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INEQUALITIES OF LIPSCHITZ TYPE FOR POWER SERIES IN BANACH ALGEBRAS

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Abstract. Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$, a Banach algebra, with $\|x\|,\|y\|<R$ we show among others that

$$\|f(y)-f(x)\|\le\|y-x\|\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt$$

where $f_a(z)=\sum_{n=0}^{\infty}|\alpha_n|z^n$.

Inequalities for the commutator such as

$$\|f(x)f(y)-f(y)f(x)\|\le 2f_a(M)f_a'(M)\|y-x\|,$$

if $\|x\|,\|y\|\le M<R$, as well as some inequalities of Hermite-Hadamard type are also provided.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further:

$$\|ab\|\le\|a\|\,\|b\|$$

for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a,b\in$Inv$\mathcal{B}$ then $ab\in$Inv$\mathcal{B}$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in$Inv$\mathcal{B}$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset$Inv$\mathcal{B}$;
(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map Inv$\mathcal{B}\ni a\longmapsto a^{-1}\in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\text{Inv}\mathcal{B}\};$$

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the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \text{Inv}\mathcal{B}$,

$$R_a(z) := (z-a)^{-1}.$$

For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

If a, b are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \le \nu(a)\nu(b) \text{ and } \nu(a+b) \le \nu(a) + \nu(b).$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have

$$a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi;$$

(v) We have

$$\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0,R)$ given by the *power series*

$$f(z) := \sum_{j=0}^{\infty} \alpha_j z^j \ \ (|z| < R).$$

If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1-a\|<1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b=-\sum_{n=1}^{\infty}\frac{1}{n}(1-a)^{n},$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b)=a$.

In this paper we establish some upper bounds for the following quantities

$$\|f(y)-f(x)\|,\ \|f(x)f(y)-f(y)f(x)\|,$$

$$\left\|f\left(\frac{x+y}{2}\right)-\int_0^1 f((1-s)x+sy)\,ds\right\|$$

and

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-s)x+sy)\,ds\right\|$$

that can naturally be associated with the analytic functions $f(z):=\sum_{j=0}^{\infty}\alpha_j z^j$ defined on the open disk $D(0,R)$ and the elements x and y of the unital Banach algebra $\mathcal{B}$. Some applications for functions of interest such as the exponential map on $\mathcal{B}$ are provided as well.

2. Lipschitz Type Inequalities

Now, by the help of power series $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z):=\sum_{n=0}^{\infty}|\alpha_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n\geq 0$, then $f_a=f$.

The following result is valid.

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ we have*

$$\|f(y)-f(x)\|\leq\|y-x\|\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt. \tag{2.1}$$

Proof. We use the identity (see for instance [1, p. 254])

$$a^n-b^n=\sum_{j=0}^{n-1}a^{n-1-j}(a-b)b^j \tag{2.2}$$

that holds for any $a,b\in\mathcal{B}$ and $n\geq 1$.

For $x,y\in\mathcal{B}$ we consider the function $\varphi:[0,1]\to\mathcal{B}$ defined by $\varphi(t)=[(1-t)x+ty]^n$. For $t\in(0,1)$ and $\varepsilon\neq 0$ with $t+\varepsilon\in(0,1)$ we have from (2.2) that

$$\begin{aligned}\varphi(t+\varepsilon)-\varphi(t)&=[(1-t-\varepsilon)x+(t+\varepsilon)y]^n-[(1-t)x+ty]^n\\&=\varepsilon\sum_{j=0}^{n-1}[(1-t-\varepsilon)x+(t+\varepsilon)y]^{n-1-j}(y-x)[(1-t)x+ty]^j\end{aligned}$$

Dividing with $\varepsilon \neq 0$ and taking the limit over $\varepsilon \to 0$ we have in the norm topology of $\mathcal{B}$ that

$$\varphi'(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\varphi(t+\varepsilon)-\varphi(t)\right] = \sum_{j=0}^{n-1}\left[(1-t)x+ty\right]^{n-1-j}(y-x)\left[(1-t)x+ty\right]^{j}. \tag{2.3}$$

Integrating on $[0,1]$ we get from (2.3) that

$$\int_0^1 \varphi'(t)\,dt = \sum_{j=0}^{n-1}\int_0^1 \left[(1-t)x+ty\right]^{n-1-j}(y-x)\left[(1-t)x+ty\right]^{j}dt$$

and since

$$\int_0^1 \varphi'(t)\,dt = \varphi(1)-\varphi(0) = y^n - x^n$$

then we get the following equality of interest

$$y^n - x^n = \sum_{j=0}^{n-1}\int_0^1 \left[(1-t)x+ty\right]^{n-1-j}(y-x)\left[(1-t)x+ty\right]^{j}dt$$

for any $x,y\in\mathcal{B}$ and $n\geq 1$.

