tag{3.8}$$

for any $x \in H$ with $\|x\| = 1$.

From a different perspective, we can state the following result as well:

Theorem 2. *Let $V_{(\cdot)} : E \to \mathcal{B}(H)$ and $p : E \to [0, \infty)$ be μ-measurable functions on E and such that $p |V|_{(\cdot)}^{2\alpha}$ and $p |V^*|_{(\cdot)}^{2(1-\alpha)}$ are Bochner integrable on E for some*

$\alpha \in [0,1]$. Then we have the inequality

$$\int_E p(t)|\langle V_t x, y\rangle| \, d\mu(t) \le \left\langle \left(\int_E p(t) |V|_t^{2\alpha} \, d\mu(t) \right) x, x \right\rangle^{1/2} \left\langle \left(\int_E p(t) |V^*|_t^{2(1-\alpha)} \, d\mu(t) \right) y, y \right\rangle^{1/2} \tag{3.9}$$

for any $x, y \in H$.

In particular, we have

$$\begin{aligned} \int_E p(t)|\langle V_t x, x\rangle| \, d\mu(t) &\le \left\langle \left(\int_E p(t) |V|_t^{2\alpha} \, d\mu(t) \right) x, x \right\rangle^{1/2} \left\langle \left(\int_E p(t) |V^*|_t^{2(1-\alpha)} \, d\mu(t) \right) x, x \right\rangle^{1/2} \\ &\le \frac{1}{2} \left\langle \left(\int_E p(t) \left[|V|_t^{2\alpha} + |V^*|_t^{2(1-\alpha)} \right] d\mu(t) \right) x, x \right\rangle \end{aligned} \tag{3.10}$$

for any $x \in H$.

Proof. Let $t \in E$. If we write Kato's inequality for the operator V_t we have

$$|\langle V_t x, y\rangle| \le \left\langle |V|_t^{2\alpha} x, x \right\rangle^{1/2} \left\langle |V^*|_t^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{3.11}$$

for any $x, y \in H$.

Now, by multiplying the inequality (3.11) with $p(t) \ge 0$, integrate over $d\mu(\cdot)$ on E and use the weighted Cauchy-Bunyakovsky-Schwarz integral inequality we get

$$\begin{aligned} &\int_E p(t)|\langle V_t x, y\rangle| \, d\mu(t) \\ \le\ &\int_E p(t) \left\langle |V|_t^{2\alpha} x, x \right\rangle^{1/2} \left\langle |V^*|_t^{2(1-\alpha)} y, y \right\rangle^{1/2} d\mu(t) \\ \le\ &\left(\int_E p(t) \left\langle |V|_t^{2\alpha} x, x \right\rangle d\mu(t) \right)^{1/2} \left(\int_E p(t) \left\langle |V^*|_t^{2(1-\alpha)} y, y \right\rangle d\mu(t) \right)^{1/2} \\ =\ &\left\langle \left(\int_E p(t) |V|_t^{2\alpha} \, d\mu(t) \right) x, x \right\rangle^{1/2} \left\langle \left(\int_E p(t) |V^*|_t^{2(1-\alpha)} \, d\mu(t) \right) y, y \right\rangle^{1/2} \end{aligned}$$

for any $x, y \in H$.

The second inequality in (3.10) follows by the arithmetic mean-geometric mean inequality.

The proof is complete. □

Remark 3. *The symmetric case for powers, namely the case $\alpha = \frac{1}{2}$ in (3.9) is of interest since will produce the simpler result*

$$\int_E p(t)|\langle V_t x, y\rangle| \, d\mu(t) \le \left\langle \left(\int_E p(t) |V|_t \, d\mu(t) \right) x, x \right\rangle^{1/2} \left\langle \left(\int_E p(t) |V^*|_t \, d\mu(t) \right) y, y \right\rangle^{1/2} \tag{3.12}$$

for any $x, y \in H$ and provided that $p|V|_{(\cdot)}$ and $p|V^*|_{(\cdot)}$ are Bochner integrable on E.

