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INEQUALITIES OF LIPSCHITZ TYPE FOR POWER SERIES OF OPERATORS IN HILBERT SPACES

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Abstract. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R), R>0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then we show among others that

$$\|f(T) - f(V)\| \le f_a'(\max\{\|T\|, \|V\|\}) \|T - V\|$$

where $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$.

If $U, V \in \mathcal{B}(H)$ are such that the numerical radii $w(T), w(V) < R$ then

$$w[f(T) - f(V)] \le 4 f_a'(\max\{w(T), w(V)\}) w(T - V).$$

Applications in connection with the Hermite-Hadamard inequality for Lipschitzian functions are provided as well.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by [14, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators $T: H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [14, p. 9]:

Theorem 1 (Equivalent norm)**.** *For any $T \in \mathcal{B}(H)$ one has*

$$w(T) \le \|T\| \le 2w(T). \tag{1.2}$$

Some improvements of (1.2) are as follows:

Theorem 2 (Kittaneh, 2003 [19])**.** *For any operator $T \in \mathcal{B}(H)$ we have the following refinement of the first inequality in (1.2)*

$$w(T) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.3}$$

From a different perspective, we have the following result as well:

Theorem 3 (Dragomir, 2007 [6])**.** *For any operator $T \in \mathcal{B}(H)$ we have*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{1.4}$$

The following general result for the product of two operators holds [14, p. 37]:

Theorem 4 (Holbrook, 1969 [16])**.** *If A, B are two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then $w(AB) \le 4w(A)w(B)$. In the case that $AB = BA$, then $w(AB) \le 2w(A)w(B)$. The constant 2 is best possible here.*

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The following results are also well known [14, p. 38].

Theorem 5 (Holbrook, 1969 [16]). *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \leq w(B). \tag{1.5}$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

For other results on numerical radius inequalities see [1], [3]-[7], [8]-[11], [13] and [17]-[23].

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$, then $f_a = f$.

In this paper we establish some Lipschitz type inequalities, i.e., bounds for the quantities

$$\|f(T) - f(V)\|, \ \ \|f(\|T\|)T - f(\|V\|)V\| \text{ and } w(f(T) - f(V)) \text{ etc.}$$

where $T, V \in \mathcal{B}(H)$, in terms of different values of f_a, $\|T - V\|$ and $w(T - V)$, respectively.

Applications in relation with the operator version of the Hermite-Hadamard inequality that compares the operators

$$\frac{f(U) + f(V)}{2}, \ \int_0^1 f((1-s)U + tV)\,ds \text{ and } f\left(\frac{U+V}{2}\right)$$

are also given. Some examples for particular functions of interest such as the exponential, logarithmic and trigonometric functions are also presented.

2. Norm Inequalities

We start with the following result that provides a quasi-Lipschitzian condition for functions defined by power series:

Theorem 6. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then*

$$\|f(T) - f(V)\| \leq f_a'(\max\{\|T\|, \|V\|\}) \|T - V\|. \tag{2.1}$$

Proof. We show first that the following power inequality holds true for any $n \in \mathbb{N}$

$$\|T^n - V^n\| \leq n (\max\{\|T\|, \|V\|\})^{n-1} \|T - V\|. \tag{2.2}$$

We prove this by induction. We observe that for $n = 0$ and $n = 1$ the inequality reduces to an equality.

Assume now that (2.2) is true for $k \in \mathbb{N}$, $k \geq 1$ and let us prove it for $k + 1$.

Utilising the properties of the operator norm, and the induction hypothesis, we have successively

$$\begin{aligned}\left\|T^{k+1}-V^{k+1}\right\| &= \left\|T^{k}\left(T-V\right)+\left(T^{k}-V^{k}\right)V\right\| \\ &\leq \left\|T^{k}\left(T-V\right)\right\|+\left\|\left(T^{k}-V^{k}\right)V\right\| \\ &\leq \left\|T^{k}\right\|\left\|T-V\right\|+\left\|T^{k}-V^{k}\right\|\left\|V\right\| \\ &\leq \left\|T\right\|^{k}\left\|T-V\right\|+k\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{k-1}\left\|T-V\right\|\left\|V\right\| \\ &\leq \max\left\{\left\|T\right\|^{k},\left\|V\right\|^{k}\right\}\left\|T-V\right\| \\ &\quad + k\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{k-1}\left\|T-V\right\|\max\left\{\left\|T\right\|,\left\|V\right\|\right\} \\ &= \left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{k}\left\|T-V\right\| \\ &\quad + k\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{k}\left\|T-V\right\| \\ &= \left(k+1\right)\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{k}\left\|T-V\right\|\end{aligned}$$

and the inequality (2.2) is proved.

Now, for any $m \geq 1$, by making use of the inequality (2.2) we have

$$\begin{aligned}\left\|\sum_{n=0}^{m}a_nT^n-\sum_{n=0}^{m}a_nV^n\right\| &\leq \sum_{n=0}^{m}\left|a_n\right|\left\|T^n-V^n\right\| \\ &\leq \left\|T-V\right\|\sum_{n=0}^{m}n\left|a_n\right|\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{n-1}.\end{aligned} \tag{2.3}$$

Since the series $\sum_{n=0}^{\infty}a_nT^n$, $\sum_{n=0}^{\infty}a_nV^n$ and $\sum_{n=0}^{\infty}n\left|a_n\right|\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{n-1}$ are convergent, then by letting $m\to\infty$ in (2.3) we get the inequality (2.1). □

Remark 1. *We observe, from the proof of the above theorem, that the inequality (2.1) remains valid in the more general setting of a Banach algebra. However the details are not considered here.*

We define the *absolute value* of an operator $A\in\mathcal{B}\left(H\right)$ defined as $\left|A\right|$ as the *square root operator* of the positive operator $A^{*}A$. With this notation, we have:

Corollary 1. *With the above assumptions for f, we have*

$$\left\|f\left(T\right)-f\left(T^{*}\right)\right\|\leq f_a'\left(\left\|T\right\|\right)\left\|T-T^{*}\right\| \tag{2.4}$$

if $T\in\mathcal{B}\left(H\right)$ with $\left\|T\right\|<R$ and

$$\left\|f\left(\left|N^{*}\right|^{2}\right)-f\left(\left|N\right|^{2}\right)\right\|\leq f_a'\left(\left\|N\right\|^{2}\right)\left\|\left|N^{*}\right|^{2}-\left|N\right|^{2}\right\| \tag{2.5}$$

if $N\in\mathcal{B}\left(H\right)$ with $\left\|N\right\|^{2}<R$.