Taking the norm and utilising the properties of Bochner integral for vector valued functions (see for instance [2, p. 21]) we have

$$\begin{aligned}\|y^n-x^n\| &\leq \sum_{j=0}^{n-1}\left\|\int_0^1 \left[(1-t)x+ty\right]^{n-1-j}(y-x)\left[(1-t)x+ty\right]^{j}dt\right\| \\ &\leq \sum_{j=0}^{n-1}\int_0^1 \left\|\left[(1-t)x+ty\right]^{n-1-j}(y-x)\left[(1-t)x+ty\right]^{j}\right\|dt \\ &\leq \sum_{j=0}^{n-1}\int_0^1 \left\|\left[(1-t)x+ty\right]^{n-1-j}\right\|\|y-x\|\left\|\left[(1-t)x+ty\right]^{j}\right\|dt \\ &\leq \sum_{j=0}^{n-1}\int_0^1 \|(1-t)x+ty\|^{n-1-j}\|y-x\|\|(1-t)x+ty\|^{j}dt \\ &= n\|y-x\|\int_0^1 \|(1-t)x+ty\|^{n-1}dt\end{aligned} \tag{2.4}$$

for any $x,y\in\mathcal{B}$ and $n\geq 1$.

Now, for any $m \geq 1$, by making use of the inequality (2.2) we have

$$\begin{aligned}(2.5)\quad \left\|\sum_{n=0}^{m}\alpha_n y^n - \sum_{n=0}^{m}\alpha_n x^n\right\| &= \left\|\sum_{n=1}^{m}\alpha_n\left(y^n - x^n\right)\right\| \\ &\leq \sum_{n=1}^{m}\left|\alpha_n\right|\left\|y^n - x^n\right\| \\ &\leq \left\|y-x\right\|\sum_{n=1}^{m} n\left|\alpha_n\right|\int_0^1 \left\|(1-t)x+ty\right\|^{n-1}dt \\ &= \left\|y-x\right\|\int_0^1\left(\sum_{n=1}^{m} n\left|\alpha_n\right|\left\|(1-t)x+ty\right\|^{n-1}\right)dt.\end{aligned}$$

Moreover, since $\|x\|, \|y\| < R$, then the series $\sum_{n=0}^{\infty}\alpha_n y^n$, $\sum_{n=0}^{\infty}\alpha_n x^n$ and

$$\sum_{n=1}^{\infty} n\left|\alpha_n\right|\left\|(1-t)x+ty\right\|^{n-1}$$

are convergent and

$$\sum_{n=0}^{\infty}\alpha_n y^n = f(y), \sum_{n=0}^{\infty}\alpha_n x^n = f(x)$$

while

$$\sum_{n=1}^{\infty} n\left|\alpha_n\right|\left\|(1-t)x+ty\right\|^{n-1} = f_a'\left(\left\|(1-t)x+ty\right\|\right).$$

Therefore, by taking the limit over $m \to \infty$ in the inequality (2.4) we deduce the desired result (2.1). □

Remark 1. *We observe that* f_a' *is monotonic nondecreasing and convex on the interval* $[0, R)$ *and since the function* $\psi(t) := \|(1-t)x+ty\|$ *is convex on* $[0,1]$ *we have that* $f_a' \circ \psi$ *is also convex on* $[0,1]$. *Utilising the Hermite-Hadamard inequality for convex functions (see for instance* [4, p. 2]*) we have the sequence of inequalities*

$$\begin{aligned}(2.6)\quad \left\|f(y)-f(x)\right\| &\leq \left\|y-x\right\|\int_0^1 f_a'\left(\left\|(1-t)x+ty\right\|\right)dt \\ &\leq \frac{1}{2}\left\|y-x\right\|\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right] \\ &\leq \frac{1}{2}\left\|y-x\right\|\left[f_a'(\|x\|)+f_a'(\|y\|)\right] \\ &\leq \left\|y-x\right\|\max\left\{f_a'(\|x\|), f_a'(\|y\|)\right\}.\end{aligned}$$

We also have

$$\|f(y)-f(x)\| \le \|y-x\| \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \tag{2.7}$$
$$\le \|y-x\| \int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt$$
$$\le \frac{1}{2}\|y-x\| \left[f_a'\left(\frac{\|x\|+\|y\|}{2}\right) + \frac{f_a'(\|x\|)+f_a'(\|y\|)}{2} \right]$$
$$\le \frac{1}{2}\|y-x\| \left[f_a'(\|x\|)+f_a'(\|y\|) \right]$$
$$\le \|y-x\| \max\{f_a'(\|x\|), f_a'(\|y\|)\}.$$

We observe that if $\|y\| \neq \|x\|$, *then by the change of variable* $s=(1-t)\|x\|+t\|y\|$ *we have*

$$\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt = \frac{1}{\|y\|-\|x\|}\int_{\|x\|}^{\|y\|} f_a'(s)\,ds = \frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|}.$$

If $\|y\| = \|x\|$, *then*

$$\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt = f_a'(\|x\|).$$

Utilising these observations we then get the following divided difference inequality for $x \neq y$

$$\frac{\|f(y)-f(x)\|}{\|y-x\|} \le \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \le \begin{cases} \frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\| \neq \|x\|, \\ f_a'(\|x\|) & \text{if } \|y\| = \|x\|. \end{cases} \tag{2.8}$$

If $\|x\|, \|y\| \le M < R$, *then from either of the inequalities (2.6) or (2.7) we have the Lipschitz type inequality*

$$\|f(y)-f(x)\| \le f_a'(M)\|y-x\|. \tag{2.9}$$

Remark 2. *We observe that the integral* $\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt$, *which might be difficult to compute in various examples of Banach algebras, has got the simpler bounds*

$$B_1(x,y) := \frac{1}{2}\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right) + \frac{f_a'(\|x\|)+f_a'(\|y\|)}{2} \right]$$

and

$$B_2(x,y) := \begin{cases} \frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\| \neq \|x\|, \\ f_a'(\|x\|) & \text{if } \|y\| = \|x\|. \end{cases}$$

It is natural then to ask which of these bounds is better?