If in this inequality we take $y = x$, then we get

$$\int_E p(t) |\langle V_t x, x\rangle| \, d\mu(t) \le \left\langle \left(\int_E p(t) |V|_t \, d\mu(t) \right) x, x \right\rangle^{1/2} \left\langle \left(\int_E p(t) |V^*|_t \, d\mu(t) \right) x, x \right\rangle^{1/2} \tag{3.13}$$

for any $x \in H$.

Moreover, if the values of the function $V_{(\cdot)}$ are normal operators for μ-a.e. $t \in E$, then the inequality (3.12) becomes

$$\int_E p(t) |\langle V_t x, y\rangle| \, d\mu(t) \le \left\langle \left(\int_E p(t) |V|_t \, d\mu(t) \right) x, x \right\rangle^{1/2} \left\langle \left(\int_E p(t) |V|_t \, d\mu(t) \right) y, y \right\rangle^{1/2} \tag{3.14}$$

for any $x, y \in H$, while the inequality (3.13) becomes

$$\int_E p(t) |\langle V_t x, x\rangle| \, d\mu(t) \le \left\langle \left(\int_E p(t) |V|_t \, d\mu(t) \right) x, x \right\rangle \tag{3.15}$$

for any $x \in H$.

4. Norm and Numerical Radius Inequalities

Let $p : E \to [0, \infty)$ be a μ-measurable function on E and such that $\int_E p(t) \, d\mu(t) = 1$. For $V_{(\cdot)} : E \to \mathcal{B}(H)$ a μ-measurable function on E and such that $p |V|_{(\cdot)}^2$ is Bochner integrable on E, we define the *s-2-p-semi-norm* by

$$\left\| V_{(\cdot)} \right\|_{s,p,2} := \sup_{\|x\|=\|y\|=1} \left(\int_E p(t) |\langle V_t x, y\rangle|^2 \, d\mu(t) \right)^{1/2}$$

and the *s-2-p-semi-numerical radius* by

$$w_{s,p,2}\left(V_{(\cdot)}\right) := \sup_{\|x\|=1} \left(\int_E p(t) |\langle V_t x, x\rangle|^2 \, d\mu(t) \right)^{1/2}.$$

If we consider the Banach space $\mathcal{L}_{2,p}(E, \mathcal{B}(H), \mu)$ of all functions $V_{(\cdot)} : E \to \mathcal{B}(H)$ that are μ-measurable on E and such that

$$\left\| V_{(\cdot)} \right\|_{p,2} := \left(\int_E p(t) \|V_t\|^2 \, d\mu(t) \right)^{1/2} < \infty,$$

we observe that $\|\cdot\|_{p,2}$ and $w(\cdot)_{p,2}$ defined on $\mathcal{L}_{2,p}(E, \mathcal{B}(H), \mu)$ are nonnegative, absolute homogeneous and satisfy the triangle inequality on this space.

If we consider the norm on $\mathcal{L}_{2,p}(E, \mathcal{B}(H), \mu)$ induced by the numerical radius on $\mathcal{B}(H)$, i.e.

$$w_{p,2}\left(V_{(\cdot)}\right) := \left(\int_E p(t) w^2(V_t) \, d\mu(t) \right)^{1/2}$$

then by taking into account the well known numerical radius-norm inequalities

$$\frac{1}{2} \|T\| \le w(T) \le \|T\|, \; T \in \mathcal{B}(H) \tag{4.1}$$

we observe that the norms $\|\cdot\|_{p,2}$ and $w_{p,2}(\cdot)$ will preserve the inequalities (4.1).

Utilising the properties of the supremum, we also observe that

$$\left\|V_{(\cdot)}\right\|_{s,p,2} \leq \left\|V_{(\cdot)}\right\|_{p,2} \text{ and } w_{s,p,2}\left(V_{(\cdot)}\right) \leq w_{p,2}\left(V_{(\cdot)}\right) \tag{4.2}$$

for any $V_{(\cdot)} \in \mathcal{L}_{2,p}\left(E, \mathcal{B}\left(H\right), \mu\right)$.

We have the following result.