Remark 2. *With the assumption of Theorem 6 we have*

$$\left\|f\left(\left|T\right|\right)-f\left(\left|V\right|\right)\right\|\leq f_a'\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)\left\|\left|T\right|-\left|V\right|\right\|$$

provided $\left\|T\right\|,\left\|V\right\|<R$, and in particular, the following power inequality that we will use below

$$\left\|\left|T\right|^{n}-\left|V\right|^{n}\right\|\leq n\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{n-1}\left\|\left|T\right|-\left|V\right|\right\| \tag{2.6}$$

that holds for any $T,V\in\mathcal{B}\left(H\right)$.

We notice that if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\ z\in D(0,1);\tag{2.7}$$
$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C};$$
$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ z\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z)=\sum_{n=1}^{\infty}\frac{1}{n!}z^n=\ln\frac{1}{1-z},\ z\in D(0,1);\tag{2.8}$$
$$g_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$
$$h_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$
$$l_a(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).$$

If $T,V\in\mathcal{B}(H)$ are such that $\|T\|,\|V\|<1$, then by (2.1) we have

$$\left\|(I\pm T)^{-1}-(I\pm V)^{-1}\right\|\le(1-\max\{\|T\|,\|V\|\})^{-2}\|T-V\|\tag{2.9}$$

and

$$\left\|\ln(I\pm T)^{-1}-\ln(I\pm V)^{-1}\right\|\le(1-\max\{\|T\|,\|V\|\})^{-1}\|T-V\|.\tag{2.10}$$

If $T,V\in\mathcal{B}(H)$, then by (2.1) we also have

$$\max\{\|\sin T-\sin V\|,\|\sinh T-\sinh V\|\}\le\cosh(\max\{\|T\|,\|V\|\})\|T-V\|\tag{2.11}$$

and

$$\max\{\|\cos T-\cos V\|,\|\cosh T-\cosh V\|\}\le\sinh(\max\{\|T\|,\|V\|\})\|T-V\|.\tag{2.12}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \ z \in \mathbb{C}; \tag{2.13}$$

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1);$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \ z \in D(0,1);$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1);$$

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0,$$
$$z \in D(0,1);$$

where Γ is *Gamma function*.

If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < 1$, then by (2.1) we have

$$\left\|\tanh^{-1} T - \tanh^{-1} V\right\| \leq \left[1 - (\max\{\|T\|, \|V\|\})^2\right]^{-1} \|T - V\| \tag{2.14}$$

and

$$\left\|\sin^{-1} T - \sin^{-1} V\right\| \leq \left[1 - (\max\{\|T\|, \|V\|\})^2\right]^{-1/2} \|T - V\|. \tag{2.15}$$

If $T, V \in \mathcal{B}(H)$, then by (2.1) we also have

$$\|\exp(T) - \exp(V)\| \leq \exp(\max\{\|T\|, \|V\|\}) \|T - V\|. \tag{2.16}$$

The following result also holds.

Theorem 7. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R), R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then*

$$\begin{aligned} &\|f(\|T\|)T - f(\|V\|)V\| \\ &\leq [f_a(\max\{\|T\|, \|V\|\}) + \max\{\|T\|, \|V\|\} f_a'(\max\{\|T\|, \|V\|\})] \\ &\times \|T - V\|. \end{aligned} \tag{2.17}$$

Proof. We show first that the following power inequality holds true for any $n \in \mathbb{N}$

$$\left\|\|T\|^n T - \|V\|^n V\right\| \leq (n+1)(\max\{\|T\|, \|V\|\})^n \|T - V\|. \tag{2.18}$$

For $n = 0$, the inequality becomes an equality.

Assume that $n \geq 1$, then we have

$$\begin{aligned} \left\|\|T\|^n T - \|V\|^n V\right\| &= \left\|\|T\|^n T - \|T\|^n V + \|T\|^n V - \|V\|^n V\right\| \\ &\leq \left\|\|T\|^n (T - V)\right\| + \left\|(\|T\|^n - \|V\|^n) V\right\| \\ &= \|T\|^n \|T - V\| + \left|\|T\|^n - \|V\|^n\right| \|V\| \\ &\leq (\max\{\|T\|, \|V\|\})^n \|T - V\| \\ &+ \left|\|T\|^n - \|V\|^n\right| \max\{\|T\|, \|V\|\}. \end{aligned} \tag{2.19}$$

On the other hand

$$\left|\|T\|^n - \|V\|^n\right| = \left|\|T\| - \|V\|\right| \left(\|T\|^{n-1} + ... + \|V\|^{n-1}\right) \leq n \|T - V\| \left(\max\{\|T\|, \|V\|\}\right)^{n-1}. \tag{2.20}$$

Using (2.19) and (2.20) we have

$$\begin{aligned} \left\|\|T\|^n T - \|V\|^n V\right\| &\leq \left(\max\{\|T\|, \|V\|\}\right)^n \|T - V\| \\ &+ n \|T - V\| \left(\max\{\|T\|, \|V\|\}\right)^n \\ &= (n+1) \left(\max\{\|T\|, \|V\|\}\right)^n \|T - V\| \end{aligned}$$

and the inequality (2.18) is proved.