Let us consider the simple examples of powers, namely $f(z)=z^m$ *with* $m \ge 1$. *Then*

$$B_1(x,y) = \frac{1}{2}m\left[\left\|\frac{x+y}{2}\right\|^{m-1} + \frac{\|x\|^{m-1}+\|y\|^{m-1}}{2} \right]$$

and

$$B_2(x,y) := \begin{cases} \|y\|^{m-1} + \|y\|^{m-2}\|x\| + ... + \|x\|^{m-1} & \text{if } \|y\| \neq \|x\|, \\ m\|x\|^{m-1} & \text{if } \|y\| = \|x\|. \end{cases}$$

If we take $y = tx$ *with* $\|x\| = 1$ *and* $|t| \neq 1$ *then we get*

$$B_1(t) = \frac{1}{2}m\left[\left|\frac{1+t}{2}\right|^{m-1} + \frac{1+|t|^{m-1}}{2}\right]$$

and

$$B_2(t) = |t|^{m-1} + ... + |t| + 1.$$

If we take $m = 4$ *and plot the difference*

$$d(t) := 2\left(\left|\frac{t+1}{2}\right|^3 + \frac{1+|t|^3}{2}\right) - \left(|t|^3 + |t|^2 + |t| + 1\right)$$

on the interval $[-8, 8]$*, then we can conclude that some time the first bound is better than the second, while other time the conclusion is the other way around.*

The plot for the function d *is depicted in the Figure 1 below:*

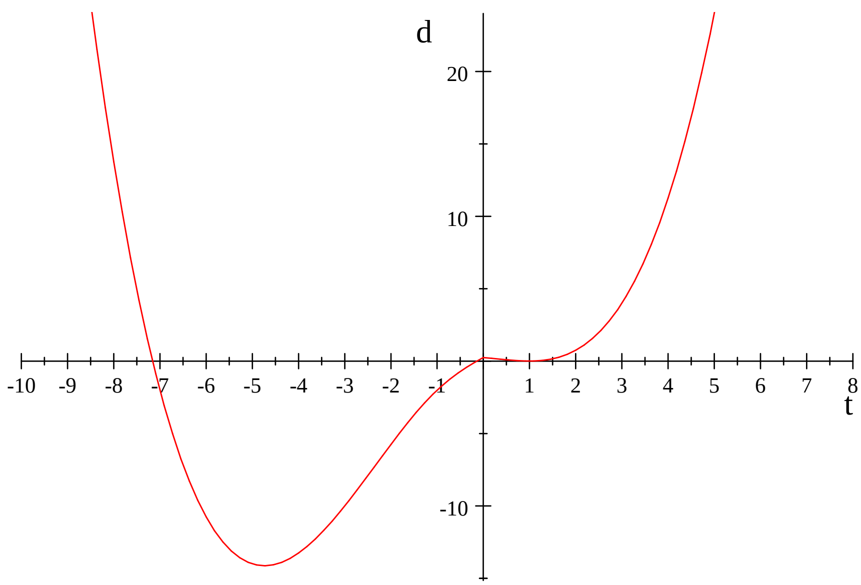

Figure 1: Variation of the difference $d(t)$ *for* $t \in [-8, 8]$.

It is natural now to consider some examples of interest.

If we consider the exponential function $\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^{n}$, then for any $x,y\in\mathcal{B}$ we have the inequalities

$$\|\exp(y)-\exp(x)\|\leq\|y-x\|\int_{0}^{1}\exp(\|(1-t)x+ty\|)\,dt \tag{2.10}$$
$$\leq\|y-x\|\begin{cases}\frac{1}{2}\left[\exp\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{\exp(\|x\|)+\exp(\|y\|)}{2}\right],\\ \\ \frac{\exp(\|y\|)-\exp(\|x\|)}{\|y\|-\|x\|}\ \text{ if }\ \|y\|\neq\|x\|,\\ \\ \exp(\|x\|)\ \text{ if }\ \|y\|=\|x\|.\end{cases}$$

Now, if we consider the functions $(1-z)^{-1}=\sum_{n=0}^{\infty}z^{n}$ and $(1+z)^{-1}=\sum_{n=0}^{\infty}(-1)^{n}z^{n}$, then for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<1$ we have the inequalities

$$\left\|(1\pm y)^{-1}-(1\pm x)^{-1}\right\| \tag{2.11}$$
$$\leq\|y-x\|\int_{0}^{1}(1-\|(1-t)x+ty\|)^{-2}\,dt$$
$$\leq\|y-x\|\begin{cases}\frac{1}{2}\left[\left(1-\left\|\frac{x+y}{2}\right\|\right)^{-2}+\frac{(1-\|x\|)^{-2}+(1-\|y\|)^{-2}}{2}\right],\\ \\ (1-\|y\|)^{-1}(1-\|x\|)^{-1}\ \text{ if }\ \|y\|\neq\|x\|,\\ \\ (1-\|x\|)^{-2}\ \text{ if }\ \|y\|=\|x\|.\end{cases}$$

3. Inequalities for Commutators

By the use of Lipschitz type inequalities obtained before we can establish some upper bounds for the *commutator*

$$f(x)g(y)-g(y)f(x)$$

where $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$.