Theorem 3. *For any $V_{(\cdot)} \in \mathcal{L}_{2,p}\left(E, \mathcal{B}\left(H\right), \mu\right)$ and $\alpha \in [0,1]$ we have the inequalities*

$$\begin{aligned} \left\|\int_E p\left(t\right) V_t d\mu\left(t\right)\right\|^2 &\leq \left\|V_{(\cdot)}\right\|_{s,p,2}^2 \\ &\leq \left\|\int_E p\left(t\right)\left|V\right|_t^2 d\mu\left(t\right)\right\|^{\alpha}\left\|\int_E p\left(t\right)\left|V^*\right|_t^2 d\mu\left(t\right)\right\|^{1-\alpha} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} & w^2\left(\int_E p\left(t\right) V_t d\mu\left(t\right)\right) \\ & \leq w_{s,p,2}^2\left(V_{(\cdot)}\right) \\ & \leq \begin{cases} \left\|\int_E p\left(t\right)\left|V\right|_t^2 d\mu\left(t\right)\right\|^{\alpha}\left\|\int_E p\left(t\right)\left|V^*\right|_t^2 d\mu\left(t\right)\right\|^{1-\alpha}, \\ \\ \left\|\int_E p\left(t\right)\left[\alpha\left|V\right|_t^2 + \left(1-\alpha\right)\left|V^*\right|_t^2\right] d\mu\left(t\right)\right\|. \end{cases} \end{aligned} \tag{4.4}$$

Proof. By the Cauchy-Bunyakovsky-Schwarz integral inequality and the inequality (3.1) we have

$$\begin{aligned} \left|\left\langle\left(\int_E p\left(t\right) V_t d\mu\left(t\right)\right) x, y\right\rangle\right|^2 &\leq \int_E p\left(t\right)\left|\left\langle V_t x, y\right\rangle\right|^2 d\mu\left(t\right) \\ \leq \left\langle\left(\int_E p\left(t\right)\left|V\right|_t^2 d\mu\left(t\right)\right) x, x\right\rangle^{\alpha} & \left\langle\left(\int_E p\left(t\right)\left|V^*\right|_t^2 d\mu\left(t\right)\right) y, y\right\rangle^{1-\alpha} \end{aligned} \tag{4.5}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Taking the supremum over $\|x\| = \|y\| = 1$ we have

$$\begin{aligned} & \left[\sup_{\|x\|=\|y\|=1}\left|\left\langle\left(\int_E p\left(t\right) V_t d\mu\left(t\right)\right) x, y\right\rangle\right|\right]^2 \\ & \leq \sup_{\|x\|=\|y\|=1} \int_E p\left(t\right)\left|\left\langle V_t x, y\right\rangle\right|^2 d\mu\left(t\right) \\ & \leq \left[\sup_{\|x\|=1}\left\langle\left(\int_E p\left(t\right)\left|V\right|_t^2 d\mu\left(t\right)\right) x, x\right\rangle\right]^{\alpha} \\ & \times \left[\sup_{\|y\|=1}\left\langle\left(\int_E p\left(t\right)\left|V^*\right|_t^2 d\mu\left(t\right)\right) y, y\right\rangle\right]^{1-\alpha} \end{aligned} \tag{4.6}$$

and since

$$\sup_{\|x\|=\|y\|=1}\left|\left\langle\left(\int_E p(t)\,V_t d\mu(t)\right)x,y\right\rangle\right|=\left\|\int_E p(t)\,V_t d\mu(t)\right\|,$$

$$\sup_{\|x\|=\|y\|=1}\int_E p(t)\,|\langle V_t x,y\rangle|^2\,d\mu(t)=\left\|V_{(\cdot)}\right\|_{s,p,2}^2,$$

and

$$\begin{aligned}\sup_{\|x\|=1}\left\langle\left(\int_E p(t)\,|V|_t^2\,d\mu(t)\right)x,x\right\rangle &= w\left(\int_E p(t)\,|V|_t^2\,d\mu(t)\right)\\ &=\left\|\int_E p(t)\,|V|_t^2\,d\mu(t)\right\|\end{aligned}$$

while

$$\begin{aligned}\sup_{\|y\|=1}\left\langle\left(\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right)y,y\right\rangle &= w\left(\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right)\\ &=\left\|\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right\|\end{aligned}$$

since the operators $\int_E p(t)\,|V|_t^2\,d\mu(t)$ and $\int_E p(t)\,|V^*|_t^2\,d\mu(t)$ are selfadjoint, then we deduce from (4.6) the desired result (4.3).