Now, for any $m \geq 1$, by making use of the inequality (2.18) we have

$$\begin{aligned} &\left\|\left(\sum_{n=0}^{m} a_n \|T\|^n\right) T - \left(\sum_{n=0}^{m} a_n \|V\|^n\right) V\right\| \\ &\leq \sum_{n=0}^{m} |a_n| \left\|\|T\|^n T - \|V\|^n V\right\| \\ &\leq \|T - V\| \sum_{n=0}^{m} (n+1) |a_n| \left(\max\{\|T\|, \|V\|\}\right)^n \\ &= \|T - V\| \left(\sum_{n=0}^{m} |a_n| \left(\max\{\|T\|, \|V\|\}\right)^n \right. \\ &\left. + \sum_{n=0}^{m} n |a_n| \left(\max\{\|T\|, \|V\|\}\right)^n\right) \\ &= \|T - V\| \left(\sum_{n=0}^{m} |a_n| \left(\max\{\|T\|, \|V\|\}\right)^n \right. \\ &\left. + \sum_{n=1}^{m} n |a_n| \left(\max\{\|T\|, \|V\|\}\right)^n\right). \end{aligned} \tag{2.21}$$

Since the following series are convergent and

$$\sum_{n=0}^{\infty} a_n \|T\|^n = f(\|T\|), \quad \sum_{n=0}^{\infty} a_n \|V\|^n = f(\|V\|),$$

$$\sum_{n=0}^{\infty} |a_n| \left(\max\{\|T\|, \|V\|\}\right)^n = f_a\left(\max\{\|T\|, \|V\|\}\right)$$

and

$$\sum_{n=1}^{\infty} n |a_n| \left(\max\{\|T\|, \|V\|\}\right)^n = \max\{\|T\|, \|V\|\} f_a'\left(\max\{\|T\|, \|V\|\}\right)$$

then by letting $m \to \infty$ in (2.21) we deduce the desired result (2.17). □

Remark 3. *We observe that, since no norm inequality for the operator product has been used in the proof of the inequality (2.17), it holds true for any norm on* $\mathcal{B}(H)$.

Corollary 2. *Let f be as in Theorem 6, then*

$$\begin{aligned}(2.22)\qquad &\|f(\|AB\|)AB - f(\|BA\|)BA\| \\ &\leq \|AB - BA\|\left[f_a(\max\{\|AB\|,\|BA\|\}) \right. \\ &\left. + \max\{\|AB\|,\|BA\|\} f'_a(\max\{\|AB\|,\|BA\|\})\right] \\ &\leq \left[f_a(\|A\|\|B\|) + \|A\|\|B\| f'_a(\|A\|\|B\|)\right]\|AB - BA\|\end{aligned}$$

for any $A, B \in \mathcal{B}(H)$ with $\|A\|\|B\| < R$.

From a different perspective we have:

Theorem 8. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R), R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then*

$$\begin{aligned}(2.23)\qquad &\|f(|T|)T - f(|V|)V\| \\ &\leq f_a(\max\{\|T\|,\|V\|\})\|T - V\| + \max\{\|T\|,\|V\|\} \\ &\times f'_a(\max\{\|T\|,\|V\|\})\||T| - |V|\|.\end{aligned}$$

Proof. Observe that for $n \geq 1$, we have

$$\begin{aligned}(2.24)\qquad \||T|^n T - |V|^n V\| &= \||T|^n T - |T|^n V + |T|^n V - |V|^n V\| \\ &\leq \||T|^n (T - V)\| + \|(|T|^n - |V|^n)V\| \\ &\leq \|T\|^n \|T - V\| + \||T|^n - |V|^n\|\|V\| \\ &\leq (\max\{\|T\|,\|V\|\})^n \|T - V\| \\ &+ \||T|^n - |V|^n\| \max\{\|T\|,\|V\|\}.\end{aligned}$$

Since by (2.6) we have

$$(2.25)\qquad \||T|^n - |V|^n\| \leq n(\max\{\|T\|,\|V\|\})^{n-1}\||T| - |V|\|,$$

then by (2.24) and (2.25) we have

$$\begin{aligned}(2.26)\qquad \||T|^n T - |V|^n V\| &\leq (\max\{\|T\|,\|V\|\})^n \|T - V\| \\ &+ n(\max\{\|T\|,\|V\|\})^n \||T| - |V|\|.\end{aligned}$$

The inequality (2.26) also holds for $n = 0$, so by employing an argument similar to the one from the proof of Theorem 6 we deduce the desired result (2.23). □

Corollary 3. *With the assumptions of Theorem 8 for f we have*

$$\begin{aligned}(2.27)\qquad &\|f(|T|)T - f(|T^*|)T^*\| \\ &\leq f_a(\|T\|)\|T - T^*\| + \|T\| f'_a(\|T\|)\||T| - |T^*|\|.\end{aligned}$$

if $T \in \mathcal{B}(H)$ with $\|T\| < R$.

Proposition 1. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R), R > 0$. If $T \in \mathcal{B}(H)$ are such that $\|T\| < R$, then*

$$(2.28)\qquad \|f(T)T^* - f(T^*)T\| \leq \left[f_a(\|T\|) + \|T\| f'_a(\|T\|)\right]\|T - T^*\|.$$

Proof. For any $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\left\| T^n T^* - (T^*)^n T \right\| \\
&= \left\| T^n T^* - T^{n+1} + T^{n+1} - (T^*)^n T \right\| \\
&= \left\| T^n (T^* - T) + (T^n - (T^*)^n) T \right\| \\
&\le \left\| T^n (T^* - T) \right\| + \left\| (T^n - (T^*)^n) T \right\| \\
&\le \|T\|^n \|T^* - T\| + \|T^n - (T^*)^n\| \|T\| \\
&\le (\max\{\|T\|, \|T^*\|\})^n \|T^* - T\| + \|T^n - (T^*)^n\| \max\{\|T\|, \|T^*\|\} \\
&= \|T\|^n \|T^* - T\| + \|T\| \|T^n - (T^*)^n\| \\
&=: J
\end{aligned}$$