Theorem 2. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_{n}z^{n}$ and $g(z)=\sum_{n=0}^{\infty}\beta_{n}z^{n}$ be two functions defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ we have*

$$\|f(x)g(y)-g(y)f(x)\| \tag{3.1}$$
$$\leq\|y-x\|\left[\min\{f_{a}(\|x\|),f_{a}(\|y\|)\}\int_{0}^{1}g_{a}'(\|(1-t)x+ty\|)\,dt\right.$$
$$\left.+\min\{g_{a}(\|x\|),g_{a}(\|y\|)\}\int_{0}^{1}f_{a}'(\|(1-t)x+ty\|)\,dt\right]$$
$$\leq\|y-x\|\left[\frac{f_{a}(\|x\|)+f_{a}(\|y\|)}{2}\int_{0}^{1}g_{a}'(\|(1-t)x+ty\|)\,dt\right.$$
$$\left.+\frac{g_{a}(\|x\|)+g_{a}(\|y\|)}{2}\int_{0}^{1}f_{a}'(\|(1-t)x+ty\|)\,dt\right].$$

Proof. Let $n,m\in\mathbb{N}$. Then we have

$$\begin{aligned}x^{n}y^{m}-y^{n}x^{m}&=x^{n}y^{m}-x^{n}x^{m}+x^{n}x^{m}-y^{n}x^{m}\\&=x^{n}(y^{m}-x^{m})+(x^{n}-y^{n})x^{m}.\end{aligned}$$

Utilising the properties of the norm, we have

$$\begin{aligned}\|x^n y^m - y^n x^m\| &\le \|x^n (y^m - x^m)\| + \|(x^n - y^n) x^m\| \\ &\le \|x^n\| \|y^m - x^m\| + \|x^n - y^n\| \|x^m\| \\ &\le \|x\|^n \|y^m - x^m\| + \|x\|^m \|x^n - y^n\|\end{aligned}$$

for any $n, m \in \mathbb{N}$.

We also have

$$x^n y^m - y^n x^m = y^n (y^m - x^m) + (x^n - y^n) y^m$$

which gives

$$\|x^n y^m - y^n x^m\| \le \|y\|^n \|y^m - x^m\| + \|y\|^m \|x^n - y^n\|$$

for any $n, m \in \mathbb{N}$.

Therefore

$$\begin{aligned}\|x^n y^m - y^n x^m\| &\le \min\{\|x\|^n, \|y\|^n\} \|y^m - x^m\| \\ &\quad + \min\{\|x\|^m, \|y\|^m\} \|y^n - x^n\|\end{aligned} \tag{3.2}$$

for any $n, m \in \mathbb{N}$.

For any $k \ge 1$ we then have

$$\begin{aligned}&\left\|\sum_{n=0}^{k} \alpha_n x^n \sum_{m=0}^{k} \beta_m y^m - \sum_{n=0}^{k} \alpha_n y^n \sum_{m=0}^{k} \beta_m x^m\right\| \\ &= \left\|\sum_{n=0}^{k} \sum_{m=0}^{k} \alpha_n \beta_m (x^n y^m - y^n x^m)\right\| \\ &\le \sum_{n=0}^{k} \sum_{m=0}^{k} |\alpha_n| |\beta_m| \|x^n y^m - y^n x^m\| \\ &\le \sum_{n=0}^{k} |\alpha_n| \min\{\|x\|^n, \|y\|^n\} \sum_{m=0}^{k} |\beta_m| \|y^m - x^m\| \\ &+ \sum_{n=0}^{k} |\beta_m| \min\{\|x\|^m, \|y\|^m\} \sum_{m=0}^{k} |\alpha_n| \|y^n - x^n\| \\ &\le \min\left\{\sum_{n=0}^{k} |\alpha_n| \|x\|^n, \sum_{n=0}^{k} |\alpha_n| \|y\|^n\right\} \sum_{m=0}^{k} |\beta_m| \|y^m - x^m\| \\ &+ \min\left\{\sum_{n=0}^{k} |\beta_m| \|x\|^m, \sum_{n=0}^{k} |\beta_m| \|y\|^m\right\} \sum_{m=0}^{k} |\alpha_n| \|y^n - x^n\|\end{aligned} \tag{3.3}$$

for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$.