From the inequality (3.6) we also have

$$\begin{aligned}&\left|\left\langle\int_E p(t)\,V_t x,x\right\rangle\right|^2 d\mu(t)\\ &\le\int_E p(t)\,|\langle V_t x,x\rangle|^2\,d\mu(t)\\ &\le\begin{cases}\left\langle\left(\int_E p(t)\,|V|_t^2\,d\mu(t)\right)x,x\right\rangle^{\alpha}\left\langle\left(\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right)x,x\right\rangle^{1-\alpha},\\ \left\langle\left(\int_E p(t)\left[\alpha\,|V|_t^2+(1-\alpha)\,|V^*|_t^2\right]d\mu(t)\right)x,x\right\rangle\end{cases}\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

Taking the supremum over $\|x\|=1$ we deduce the desired inequality (4.4). The details are omitted. □

Remark 4. *Since by the integral triangle inequality for the norm we have*

$$\begin{aligned}\left\|\int_E p(t)\,|V|_t^2\,d\mu(t)\right\| &\le \int_E p(t)\left\||V|_t^2\right\|d\mu(t)\\ &= \int_E p(t)\,\|V_t\|^2\,d\mu(t)=\left\|V_{(\cdot)}\right\|_{p,2}^2\end{aligned}$$

and

$$\begin{aligned}\left\|\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right\| &\le \int_E p(t)\left\||V^*|_t^2\right\|d\mu(t)\\ &= \int_E p(t)\,\|V_t\|^2\,d\mu(t)=\left\|V_{(\cdot)}\right\|_{p,2}^2,\end{aligned}$$

then we have from (4.3) the following sequence of inequalities

$$\left\|\int_E p(t)\,V_t d\mu(t)\right\|^2 \le \left\|V_{(\cdot)}\right\|_{s,p,2}^2 \le \left\|\int_E p(t)\,|V|_t^2\,d\mu(t)\right\|^{\alpha}\left\|\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right\|^{1-\alpha} \le \alpha\left\|\int_E p(t)\,|V|_t^2\,d\mu(t)\right\| + (1-\alpha)\left\|\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right\| \le \left\|V_{(\cdot)}\right\|_{p,2}^2 \tag{4.7}$$

for any $V_{(\cdot)} \in \mathcal{L}_{2,p}(E,\mathcal{B}(H),\mu)$ *and* $\alpha \in [0,1]$.

From (4.4) we also have

$$w^2\left(\int_E p(t)\,V_t d\mu(t)\right) \le w_{s,p,2}^2\left(V_{(\cdot)}\right) \le \begin{cases} \left\|\int_E p(t)\,|V|_t^2\,d\mu(t)\right\|^{\alpha}\left\|\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right\|^{1-\alpha} \\ \left\|\int_E p(t)\left[\alpha\,|V|_t^2 + (1-\alpha)\,|V^*|_t^2\right]d\mu(t)\right\| \end{cases} \le \alpha\left\|\int_E p(t)\,|V|_t^2\,d\mu(t)\right\| + (1-\alpha)\left\|\int_E p(t)\,|V^*|_t^2\,d\mu(t)\right\| \le \left\|V_{(\cdot)}\right\|_{p,2}^2 \tag{4.8}$$

for any $V_{(\cdot)} \in \mathcal{L}_{2,p}(E,\mathcal{B}(H),\mu)$ *and* $\alpha \in [0,1]$.