Utilising the inequality (2.4) for the power function we have

$$\|T^n - (T^*)^n\| \le n \|T\|^{n-1} \|T - T^*\| . \tag{2.29}$$

By (2.29) we have

$$\begin{aligned}
J &\le \|T\|^n \|T^* - T\| + \|T\| \|T^n - (T^*)^n\| \\
&\le \|T\|^n \|T^* - T\| + n \|T\|^n \|T - T^*\| \\
&= (n+1) \|T\|^n \|T^* - T\| .
\end{aligned}$$

Therefore

$$\|T^n T^* - (T^*)^n T\| \le (n+1) \|T\|^n \|T^* - T\| ,$$

for any $n \in \mathbb{N}$.

On making use of similar argument to that in Theorem 7 we obtain the desired result (2.28). □

Remark 4. *By employing the examples of power series presented before the reader may state other operator inequalities. However the details are omitted.*

3. Numerical Radius Inequalities

The following result holds:

Theorem 9. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $w(T), w(V) < R$, then*

$$w(f(T) - f(V)) \le 4 f_a' (\max\{w(T), w(V)\}) \, w(T - V) . \tag{3.1}$$

Moreover, if T and V are commutative, then we have a better inequality

$$w(f(T) - f(V)) \le 2 f_a' (\max\{w(T), w(V)\}) \, w(T - V) . \tag{3.2}$$

Proof. We show first that the following power inequality holds true for any $n \in \mathbb{N}$

$$w(T^n - V^n) \le 4n (\max\{w(T), w(V)\})^{n-1} w(T - V) . \tag{3.3}$$

We prove this by induction. We observe that for $n = 0$ and $n = 1$ the inequality reduces to an equality.

Assume now that (3.3) is true for $k \in \mathbb{N}$, $k \ge 1$ and let us prove it for $k + 1$.

Utilising the properties of the numerical radius, and the induction hypothesis, we have successively

$$
\begin{aligned}
w\left(T^{k+1}-V^{k+1}\right) &= w\left[T^{k}(T-V)+\left(T^{k}-V^{k}\right)V\right]\\
&\leq w\left[T^{k}(T-V)\right]+w\left[\left(T^{k}-V^{k}\right)V\right]\\
&\leq 4w\left(T^{k}\right)w(T-V)+4w\left(T^{k}-V^{k}\right)w(V)\\
&\leq 4w(T)^{k}w(T-V)\\
&+4k\left(\max\{w(T),w(V)\}\right)^{k-1}w(T-V)\,w(V)\\
&\leq 4\max\left\{w(T)^{k},w(V)^{k}\right\}w(T-V)\\
&+4k\left(\max\{w(T),w(V)\}\right)^{k-1}w(T-V)\max\{w(T),w(V)\}\\
&=4\left(\max\{w(T),w(V)\}\right)^{k}w(T-V)\\
&+4k\left(\max\{w(T),w(V)\}\right)^{k}w(T-V)\\
&=4(k+1)\left(\max\{w(T),w(V)\}\right)^{k}w(T-V)
\end{aligned}
$$

and the inequality (3.3) is proved.

Now, for any $m\geq 1$, by making use of the inequality (3.3) we have

$$
\begin{aligned}
&w\left(\sum_{n=0}^{m}a_{n}T^{n}-\sum_{n=0}^{m}a_{n}V^{n}\right)\\
&\leq\sum_{n=0}^{m}|a_{n}|\,w\left(T^{n}-V^{n}\right)\\
&\leq 4w(T-V)\sum_{n=0}^{m}n\,|a_{n}|\left(\max\{w(T),w(V)\}\right)^{n-1}.
\end{aligned}\tag{3.4}
$$

Since the series $\sum_{n=0}^{\infty}a_{n}T^{n}$, $\sum_{n=0}^{\infty}a_{n}V^{n}$ and $\sum_{n=0}^{\infty}n\,|a_{n}|\left(\max\{\|T\|,\|V\|\}\right)^{n-1}$ are convergent, then by letting $m\to\infty$ in (3.4) we get the inequality (3.1).

Now, if T and V are commutative, then T^{k} and $T-V$ as well as $T^{k}-V^{k}$ and V are commutative and we have

$$
\begin{aligned}
&w\left[T^{k}(T-V)\right]+w\left[\left(T^{k}-V^{k}\right)V\right]\\
&\leq 2w\left(T^{k}\right)w(T-V)+2w\left(T^{k}-V^{k}\right)w(V)
\end{aligned}
$$

and performing an argument similar with the one above, we deduce the inequality

$$
w\left(T^{n}-V^{n}\right)\leq 2n\left(\max\{w(T),w(V)\}\right)^{n-1}w(T-V), \tag{3.5}
$$

for any $n\in\mathbb{N}$.

This implies the desired inequality (3.2). The details are omitted. □

Taking into account Remark 3 we can state the following result for the numerical radius:

Proposition 2. *Let $f(z):=\sum_{n=0}^{\infty}a_{n}z^{n}$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $T,V\in\mathcal{B}(H)$ are such that*

$w(T), w(V) < R$, *then*

$$w(f(w(T))T - f(w(V))V) \leq [f_a(\max\{w(T), w(V)\}) + \max\{w(T), w(V)\} f_a'(\max\{w(T), w(V)\})] \times w(T-V). \tag{3.6}$$

For unitary commuting operators we have the following result of interest as well:

Proposition 3. *Let* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R), R > 0$. *If* $T, V \in \mathcal{B}(H)$ *are two commuting unitary operators and* $z \in D(0,R)$, *then*

$$w(f(zT) - f(zV)) \leq |z| f_a'(|z|) w(T-V). \tag{3.7}$$

Proof. We show first that the following power inequality holds true for any $n \in \mathbb{N}$

$$w(T^n - V^n) \leq nw(T-V). \tag{3.8}$$

We prove this by induction. We observe that for $n = 0$ and $n = 1$ the inequality reduces to an equality.