From (2.4) we have

$$\begin{aligned}\sum_{m=0}^{k} |\beta_m| \|y^m - x^m\| &= \sum_{m=1}^{k} |\beta_m| \|y^m - x^m\| \\ &\le \|y - x\| \int_0^1 \left(\sum_{m=1}^{k} m |\beta_m| \|(1-t)x + ty\|^{m-1}\right) dt\end{aligned} \tag{3.4}$$

and

$$\sum_{m=0}^{k}|\alpha_n|\|y^n-x^n\| \le \|y-x\|\int_0^1\left(\sum_{n=1}^{k} n|\beta_n|\|(1-t)x+ty\|^{n-1}\right)dt \tag{3.5}$$

for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$.

From (3.3)-(3.5) we have

$$\left\|\sum_{n=0}^{k}\alpha_n x^n\sum_{m=0}^{k}\beta_m y^m-\sum_{n=0}^{k}\alpha_n y^n\sum_{m=0}^{k}\beta_m x^m\right\| \le \|y-x\| \tag{3.6}$$
$$\times\left\{\min\left\{\sum_{n=0}^{k}|\alpha_n|\|x\|^n,\sum_{n=0}^{k}|\alpha_n|\|y\|^n\right\}\int_0^1\left(\sum_{m=1}^{k} m|\beta_m|\|(1-t)x+ty\|^{m-1}\right)dt\right.$$
$$\left.+\min\left\{\sum_{n=0}^{k}|\beta_m|\|x\|^m,\sum_{n=0}^{k}|\beta_m|\|y\|^m\right\}\int_0^1\left(\sum_{n=1}^{k} n|\beta_n|\|(1-t)x+ty\|^{n-1}\right)dt\right\}$$

for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ and $k\ge 1$.

Since all the series whose partial sums are involved in (3.6) are convergent, then by letting $m\to\infty$ in (3.6) we deduce the first inequality in (3.1).

The second inequality is obvious. □

Remark 3. *If $g=f$ in (3.1), then we have the following sequence of inequalities*

$$\begin{aligned}
&\|f(x)f(y)-f(y)f(x)\| \\
&\le 2\|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\}\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \\
&\le \|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\} \\
&\times\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right] \\
&\le \|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\}[f_a'(\|x\|)+f_a'(\|y\|)] \\
&\le 2\|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\}\max\{f_a'(\|x\|),f_a'(\|y\|)\}
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
&\|f(x)f(y)-f(y)f(x)\| \\
&\le 2\|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\}\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt \\
&\le 2\|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\}\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt \\
&\le \|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\} \\
&\times\left[f_a'\left(\frac{\|x\|+\|y\|}{2}\right)+\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right] \\
&\le \|y-x\|\min\{f_a(\|x\|),f_a(\|y\|)\}[f_a'(\|x\|)+f_a'(\|y\|)] \\
&\le 2\|y-x\|\min\{f_a'(\|x\|),f_a'(\|y\|)\}\max\{f_a'(\|x\|),f_a'(\|y\|)\}.
\end{aligned}\tag{3.8}$$

for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$.

Since

$$\min\{f_a(\|x\|),f_a(\|y\|)\}\le\frac{f_a(\|x\|)+f_a(\|y\|)}{2}$$

and

$$\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt=\begin{cases}\frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\|\neq\|x\|,\\[2ex] f_a'(\|x\|) & \text{if } \|y\|=\|x\|,\end{cases}$$

then by the second inequality in (3.8) we have for $x\neq y$ that

$$\begin{aligned}&\frac{\|f(x)f(y)-f(y)f(x)\|}{\|y-x\|}\\ &\leq 2\min\{f_a(\|x\|),f_a(\|y\|)\}\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt\\ &\leq\begin{cases}\frac{f_a^2(\|y\|)-f_a^2(\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\|\neq\|x\|,\\[2ex] 2f_a(\|x\|)f_a'(\|x\|) & \text{if } \|y\|=\|x\|.\end{cases}\end{aligned}\tag{3.9}$$

If $\|x\|,\|y\|\leq M<R$, then we can state the simpler inequality

$$\|f(x)f(y)-f(y)f(x)\|\leq 2f_a(M)f_a'(M)\|y-x\|.\tag{3.10}$$

Now, if for instance we use the first part of the inequality (3.7) for the exponential function, then we get

$$\begin{aligned}&\|\exp(x)\exp(y)-\exp(y)\exp(x)\|\\ &\leq 2\|y-x\|\min\{\exp(\|x\|),\exp(\|y\|)\}\int_0^1\exp(\|(1-t)x+ty\|)\,dt\\ &\leq\|y-x\|\min\{\exp(\|x\|),\exp(\|y\|)\}\\ &\times\left[\exp\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{\exp(\|x\|)+\exp(\|y\|)}{2}\right]\end{aligned}\tag{3.11}$$

while from the first part of (3.12) we have

$$\begin{aligned}&\|\exp(x)\exp(y)-\exp(y)\exp(x)\|\\ &\leq 2\|y-x\|\min\{\exp(\|x\|),\exp(\|y\|)\}\int_0^1\exp(\|(1-t)x+ty\|)\,dt\\ &\leq\|y-x\|\begin{cases}\frac{\exp(2\|y\|)-\exp(2\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\|\neq\|x\|,\\[2ex] 2\exp(2\|x\|) & \text{if } \|y\|=\|x\|,\end{cases}\end{aligned}\tag{3.12}$$

for any $x,y\in\mathcal{B}$.