Now, we can consider the following Banach space $\mathcal{L}_{1,p}(E,\mathcal{B}(H),\mu)$ of all functions $V_{(\cdot)} : E \to \mathcal{B}(H)$ that are μ-measurable on E and such that

$$\left\|V_{(\cdot)}\right\|_{p,1} := \int_E p(t)\,\|V_t\|\,d\mu(t) < \infty.$$

We can consider in this space the following numerical radius

$$w_{p,1}\left(V_{(\cdot)}\right) := \int_E p(t)\,w(V_t)\,d\mu(t)$$

and taking into account the inequality (4.1) we can state that

$$\frac{1}{2}\left\|V_{(\cdot)}\right\|_{p,1} \le w_{p,1}\left(V_{(\cdot)}\right) \le \left\|V_{(\cdot)}\right\|_{p,1} \tag{4.9}$$

for any $V_{(\cdot)} \in \mathcal{L}_{1,p}(E,\mathcal{B}(H),\mu)$.

We can define the s-1-p-*semi-norm* by

$$\left\|V_{(\cdot)}\right\|_{s,p,1} := \sup_{\|x\|=\|y\|=1}\left(\int_E p(t)\,|\langle V_t x, y\rangle|\,d\mu(t)\right)$$

and the s-1-p-*semi-numerical radius* by

$$w_{s,p,1}\left(V_{(\cdot)}\right) := \sup_{\|x\|=1}\left(\int_E p(t)\left|\langle V_t x, x\rangle\right| d\mu(t)\right),$$

where $V_{(\cdot)} \in \mathcal{L}_{1,p}(E, \mathcal{B}(H), \mu)$.

Utilising the supremum properties we also have

$$\left\|V_{(\cdot)}\right\|_{s,p,1} \leq \left\|V_{(\cdot)}\right\|_{p,1} \text{ and } w_{s,p,1}\left(V_{(\cdot)}\right) \leq w_{p,1}\left(V_{(\cdot)}\right) \tag{4.10}$$

for any $V_{(\cdot)} \in \mathcal{L}_{1,p}(E, \mathcal{B}(H), \mu)$.

More related results are incorporated in the following theorem.

Theorem 4. *If $|V|_{(\cdot)}^{2\alpha}$ and $|V^*|_{(\cdot)}^{2(1-\alpha)}$ belong to $\mathcal{L}_{1,p}(E, \mathcal{B}(H), \mu)$ for some $\alpha \in [0,1]$, then we have*

$$\begin{aligned}\left\|\int_E p(t) V_t d\mu(t)\right\| &\leq \left\|V_{(\cdot)}\right\|_{s,p,1} \\ &\leq \left\|\int_E p(t)|V|_t^{2\alpha} d\mu(t)\right\|^{1/2}\left\|\int_E p(t)|V^*|_t^{2(1-\alpha)} d\mu(t)\right\|^{1/2}\end{aligned} \tag{4.11}$$

and

$$\begin{aligned} w\left(\int_E p(t) V_t d\mu(t)\right) &\leq w_{s,p,1}\left(V_{(\cdot)}\right) \\ &\leq \begin{cases}\left\|\int_E p(t)|V|_t^{2\alpha} d\mu(t)\right\|^{1/2}\left\|\int_E p(t)|V^*|_t^{2(1-\alpha)} d\mu(t)\right\|^{1/2}, \\ \frac{1}{2}\left\|\int_E p(t)\left[|V|_t^{2\alpha}+|V^*|_t^{2(1-\alpha)}\right] d\mu(t)\right\|.\end{cases}\end{aligned} \tag{4.12}$$

Proof. By the modulus properties and the inequality (3.9) we have

$$\begin{aligned}&\left|\left\langle\left(\int_E p(t) V_t d\mu(t)\right) x, y\right\rangle\right| \\ &\leq \int_E p(t)\left|\langle V_t x, y\rangle\right| d\mu(t) \\ &\leq \left\langle\left(\int_E p(t)|V|_t^{2\alpha} d\mu(t)\right) x, x\right\rangle^{1/2}\left\langle\left(\int_E p(t)|V^*|_t^{2(1-\alpha)} d\mu(t)\right) y, y\right\rangle^{1/2},\end{aligned}$$

for any $x, y \in H$.

Taking the supremum over $\|x\| = \|y\| = 1$ we deduce (4.11).