Assume now that (3.3) is true for $k \in \mathbb{N}$, $k \geq 1$ and let us prove it for $k+1$.

Utilising the properties of the numerical radius we have

$$\begin{aligned} w\left(T^{k+1} - V^{k+1}\right) &= w\left[T^k(T-V) + \left(T^k - V^k\right)V\right] \\ &\leq \left(w\left[T^k(T-V)\right] + w\left[\left(T^k - V^k\right)V\right]\right) := \Psi. \end{aligned}$$

Now, since T is a unitary operator, then T^k is also unitary for each $k \in \mathbb{N}$. Since T and V are commuting operators then T^k and $T - V$ are also commuting and by Theorem 5 we have

$$w\left[T^k(T-V)\right] \leq w(T-V),$$

for each $k \in \mathbb{N}$.

Since V is a unitary operator and $T^k - V^k$ and V are also commuting, then by the same Theorem 5 we have

$$w\left[\left(T^k - V^k\right)V\right] \leq w\left(T^k - V^k\right)$$

for each $k \in \mathbb{N}$.

Utilising these two inequalities and the induction hypothesis we have

$$\begin{aligned} \Psi &\leq w(T-V) + w\left(T^k - V^k\right) \leq w(T-V) + kw(T-V) \\ &= (k+1)w(T-V) \end{aligned}$$

and the inequality (3.8) is thus proved for each $n \in \mathbb{N}$.

Now, for any $m \geq 1$, by making use of the inequality (3.8) we have

$$\begin{aligned} w\left(\sum_{n=0}^{m} a_n z^n T^n - \sum_{n=0}^{m} a_n z^n V^n\right) &\leq \sum_{n=0}^{m} |a_n| |z|^n w(T^n - V^n) \\ &\leq w(T-V) \sum_{n=0}^{m} n |a_n| |z|^n \\ &= w(T-V) |z| \sum_{n=0}^{m} n |a_n| |z|^{n-1}. \end{aligned} \tag{3.9}$$

Since the series $\sum_{n=0}^{\infty} a_n z^n T^n, \sum_{n=0}^{\infty} a_n z^n V^n$ and $\sum_{n=0}^{\infty} n |a_n| |z|^{n-1}$ are convergent, then by letting $m \to \infty$ in (3.9) we get the inequality (3.7). □

Remark 5. *If in the above Proposition 3 we assume that $R>1$, then we have the inequality*

$$w\left(f\left(T\right)-f\left(V\right)\right)\leq f_{a}'\left(1\right)w\left(T-V\right)$$

for any two commuting unitary operators $T,V\in\mathcal{B}\left(H\right)$.

Remark 6. *We observe that, as particular cases of interest, we have for any two commuting unitary operators $T,V\in\mathcal{B}\left(H\right)$ that*

$$w\left(\left(I-uT\right)^{-1}-\left(I-uV\right)^{-1}\right)\leq\left|u\right|\left(1-\left|u\right|\right)^{-2}w\left(T-V\right)$$

for any $u\in D\left(0,1\right)$ and

$$w\left(\exp\left(vT\right)-\exp\left(vV\right)\right)\leq\left|v\right|\exp\left(\left|v\right|\right)w\left(T-V\right)$$

for any $v\in\mathbb{C}$.

We have also the following result that provides an upper bound for $w\left(f\left(zT\right)-f\left(zV\right)\right)$ in terms of $\left\|T-V\right\|$.

Theorem 10. *Let $f\left(z\right):=\sum_{n=0}^{\infty}a_{n}z^{n}$ be a power series with complex coefficients and convergent on the open disk $D\left(0,R\right),R>0$. If $T,V\in\mathcal{B}\left(H\right)\setminus\left\{0\right\}$ and $z\in\mathbb{C}$ are such that $\left\|T\right\|^{2},\left\|V\right\|^{2},\left|z\right|^{2}<R$, then*

$$\begin{aligned}&w\left(f\left(zT\right)-f\left(zV\right)\right)\\&\leq\frac{1}{2}\left\|T-V\right\|\left\{\left|z\right|f_{a}'\left(\left|z\right|\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)+\left[f_{a}'\left(\left|z\right|^{2}\right)\right]^{1/2}\right.\\&\times\left.\left[f_{a}'\left(\max\left\{\left\|T\right\|^{2},\left\|V\right\|^{2}\right\}\right)+\max\left\{\left\|T\right\|^{2},\left\|V\right\|^{2}\right\}f_{a}''\left(\max\left\{\left\|T\right\|^{2},\left\|V\right\|^{2}\right\}\right)\right]^{1/2}\right\}.\end{aligned}\tag{3.10}$$

Proof. Observe that, for any $m\geq1$, we have

$$w\left(\sum_{n=0}^{m}a_{n}z^{n}T^{n}-\sum_{n=0}^{m}a_{n}z^{n}V^{n}\right)\leq\sum_{n=0}^{m}\left|a_{n}\right|\left|z\right|^{n}w\left(T^{n}-V^{n}\right).\tag{3.11}$$

By Kittaneh's inequality (1.3) and by (2.2) we have

$$\begin{aligned}w\left(T^{n}-V^{n}\right)&\leq\frac{1}{2}\left[\left\|T^{n}-V^{n}\right\|+\left\|\left(T^{n}-V^{n}\right)^{2}\right\|^{1/2}\right]\\&\leq\frac{1}{2}\left[n\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{n-1}\left\|T-V\right\|\right.\\&\quad\left.+\left\|\left(T^{n}-V^{n}\right)^{2}\right\|^{1/2}\right]\end{aligned}\tag{3.12}$$

for any $n\geq0$.