4. Applications for Hermite-Hadamard Type Inequalities

The following result is well known in the Theory of Inequalities as the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}$$

for any *convex function* $f:[a,b]\to\mathbb{R}$. For numerous results related to this inequality, see the monograph [4].

The distance between the middle and the left term for *Lipschitzian functions* with the constant $L>0$ has been estimated in [3] to be

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{4}L(b-a) \tag{4.1}$$

while the distance between the right term and the middle term satisfies the inequality [5]

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4}L(b-a). \tag{4.2}$$

In the following, we use the inequality

$$\|f(y)-f(x)\| \le \frac{1}{2}\|y-x\|\left[f_a'(\|x\|)+f_a'(\|y\|)\right] \tag{4.3}$$

to derive some simple refinements of the Hermite-Hadamard type inequalities (4.1) and (4.2) for power series of elements in a unital Banach algebra.

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ we have*

$$\begin{aligned}
&\left\|f\left(\frac{x+y}{2}\right)-\int_0^1 f((1-s)x+sy)\,ds\right\| \\
&\le \frac{1}{2}\|y-x\|\left[\frac{1}{4}f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\int_0^1\left|s-\frac{1}{2}\right|f_a'(\|(1-s)x+sy\|)\,ds\right] \\
&\le \frac{1}{8}\|y-x\|\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{1}{2}\left[f_a'(\|x\|)+f_a'(\|y\|)\right]\right] \\
&\le \frac{1}{8}\|y-x\|\left[f_a'(\|x\|)+f_a'(\|y\|)\right] \le \frac{1}{4}\|y-x\|\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}.
\end{aligned} \tag{4.4}$$

Proof. From the inequality (4.3) we have

$$\begin{aligned}
&\left\|f\left(\frac{x+y}{2}\right)-f((1-s)x+sy)\right\| \\
&\le \frac{1}{2}\left|s-\frac{1}{2}\right|\|y-x\|\left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)+f_a'(\|(1-s)x+sy\|)\right]
\end{aligned} \tag{4.5}$$

for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ and $s\in[0,1]$.

Integrating on $[0,1]$ we have

$$\begin{aligned}(4.6)\quad &\left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f\left((1-s)x+sy\right)ds \right\| \\ &\le \int_0^1 \left\| f\left(\frac{x+y}{2}\right) - f\left((1-s)x+sy\right)\right\| ds \\ &\le \frac{1}{2}\|y-x\| \\ &\times \left[f_a'\left(\left\|\frac{x+y}{2}\right\|\right)\int_0^1 \left|s-\frac{1}{2}\right| ds + \int_0^1 \left|s-\frac{1}{2}\right| f_a'\left(\|(1-s)x+sy\|\right)ds\right] \\ &= \frac{1}{2}\|y-x\| \\ &\times \left[\frac{1}{4} f_a'\left(\left\|\frac{x+y}{2}\right\|\right) + \int_0^1 \left|s-\frac{1}{2}\right| f_a'\left(\|(1-s)x+sy\|\right)ds\right]\end{aligned}$$

and the first inequality in (4.4) is proved.

Since the function $g(s) := f_a'(\|(1-s)x+sy\|)$ is convex on $[0,1]$, then we have

$$\begin{aligned}(4.7)\quad &\int_0^1 \left|s-\frac{1}{2}\right| f_a'\left(\|(1-s)x+sy\|\right)ds \\ &\le \int_0^1 \left|s-\frac{1}{2}\right| \left[(1-s) f_a'(\|x\|) + s f_a'(\|y\|)\right] ds \\ &= f_a'(\|x\|)\int_0^1 \left|s-\frac{1}{2}\right|(1-s)\,ds + f_a'(\|y\|)\int_0^1 \left|s-\frac{1}{2}\right| s ds.\end{aligned}$$

Since

$$\int_0^1 \left|s-\frac{1}{2}\right|(1-s)\,ds = \int_0^1 \left|s-\frac{1}{2}\right| s ds = \frac{1}{8},$$

then by (4.7) we have

$$\int_0^1 \left|s-\frac{1}{2}\right| f_a'\left(\|(1-s)x+sy\|\right)ds \le \frac{1}{8}\left[f_a'(\|x\|) + f_a'(\|y\|)\right]$$

and the second inequality in (4.4) is proved.

The last part is obvious. □

Remark 4. *If $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \le M < R$, then by the inequality (4.4) we have*

$$(4.8)\quad \left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f\left((1-s)x+sy\right)ds \right\| \le \frac{1}{4} f_a'(M)\|y-x\|.$$

The trapezoidal version is as follows:

Theorem 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. For any*

$x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ we have

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-s)x+sy)\,ds\right\| \tag{4.9}$$
$$\le\frac{1}{8}\|y-x\|\left[\frac{f_a'(\|x\|)+f_a'(\|y\|)}{2}\right.$$
$$\left.+\int_0^1 s\left[f_a'\left(\left\|(1-s)x+s\frac{x+y}{2}\right\|\right)+f_a'\left(\left\|s\frac{x+y}{2}+(1-s)y\right\|\right)\right]ds\right.$$
$$\le\frac{1}{4}\|y-x\|\left[\frac{f_a'(\|x\|)+f_a'(\|y\|)}{3}+\frac{1}{3}f_a'\left(\left\|\frac{x+y}{2}\right\|\right)\right]$$
$$\le\frac{1}{8}\|y-x\|\left[f_a'(\|x\|)+f_a'(\|y\|)\right]\le\frac{1}{4}\|y-x\|\max\left\{f_a'(\|x\|),f_a'(\|y\|)\right\}.$$