The second inequality follows by (3.10) and the details are omitted. □

Remark 5. *Since by the integral triangle inequality for the norm we have*

$$\left\|\int_E p(t)|V|_t^{2\alpha} d\mu(t)\right\| \leq \int_E p(t)\left\||V|_t^{2\alpha}\right\| d\mu(t) = \int_E p(t)\|V_t\|^{2\alpha} d\mu(t)$$

and

$$\left\Vert \int_E p(t)\,|V^*|_t^{2(1-\alpha)}\,d\mu(t)\right\Vert \leq \int_E p(t)\left\Vert |V^*|_t^{2(1-\alpha)}\right\Vert d\mu(t) = \int_E p(t)\,\|V_t^*\|^{2(1-\alpha)}\,d\mu(t)$$

then by (4.11) we have the following sequence of inequalities

$$\begin{aligned}(4.13)\quad & \left\Vert \int_E p(t)\,V_t d\mu(t)\right\Vert \\ & \leq \left\Vert V_{(\cdot)}\right\Vert_{s,p,1} \\ & \leq \left\Vert \int_E p(t)\,|V|_t^{2\alpha}\,d\mu(t)\right\Vert^{1/2}\left\Vert \int_E p(t)\,|V^*|_t^{2(1-\alpha)}\,d\mu(t)\right\Vert^{1/2} \\ & \leq \left(\int_E p(t)\,\|V_t\|^{2\alpha}\,d\mu(t)\right)^{1/2}\left(\int_E p(t)\,\|V_t\|^{2(1-\alpha)}\,d\mu(t)\right)^{1/2} \\ & \leq \frac{1}{2}\int_E p(t)\left[\|V_t\|^{2\alpha}+\|V_t\|^{2(1-\alpha)}\right]d\mu(t)\end{aligned}$$

provided that $|V|_{(\cdot)}^{2\alpha}$ *and* $|V^*|_{(\cdot)}^{2(1-\alpha)}$ *belong to* $\mathcal{L}_{1,p}(E,\mathcal{B}(H),\mu)$ *for some* $\alpha\in[0,1]$.

Under the same assumptions we also have

$$\begin{aligned}(4.14)\quad & w\left(\int_E p(t)\,V_t d\mu(t)\right) \\ & \leq w_{s,p,1}\left(V_{(\cdot)}\right) \\ & \leq \begin{cases}\left\Vert \int_E p(t)\,|V|_t^{2\alpha}\,d\mu(t)\right\Vert^{1/2}\left\Vert \int_E p(t)\,|V^*|_t^{2(1-\alpha)}\,d\mu(t)\right\Vert^{1/2} \\ \frac{1}{2}\left\Vert \int_E p(t)\left[|V|_t^{2\alpha}+|V^*|_t^{2(1-\alpha)}\right]d\mu(t)\right\Vert\end{cases} \\ & \leq \begin{cases}\left(\int_E p(t)\,\|V_t\|^{2\alpha}\,d\mu(t)\right)^{1/2}\left(\int_E p(t)\,\|V_t\|^{2(1-\alpha)}\,d\mu(t)\right)^{1/2} \\ \frac{1}{2}\int_E p(t)\left\Vert\left[|V|_t^{2\alpha}+|V^*|_t^{2(1-\alpha)}\right]\right\Vert d\mu(t)\end{cases} \\ & \leq \frac{1}{2}\int_E p(t)\left[\|V_t\|^{2\alpha}+\|V_t\|^{2(1-\alpha)}\right]d\mu(t).\end{aligned}$$

Remark 6. *The case* $\alpha = \frac{1}{2}$ *is of interest since it generates from (4.14) the following inequalities*

$$w_{s,p,1}\left(V_{(\cdot)}\right) \le \begin{cases} \left\|\int_E p(t)\,|V|_t\,d\mu(t)\right\|^{1/2} \left\|\int_E p(t)\,|V^*|_t\,d\mu(t)\right\|^{1/2} \\ \frac{1}{2}\left\|\int_E p(t)\left[|V|_t + |V^*|_t\right]d\mu(t)\right\| \end{cases} \le \begin{cases} \left\|V_{(\cdot)}\right\|_{p,1} \\ \frac{1}{2}\int_E p(t)\left\|\left[|V|_t + |V^*|_t\right]\right\| d\mu(t) \end{cases} \left(\le \left\|V_{(\cdot)}\right\|_{p,1}\right) \tag{4.15}$$

for any $V_{(\cdot)} \in \mathcal{L}_{1,p}(E, \mathcal{B}(H), \mu)$.