If we multiply the inequality (3.12) by $\left|a_{n}\right|\left|z\right|^{n}\geq0$ and sum over n from 0 to m, we have

$$\begin{aligned}&\sum_{n=0}^{m}\left|a_{n}\right|\left|z\right|^{n}w\left(T^{n}-V^{n}\right)\\&\leq\frac{1}{2}\left\|T-V\right\|\left|z\right|\sum_{n=0}^{m}n\left|a_{n}\right|\left|z\right|^{n-1}\left(\max\left\{\left\|T\right\|,\left\|V\right\|\right\}\right)^{n-1}\\&+\frac{1}{2}\sum_{n=0}^{m}\left|a_{n}\right|\left|z\right|^{n}\left\|\left(T^{n}-V^{n}\right)^{2}\right\|^{1/2}\end{aligned}\tag{3.13}$$

for any $m \geq 1$.

If we use the Cauchy-Bunyakovsky-Schwarz weighted inequality we have

$$\sum_{n=0}^{m} |a_n| |z|^n \left\| (T^n - V^n)^2 \right\|^{1/2} \tag{3.14}$$
$$\leq \left(\sum_{n=0}^{m} |a_n| |z|^{2n} \right)^{1/2} \left(\sum_{n=0}^{m} |a_n| \left\| (T^n - V^n)^2 \right\| \right)^{1/2}$$
$$\leq \left(\sum_{n=0}^{m} |a_n| |z|^{2n} \right)^{1/2} \left(\sum_{n=0}^{m} |a_n| \|T^n - V^n\|^2 \right)^{1/2}$$
$$\leq \left(\sum_{n=0}^{m} |a_n| |z|^{2n} \right)^{1/2} \left(\|T - V\|^2 \sum_{n=0}^{m} n^2 |a_n| (\max\{\|T\|, \|V\|\})^{2(n-1)} \right)^{1/2}$$

where for the last inequality, we also used (2.2).

Consequently, by (3.11)-(3.14) we have

$$w\left(\sum_{n=0}^{m} a_n z^n T^n - \sum_{n=0}^{m} a_n z^n V^n \right) \tag{3.15}$$
$$\leq \frac{1}{2} \|T - V\| \left[|z| \sum_{n=0}^{m} n |a_n| |z|^{n-1} (\max\{\|T\|, \|V\|\})^{n-1} \right.$$
$$\left. + \left(\sum_{n=0}^{m} |a_n| |z|^{2n} \right)^{1/2} \left(\sum_{n=0}^{m} n^2 |a_n| (\max\{\|T\|, \|V\|\})^{2(n-1)} \right)^{1/2} \right]$$

for any $m \geq 1$.

Since $\|T\|^2, \|V\|^2, |z|^2 < R$ then the series

$$\sum_{n=0}^{\infty} n |a_n| |z|^{n-1} (\max\{\|T\|, \|V\|\})^{n-1}$$

and

$$\sum_{n=0}^{\infty} n^2 |a_n| (\max\{\|T\|, \|V\|\})^{2(n-1)}$$

are convergent and, as above

$$\sum_{n=0}^{\infty} n |a_n| |z|^{n-1} (\max\{\|T\|, \|V\|\})^{n-1} = f'(|z| \max\{\|T\|, \|V\|\}).$$

We need to find the representation for the second series.

Consider, for $u \neq 0$, the series

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^{n-1} = \frac{1}{u} \sum_{n=0}^{\infty} n^2 \alpha_n u^n.$$

If we denote $g(u) := \sum_{n=0}^{\infty} \alpha_n u^n$, then

$$u g'(u) = \sum_{n=0}^{\infty} n \alpha_n u^n$$

and

$$u\left(ug'\left(u\right)\right)' = \sum_{n=0}^{\infty} n^2 \alpha_n u^n.$$

However

$$u\left(ug'\left(u\right)\right)' = ug'\left(u\right) + u^2 g''\left(u\right)$$

and then

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^{n-1} = g'\left(u\right) + ug''\left(u\right)$$

for $u \neq 0$.

Utilising these calculations we can state that

$$\sum_{n=0}^{\infty} n^2 \left|a_n\right| \left(\max\left\{\|T\|, \|V\|\right\}\right)^{2(n-1)} = f_a'\left(\max\left\{\|T\|^2, \|V\|^2\right\}\right) + \max\left\{\|T\|^2, \|V\|^2\right\} f_a''\left(\max\left\{\|T\|^2, \|V\|^2\right\}\right)$$

for $T, V \neq 0$.

Finally, by taking $m \to \infty$ in (3.15) we deduce the desired result (3.10). □

4. Applications for Hermite-Hadamard Type Inequalities

The following result is well known in the Theory of Inequalities as the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(t\right) dt \leq \frac{f\left(a\right) + f\left(b\right)}{2}$$

for any convex function $f : [a, b] \to \mathbb{R}$.

The distance between the middle and the left term for Lipschitzian functions with the constant $L > 0$ has been estimated in [12] to be

$$\left|\frac{1}{b-a}\int_a^b f\left(t\right) dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{4} L\left(b-a\right) \tag{4.1}$$

while the distance between the right term and the middle term satisfies the inequality [21]

$$\left|\frac{f\left(a\right) + f\left(b\right)}{2} - \frac{1}{b-a}\int_a^b f\left(t\right) dt\right| \leq \frac{1}{4} L\left(b-a\right). \tag{4.2}$$

In order to extend these results to functions of operators we need the following lemma that is of interest in itself as well:

Lemma 1. *Let $f : \mathcal{C} \subset \mathcal{B}\left(H\right) \to \mathcal{B}\left(H\right)$ be an L-Lipschitzian function on the convex set $\mathcal{C}$, i.e. it satisfies*

$$\left\|f\left(U\right) - f\left(V\right)\right\| \leq L\left\|U - V\right\| \text{ for any } U, V \in \mathcal{C}.$$