Proof. From the inequality (4.3) we have

$$\left\|f(x)-f\left((1-s)x+s\frac{x+y}{2}\right)\right\| \tag{4.10}$$
$$\le\frac{1}{4}s\|y-x\|\left[f_a'(\|x\|)+f_a'\left(\left\|(1-s)x+s\frac{x+y}{2}\right\|\right)\right]$$

and

$$\left\|f(y)-f\left(s\frac{x+y}{2}+(1-s)y\right)\right\| \tag{4.11}$$
$$\le\frac{1}{4}s\|y-x\|\left[f_a'(\|y\|)+f_a'\left(\left\|s\frac{x+y}{2}+(1-s)y\right\|\right)\right]$$

for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ and $s\in[0,1]$.

Utilising the triangle inequality, (4.10) and (4.11) we have

$$\left\|\frac{f(x)+f(y)}{2}-\frac{1}{2}\left[f\left((1-s)x+s\frac{x+y}{2}\right)+f\left(s\frac{x+y}{2}+(1-s)y\right)\right]\right\| \tag{4.12}$$
$$\le\frac{1}{2}\left\|f(x)-f\left((1-s)x+s\frac{x+y}{2}\right)\right\|$$
$$+\frac{1}{2}\left\|f(y)-f\left(s\frac{x+y}{2}+(1-s)y\right)\right\|$$
$$\le\frac{1}{8}s\|y-x\|\left[f_a'(\|x\|)+f_a'\left(\left\|(1-s)x+s\frac{x+y}{2}\right\|\right)\right]$$
$$+\frac{1}{8}s\|y-x\|\left[f_a'(\|y\|)+f_a'\left(\left\|s\frac{x+y}{2}+(1-s)y\right\|\right)\right]$$

for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ and $s\in[0,1]$.

Integrating on $[0,1]$ we have

$$\left\| \frac{f(x)+f(y)}{2} - \frac{1}{2}\int_0^1 \left[f\left((1-s)x + s\frac{x+y}{2}\right) + f\left(s\frac{x+y}{2} + (1-s)y\right)\right] ds \right\| \tag{4.13}$$
$$\le \int_0^1 \left\| \frac{f(x)+f(y)}{2} - \frac{1}{2}\left[f\left((1-s)x + s\frac{x+y}{2}\right) + f\left(s\frac{x+y}{2} + (1-s)y\right)\right] \right\| ds$$
$$\le \frac{1}{8}\|y-x\| \left[\frac{f_a'(\|x\|) + f_a'(\|y\|)}{2} + \int_0^1 s\left[f_a'\left(\left\|(1-s)x + s\frac{x+y}{2}\right\|\right) + f_a'\left(\left\|s\frac{x+y}{2} + (1-s)y\right\|\right)\right] ds \right] =: J.$$

By the convexity of the functions $h(s) := f_a'\left(\left\|(1-s)x + s\frac{x+y}{2}\right\|\right)$ and $k(s) := f_a'\left(\left\|s\frac{x+y}{2} + (1-s)y\right\|\right)$ on the interval $[0,1]$ we have

$$\int_0^1 s\left[f_a'\left(\left\|(1-s)x + s\frac{x+y}{2}\right\|\right) + f_a'\left(\left\|s\frac{x+y}{2} + (1-s)y\right\|\right)\right] ds \tag{4.14}$$
$$\le \int_0^1 s\left[(1-s) f_a'(\|x\|) + s f_a'\left(\left\|\frac{x+y}{2}\right\|\right)\right] ds + \int_0^1 s\left[s f_a'\left(\left\|\frac{x+y}{2}\right\|\right) + (1-s) f_a'(\|y\|)\right] ds$$
$$= f_a'(\|x\|)\int_0^1 s(1-s)\,ds + 2f_a'\left(\left\|\frac{x+y}{2}\right\|\right)\int_0^1 s^2 ds + f_a'(\|y\|)\int_0^1 s(1-s)\,ds$$
$$= \frac{1}{6}\left[f_a'(\|x\|) + f_a'(\|y\|)\right] + \frac{2}{3} f_a'\left(\left\|\frac{x+y}{2}\right\|\right).$$

Therefore

$$J \le \frac{1}{4}\|y-x\| \left[\frac{f_a'(\|x\|) + f_a'(\|y\|)}{3} + \frac{1}{3} f_a'\left(\left\|\frac{x+y}{2}\right\|\right)\right]. \tag{4.15}$$

Now, using the change of variable $t = 2s$ we have

$$\frac{1}{2}\int_0^1 f\left((1-t)x + t\frac{x+y}{2}\right) dt = \int_0^{1/2} f((1-s)x + sy)\,ds$$

and by the change of variable $t = 1 - v$ we have

$$\frac{1}{2}\int_0^1 f\left(t\frac{x+y}{2} + (1-t)x\right) dt = \frac{1}{2}\int_0^1 f\left((1-v)\frac{x+y}{2} + vy\right) dv.$$