From (4.11), we also have for $\alpha = \frac{1}{2}$ *the following refinement of the integral triangle inequality for norm:*

$$\begin{aligned} \left\|\int_E p(t)\,V_t d\mu(t)\right\| &\le \left\|V_{(\cdot)}\right\|_{s,p,1} \\ &\le \left\|\int_E p(t)\,|V|_t\,d\mu(t)\right\|^{1/2} \left\|\int_E p(t)\,|V^*|_t\,d\mu(t)\right\|^{1/2} \\ &\le \frac{1}{2}\left[\left\|\int_E p(t)\,|V|_t\,d\mu(t)\right\| + \left\|\int_E p(t)\,|V^*|_t\,d\mu(t)\right\|\right] \le \left\|V_{(\cdot)}\right\|_{p,1} \end{aligned} \tag{4.16}$$

for any $V_{(\cdot)} \in \mathcal{L}_{1,p}(E, \mathcal{B}(H), \mu)$.

Moreover, if the values of the function $V_{(\cdot)}$ *are normal operators for* μ*-a.e.* $t \in E$, *then the inequality (4.16) becomes*

$$\left\|\int_E p(t)\,V_t d\mu(t)\right\| \le \left\|V_{(\cdot)}\right\|_{s,p,1} \le \left\|\int_E p(t)\,|V|_t\,d\mu(t)\right\| \le \left\|V_{(\cdot)}\right\|_{p,1}. \tag{4.17}$$

5. Applications for the Operator Exponential

It is known that if U and V are commuting operators, then the operator *exponential function* $\exp : \mathcal{B}(H) \to \mathcal{B}(H)$ given by

$$\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if A is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tA)\,dt = A^{-1}\left[\exp(bA) - \exp(aA)\right].$$

We observe that if the values of the function $V_{(\cdot)}$ are normal operators for μ-a.e. $t \in E$, i.e., $|V|_t^2 = |V^*|_t^2$ for μ-a.e. $t \in E$, then we have from (3.7) that

$$\left|\left\langle\left(\int_E p(t) V_t d\mu(t)\right) x, y\right\rangle\right|^2 \leq \left\langle\left(\int_E p(t) |V|_t^2 d\mu(t)\right) x, x\right\rangle^{\alpha} \left\langle\left(\int_E p(t) |V|_t^2 d\mu(t)\right) y, y\right\rangle^{1-\alpha} \tag{5.1}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

If N is a normal operator, then for any real number t we have

$$|\exp(tN)|^2 = (\exp(tN))^* (\exp(tN)) = \exp(tN^*)(\exp(tN)) = \exp[t(N^* + N)].$$

Proposition 1. *Let N be an invertible normal operator and such that $N^* + N$ is also invertible. Then for any $a, b \in \mathbb{R}$ with $a < b$ we have*

$$\begin{aligned} &\left|\left\langle N^{-1}[\exp(bN) - \exp(aN)] x, y\right\rangle\right|^2 \\ &\leq \left\langle (N^* + N)^{-1}[\exp(b(N^* + N)) - \exp(a(N^* + N))] x, x\right\rangle^{\alpha} \\ &\times \left\langle (N^* + N)^{-1}[\exp(b(N^* + N)) - \exp(a(N^* + N))] y, y\right\rangle^{1-\alpha} \end{aligned} \tag{5.2}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Proof. Follows from (5.1) applied for $E = [a, b]$, $p(t) = \frac{1}{b-a}$, $V_t = \exp(tN)$ and $d\mu(t) = dt$, the usual Lebesgue measure. □