For $U, V \in \mathcal{C}$, define the function $\varphi_{U,V} : [0, 1] \to \mathcal{B}\left(H\right)$ by

$$\begin{aligned}\varphi_{U,V}\left(t\right) &:= \frac{1}{2}\left[f\left(\left(1-t\right)U + t\frac{U+V}{2}\right) + f\left(t\frac{U+V}{2} + \left(1-t\right)V\right)\right] \\ &= \frac{1}{2}\left[f\left(\left(1-\frac{t}{2}\right)U + \frac{t}{2}V\right) + f\left(\frac{t}{2}U + \left(1-\frac{t}{2}\right)V\right)\right].\end{aligned}$$

Then for any $t_1, t_2 \in [0,1]$ *we have the inequality*

$$\left\|\varphi_{U,V}(t_2) - \varphi_{U,V}(t_1)\right\| \leq \frac{1}{2} L \|U - V\| |t_2 - t_1|, \tag{4.3}$$

i.e., the function $\varphi_{U,V}$ *is Lipschitzian with the constant* $\frac{1}{2} L \|U - V\|$.

In particular, we have the inequalities

$$\left\| f\left(\frac{U+V}{2}\right) - \varphi_{U,V}(t)\right\| \leq \frac{1}{2} L \|U - V\| (1 - t), \tag{4.4}$$

$$\left\| \frac{f(U) + f(V)}{2} - \varphi_{U,V}(t)\right\| \leq \frac{1}{2} L \|U - V\| t \tag{4.5}$$

and

$$\left\| \frac{1}{2}\left[f\left(\frac{3U+V}{2}\right) + f\left(\frac{U+3V}{2}\right)\right] - \varphi_{U,V}(t)\right\| \leq \frac{1}{2} L \|U - V\| \left| t - \frac{1}{2}\right| \tag{4.6}$$

for any $t \in [0,1]$.

Proof. We have

$$\begin{aligned}
&\left\|\varphi_{U,V}(t_2) - \varphi_{U,V}(t_1)\right\| \\
&= \frac{1}{2}\left\| f\left((1-t_2)U + t_2\frac{U+V}{2}\right) + f\left(t_2\frac{U+V}{2} + (1-t_2)V\right)\right. \\
&\left. - f\left((1-t_1)U + t_1\frac{U+V}{2}\right) - f\left(t_1\frac{U+V}{2} + (1-t_1)V\right)\right\| \\
&\leq \frac{1}{2}\left\| f\left((1-t_2)U + t_2\frac{U+V}{2}\right) - f\left((1-t_1)U + t_1\frac{U+V}{2}\right)\right\| \\
&+ \left\| f\left(t_2\frac{U+V}{2} + (1-t_2)V\right) - f\left((1-t_1)U + t_1\frac{U+V}{2}\right)\right\| \\
&\leq \frac{1}{2} L \left\|(1-t_2)U + t_2\frac{U+V}{2} - (1-t_1)U - t_1\frac{U+V}{2}\right\| \\
&+ \frac{1}{2} L \left\| t_2\frac{U+V}{2} + (1-t_2)V - (1-t_1)U - t_1\frac{U+V}{2}\right\| \\
&= \frac{1}{4} L \|U - V\| |t_2 - t_1| + \frac{1}{4} L \|U - V\| |t_2 - t_1| = \frac{1}{2} L \|U - V\| |t_2 - t_1|
\end{aligned}$$

for any $t_1, t_2 \in [0,1]$, which proves (4.3). □

We can prove now the following Hermite-Hadamard type inequalities for Lipschitzian functions of operators.

Theorem 11. *Let* $f : \mathcal{C} \subset \mathcal{B}(H) \to \mathcal{B}(H)$ *be an* L *Lipschitzian function on the convex set* $\mathcal{C}$. *Then we have the inequalities*

$$\left\| f\left(\frac{U+V}{2}\right) - \int_0^1 f((1-s)U + sV)\, dt\right\| \leq \frac{1}{4} L \|U - V\|, \tag{4.7}$$

$$\left\| \frac{f(U) + f(V)}{2} - \int_0^1 f((1-s)U + tV)\, ds\right\| \leq \frac{1}{4} L \|U - V\| \tag{4.8}$$

and

$$\left\| \frac{1}{2}\left[f\left(\frac{3U+V}{2}\right) + f\left(\frac{U+3V}{2}\right) \right] - \int_0^1 f\left((1-s)U + sV \right) ds \right\| \tag{4.9}$$
$$\leq \frac{1}{8} L \left\| U - V \right\| .$$

Proof. First, observe that $f:\mathcal{C}\subset\mathcal{B}(H)\to\mathcal{B}(H)$ is continuous in the norm topology of $\mathcal{B}(H)$, therefore the integral $\int_0^1 f((1-t)U+tV)\,dt$ exists for any $U,V\in\mathcal{C}$.

Utilising the inequality (4.4) and the norm inequality for norm, we have

$$\left\| f\left(\frac{U+V}{2}\right) - \int_0^1 \varphi_{U,V}(t)\,dt \right\| \leq \int_0^1 \left\| f\left(\frac{U+V}{2}\right) - \varphi_{U,V}(t) \right\| dt \tag{4.10}$$
$$\leq \frac{1}{2} L \left\| U-V \right\| \int_0^1 (1-t)\,dt$$
$$= \frac{1}{4} L \left\| U - V \right\|$$

for any $U,V\in\mathcal{C}$.