Moreover, if we make the change of variable $v = 2s - 1$ we also have

$$\frac{1}{2}\int_0^1 f\left((1-v)\frac{x+y}{2} + vy\right) dv = \int_{1/2}^1 f((1-s)x + sy)\,ds.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[f\left((1-s)\,x+s\frac{x+y}{2}\right)+f\left(s\frac{x+y}{2}+(1-s)\,y\right)\right]ds\\
&=\int_0^{1/2} f\left((1-s)\,x+sy\right)dt+\int_{1/2}^1 f\left((1-s)\,x+sy\right)ds\\
&=\int_0^1 f\left((1-s)\,x+sy\right)dt.
\end{aligned}$$

Utilising (4.13) and (4.15) we deduce the first two inequalities in (4.9). The rest is obvious. □

Remark 5. *If $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \le M < R$, then by the inequality (4.9) we have*

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f\left((1-s)\,x+sy\right)ds\right\| \le \frac{1}{4}f_a'(M)\,\|y-x\|. \tag{4.16}$$

For any $x, y \in \mathcal{B}$ we have the following sequence of midpoint inequalities for the exponential function

$$\begin{aligned}
&\left\|\exp\left(\frac{x+y}{2}\right)-\int_0^1 \exp\left((1-s)\,x+sy\right)ds\right\| \\
&\le \frac{1}{2}\|y-x\|\left[\frac{1}{4}\exp\left(\left\|\frac{x+y}{2}\right\|\right)+\int_0^1\left|s-\frac{1}{2}\right|\exp\left(\|(1-s)\,x+sy\|\right)ds\right]\\
&\le \frac{1}{8}\|y-x\|\left[\exp\left(\left\|\frac{x+y}{2}\right\|\right)+\frac{1}{2}\left[\exp\left(\|x\|\right)+\exp\left(\|y\|\right)\right]\right]\\
&\le \frac{1}{8}\|y-x\|\left[\exp\left(\|x\|\right)+\exp\left(\|y\|\right)\right]\\
&\le \frac{1}{4}\|y-x\|\max\left\{\exp\left(\|x\|\right),\exp\left(\|y\|\right)\right\}.
\end{aligned} \tag{4.17}$$

We also have the trapezoid type inequalities

$$\begin{aligned}
&\left\|\frac{\exp(x)+\exp(y)}{2}-\int_0^1 \exp\left((1-s)\,x+sy\right)ds\right\| \\
&\le \frac{1}{8}\|y-x\|\left[\frac{\exp\left(\|x\|\right)+\exp\left(\|y\|\right)}{2}\right.\\
&\left.+\int_0^1 s\left[\exp\left(\left\|(1-s)\,x+s\frac{x+y}{2}\right\|\right)+\exp\left(\left\|s\frac{x+y}{2}+(1-s)\,y\right\|\right)\right]ds\right.\\
&\le \frac{1}{4}\|y-x\|\left[\frac{\exp\left(\|x\|\right)+\exp\left(\|y\|\right)}{3}+\frac{1}{3}\exp\left(\left\|\frac{x+y}{2}\right\|\right)\right]\\
&\le \frac{1}{8}\|y-x\|\left[\exp\left(\|x\|\right)+\exp\left(\|y\|\right)\right]\\
&\le \frac{1}{4}\|y-x\|\max\left\{\exp\left(\|x\|\right),\exp\left(\|y\|\right)\right\}.
\end{aligned} \tag{4.18}$$

It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y)=\exp(y)\exp(x)=\exp(x+y).$$

Also, if x is invertible and $a,b\in\mathbb{R}$ with $a<b$ then

$$\int_a^b \exp(tx)\,dt = x^{-1}\left[\exp(bx)-\exp(ax)\right].$$

Moreover, if x and y are commuting and $y-x$ is invertible, then

$$\begin{aligned}\int_0^1 \exp\left((1-s)x+sy\right)ds &= \int_0^1 \exp\left(s(y-x)\right)\exp(x)\,ds\\ &= \left(\int_0^1 \exp\left(s(y-x)\right)ds\right)\exp(x)\\ &= (y-x)^{-1}\left[\exp(y-x)-I\right]\exp(x)\\ &= (y-x)^{-1}\left[\exp(y)-\exp(x)\right].\end{aligned}$$

In this case the first term in (4.17) may be replaced by

$$\left\|\exp\left(\frac{x+y}{2}\right)-(y-x)^{-1}\left[\exp(y)-\exp(x)\right]\right\|$$

while the first term in (4.18), by

$$\left\|\frac{\exp(x)+\exp(y)}{2}-(y-x)^{-1}\left[\exp(y)-\exp(x)\right]\right\|.$$

The interested reader may apply the above inequalities to other important functions such as $(1-z)^{-1}=\sum_{n=0}^{\infty} z^n$ and $(1+z)^{-1}=\sum_{n=0}^{\infty}(-1)^n z^n$ defined on $D(0,1)$. However, the details are omitted.

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