If S is an invertible selfadjoint operator, then from (5.2) we get for any $a, b \in \mathbb{R}$ with $a < b$ that

$$\begin{aligned} &\left|\left\langle S^{-1}[\exp(bS) - \exp(aS)] x, y\right\rangle\right|^2 \\ &\leq \frac{1}{2}\left\langle S^{-1}[\exp(2bS) - \exp(2aS)] x, x\right\rangle^{\alpha} \\ &\times \left\langle S^{-1}[\exp(2bS) - \exp(2aS)] y, y\right\rangle^{1-\alpha} \end{aligned} \tag{5.3}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

In particular, we have from (5.3) that

$$\left|\left\langle S^{-1}[\exp(S) - I] x, y\right\rangle\right|^2 \leq \frac{1}{2}\left\langle S^{-1}[\exp(2S) - I] x, x\right\rangle^{\alpha} \left\langle S^{-1}[\exp(2S) - I] y, y\right\rangle^{1-\alpha} \tag{5.4}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

If we use the inequality (3.9) for functions $V_{(\cdot)}$ that are normal operators for μ-a.e. $t \in E$, then we have

$$\left|\left\langle\left(\int_E p(t) V_t d\mu(t)\right) x, y\right\rangle\right| \leq \left\langle\left(\int_E p(t) |V|_t^{2\alpha} d\mu(t)\right) x, x\right\rangle^{1/2} \left\langle\left(\int_E p(t) |V|_t^{2(1-\alpha)} d\mu(t)\right) y, y\right\rangle^{1/2} \tag{5.5}$$

for any $x, y \in H$ and $\alpha \in [0,1]$.

Utilising the inequality (5.5) we can state:

Proposition 2. *With the assumptions of Proposition 1, we have*

$$\left|\left\langle N^{-1}\left[\exp\left(bN\right)-\exp\left(aN\right)\right]x,y\right\rangle\right| \tag{5.6}$$
$$\leq\left\langle \alpha^{-1}\left(N^{*}+N\right)^{-1}\left[\exp\left(b\alpha\left(N^{*}+N\right)\right)-\exp\left(a\alpha\left(N^{*}+N\right)\right)\right]x,x\right\rangle^{1/2}$$
$$\times\left\langle \left(1-\alpha\right)^{-1}\left(N^{*}+N\right)^{-1}\right.$$
$$\times\left.\left[\exp\left(b\left(1-\alpha\right)\left(N^{*}+N\right)\right)-\exp\left(a\left(1-\alpha\right)\left(N^{*}+N\right)\right)\right]y,y\right\rangle^{1/2}$$

for any $x,y\in H$ *and* $\alpha\in(0,1)$.

If S is an invertible selfadjoint operator, then from (5.6) we get for any $a,b\in\mathbb{R}$ with $a<b$ that

$$\left|\left\langle S^{-1}\left[\exp\left(bS\right)-\exp\left(aS\right)\right]x,y\right\rangle\right| \tag{5.7}$$
$$\leq\frac{1}{2\sqrt{\alpha\left(1-\alpha\right)}}\left\langle S^{-1}\left[\exp\left(2b\alpha S\right)-\exp\left(2a\alpha S\right)\right]x,x\right\rangle^{1/2}$$
$$\times\left\langle S^{-1}\left[\exp\left(2b\left(1-\alpha\right)S\right)-\exp\left(2a\left(1-\alpha\right)S\right)\right]y,y\right\rangle^{1/2}$$

for any $x,y\in H$ and $\alpha\in(0,1)$.

In particular, we have from (5.7) that

$$\left|\left\langle S^{-1}\left[\exp\left(S\right)-I\right]x,y\right\rangle\right| \tag{5.8}$$
$$\leq\frac{1}{2\sqrt{\alpha\left(1-\alpha\right)}}\left\langle S^{-1}\left[\exp\left(2\alpha S\right)-I\right]x,x\right\rangle^{1/2}$$
$$\times\left\langle S^{-1}\left[\exp\left(2\left(1-\alpha\right)S\right)-I\right]y,y\right\rangle^{1/2}$$

for any $x,y\in H$ and $\alpha\in(0,1)$.

The interested reader may state the corresponding norm inequalities. However the details are omitted here.

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