By the definition of $\varphi_{U,V}$ we have

$$\int_0^1 \varphi_{U,V}(t)\,dt$$
$$= \frac{1}{2}\left[\int_0^1 f\left((1-t)U + t\frac{U+V}{2} \right) dt + \int_0^1 f\left(t\frac{U+V}{2} + (1-t)V \right) dt \right].$$

Now, using the change of variable $t=2s$ we have

$$\frac{1}{2}\int_0^1 f\left((1-t)U + t\frac{U+V}{2}\right) dt = \int_0^{1/2} f\left((1-s)U+sV\right) ds$$

and by the change of variable $t = 1-v$ we have

$$\frac{1}{2}\int_0^1 f\left(t\frac{U+V}{2} + (1-t)V \right) dt = \frac{1}{2}\int_0^1 f\left((1-v)\frac{U+V}{2} + vV \right) dv.$$

Moreover, if we make the change of variable $v=2s-1$ we also have

$$\frac{1}{2}\int_0^1 f\left((1-v)\frac{U+V}{2} + vV \right) dv = \int_{1/2}^1 f\left((1-s)U+sV\right) ds.$$

Therefore

$$\int_0^1 \varphi_{U,V}(t)\,dt = \int_0^{1/2} f\left((1-s)U+sV\right) dt + \int_{1/2}^1 f\left((1-s)U+sV\right) ds$$
$$= \int_0^1 f\left((1-s)U+sV\right) dt$$

and by (4.10) we deduce (4.7).

The other inequalities (4.8) and (4.9) follow in a similar way and the details are omitted. □

Remark 7. *The inequalities (4.7) and (4.8) provide operator generalizations for the known inequalities (4.1) and (4.2). Moreover the technique employed above by utilizing the auxiliary function* $\varphi_{U,V}$ *is different from the original proofs in* [12] *and*

[21]. *It also offers an unifying tool which allows to obtain a better approximation of the integral* $\int_0^1 f((1-s)U+sV)\,ds$ *as provided by the inequality (4.9).*

Corollary 4. *Let* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R>0$. *If* $U,V \in \mathcal{B}(H)$ *are such that* $\|U\|, \|V\| \le M < R$, *then*

$$\left\| f\left(\frac{U+V}{2}\right) - \int_0^1 f((1-s)U+sV)\,ds \right\| \le \frac{1}{4} f_a'(M)\|U-V\|, \tag{4.11}$$

$$\left\| \frac{f(U)+f(V)}{2} - \int_0^1 f((1-s)U+tV)\,ds \right\| \le \frac{1}{4} f_a'(M)\|U-V\| \tag{4.12}$$

and

$$\begin{aligned}\left\| \frac{1}{2}\left[f\left(\frac{3U+V}{2}\right) + f\left(\frac{U+3V}{2}\right)\right] - \int_0^1 f((1-s)U+sV)\,ds \right\| \\ \le \frac{1}{8} f_a'(M)\|U-V\|.\end{aligned} \tag{4.13}$$

Corollary 5. *With the assumptions of Corollary 4 we have*

$$\begin{aligned}&\left\| f\left(\left\|\frac{U+V}{2}\right\|\right)\frac{U+V}{2} \right. \\ &\qquad \left. - \int_0^1 f(\|(1-s)U+sV\|)((1-s)U+sV)\,ds \right\| \\ &\qquad\qquad \le \frac{1}{4}\left[f_a(M) + Mf_a'(M)\right]\|U-V\|,\end{aligned} \tag{4.14}$$

$$\begin{aligned}&\left\| \frac{1}{2}\left[f(\|U\|)U + f(\|V\|)V\right] \right. \\ &\qquad \left. - \int_0^1 f(\|(1-s)U+sV\|)((1-s)U+sV)\,ds \right\| \\ &\qquad\qquad \le \frac{1}{4}\left[f_a(M) + Mf_a'(M)\right]\|U-V\|\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&\left\| \frac{1}{2}\left[f\left(\left\|\frac{3U+V}{2}\right\|\right)\frac{3U+V}{2} + f\left(\left\|\frac{U+3V}{2}\right\|\right)\frac{U+3V}{2}\right] \right. \\ &\qquad \left. - \int_0^1 f(\|(1-s)U+sV\|)((1-s)U+sV)\,ds \right\| \\ &\qquad\qquad \le \frac{1}{8}\left[f_a(M) + Mf_a'(M)\right]\|U-V\|\end{aligned} \tag{4.16}$$

for $U,V \in \mathcal{B}(H)$ *such that* $\|U\|, \|V\| \le M < R$.

Remark 8. *Similar results may be stated if one uses the numerical radius inequalities obtained above. However the details are omitted.*

It is known that if U and V are commuting operators, then the *operator exponential function* $\exp : \mathcal{B}(H) \to \mathcal{B}(H)$ given by

$$\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if A is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tA)\, dt = A^{-1}[\exp(bA) - \exp(aA)].$$

Proposition 4. *Let U and V be commuting operators with $\|U\|, \|V\| \le M$ and such that $V-U$ is invertible. Then we have the inequalities*

$$\left\| \exp\left(\frac{U+V}{2}\right) - (V-U)^{-1}[\exp(V) - \exp(U)] \right\| \le \frac{1}{4}\|U-V\|\exp(M), \tag{4.17}$$

$$\left\| \frac{\exp(U)+\exp(V)}{2} - (V-U)^{-1}[\exp(V) - \exp(U)] \right\| \le \frac{1}{4}\|U-V\|\exp(M) \tag{4.18}$$

and

$$\left\| \frac{1}{2}\left[\exp\left(\frac{3U+V}{2}\right) + \exp\left(\frac{U+3V}{2}\right)\right] - (V-U)^{-1}[\exp(V) - \exp(U)] \right\| \le \frac{1}{8}\|U-V\|\exp(M). \tag{4.19}$$

Proof. Follows by Corollary 4 on observing that

$$\begin{aligned}
\int_0^1 \exp((1-s)U + sV)\, ds &= \int_0^1 \exp(s(V-U))\exp(U)\, ds \\
&= \left(\int_0^1 \exp(s(V-U))\, ds\right)\exp(U) \\
&= (V-U)^{-1}[\exp(V-U) - I]\exp(U) \\
&= (V-U)^{-1}[\exp(V) - \exp(U)].
\end{aligned}$$

□

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